

Chapter 16

Multiple Integration

16.1 Double Integrals over Rectangular Regions

16.1.1 $\int_0^2 \int_1^3 xy \, dy \, dx$ or $\int_1^3 \int_0^2 xy \, dx \, dy$.

16.1.2 Two solutions: $\int_0^5 \int_{-2}^4 10 \, dy \, dx$ or $\int_{-2}^4 \int_0^5 10 \, dx \, dy$.

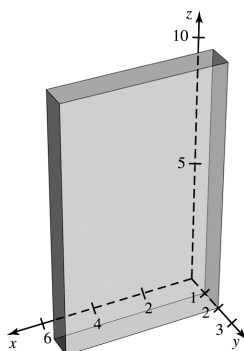
16.1.3 With respect to x first: $\int_1^5 \int_{-2}^4 f(x, y) \, dx \, dy$; with respect to y first: $\int_{-2}^4 \int_1^5 f(x, y) \, dy \, dx$.

16.1.4 y is the first (inner) variable and has limits $-1 \leq y \leq 1$, x is the second (outer) variable and has limits $1 \leq x \leq 3$.

16.1.5 Note that $\Delta A_k = 4$. We have a value for the Riemann sum of

$$4(f(1, 1) + f(1, 3) + f(1, 5) + f(3, 5) + f(3, 3) + f(3, 1)) = 4(0.5 + 1.5 + 2.5 + 3.5 + 2.5 + 1.5) = 48.$$

16.1.6 $V = \int_0^6 \int_1^2 10 \, dy \, dx = \int_0^6 (10y) \Big|_1^2 \, dx = \int_0^6 10 \, dx = (10x) \Big|_0^6 = 60$.



16.1.7 $\int_0^2 \int_0^1 (4xy) \, dx \, dy = \int_0^2 (2x^2y) \Big|_0^1 \, dy = \int_0^2 (2y) \, dy = (y^2) \Big|_0^2 = 4$.

16.1.8 $\int_1^2 \int_0^1 (3x^2 + 4y^3) \, dy \, dx = \int_1^2 (3x^2y + y^4) \Big|_0^1 \, dx = \int_1^2 (3x^2 + 1) \, dx = (x^3 + x) \Big|_1^2 = 10 - 2 = 8$.

16.1.9 $\int_1^3 \int_0^2 x^2 y \, dx \, dy = \int_1^3 \left(\frac{x^3 y}{3} \right) \Big|_0^2 \, dy = \frac{8}{3} \int_1^3 y \, dy = \frac{8}{3} \left(\frac{y^2}{2} \right) \Big|_1^3 = \frac{32}{3}$.

$$16.1.10 \int_0^3 \int_{-2}^1 (2x + 3y) \, dx \, dy = \int_0^3 (x^2 + 3xy) \Big|_{-2}^1 \, dy = \int_0^3 (9y - 3) \, dy = \left(\frac{9}{2}y^2 - 3y \right) \Big|_0^3 = \frac{63}{2}.$$

$$16.1.11 \int_1^3 \int_0^{\pi/2} x \sin y \, dy \, dx = \int_1^3 (-x \cos y) \Big|_0^{\pi/2} \, dx = \int_1^3 x \, dx = \left(\frac{x^2}{2} \right) \Big|_1^3 = 4.$$

$$16.1.12 \int_1^3 \int_1^2 (y^2 + y) \, dx \, dy = \int_1^3 (xy^2 + xy) \Big|_1^2 \, dy = \int_1^3 (y^2 + y) \, dy = \left(\frac{y^3}{3} + \frac{y^2}{2} \right) \Big|_1^3 = \frac{38}{3}.$$

$$16.1.13 \int_1^4 \int_0^4 \sqrt{uv} \, du \, dv = \int_1^4 \left(\frac{2}{3} u^{3/2} v^{1/2} \right) \Big|_{u=0}^{u=4} \, dv = \int_1^4 \left(\frac{16}{3} v^{1/2} \right) \, dv = \left(\frac{32}{9} v^{3/2} \right) \Big|_{v=1}^{v=4} = \frac{224}{9}.$$

$$16.1.14 \int_0^{\pi/4} \int_0^3 (\sec \theta) r \, dr \, d\theta = \int_0^{\pi/4} \left((\sec \theta) \frac{r^2}{2} \right) \Big|_{r=0}^{r=3} \, d\theta = \int_0^{\pi/4} \left(\frac{9}{2} \sec \theta \right) \, d\theta = \left(\frac{9}{2} \cdot \ln \left| \sec \theta + \tan \theta \right| \right) \Big|_{\theta=0}^{\theta=\pi/4} = \frac{9}{2} \left(\ln \left| \sqrt{2} + 1 \right| - \ln \left| 1 + 0 \right| \right) = \frac{9}{2} \ln (\sqrt{2} + 1).$$

$$16.1.15 \int_1^{\ln 5} \int_0^{\ln 3} e^{x+y} \, dx \, dy = \int_1^{\ln 5} (e^x) (e^y) \, dx \, dy = \int_1^{\ln 5} (e^x \cdot e^y) \Big|_0^{\ln 3} \, dy = \int_1^{\ln 5} (2e^y) \, dy = (2e^y) \Big|_1^{\ln 5} = 10 - 2e.$$

$$16.1.16 \text{ Let } w = u^2v. \text{ Then } dw = 2uv \, du. \text{ Then we have } \frac{1}{2} \int_0^{\pi/2} \int_0^v \cos w \, dw \, dv = \frac{1}{2} \int_0^{\pi/2} \sin v \, dv = -\frac{1}{2} (\cos \pi/2 - \cos 0) = \frac{1}{2}.$$

$$16.1.17 \text{ We have } \int_0^1 te^{st} \Big|_{s=0}^{s=1} \, dt = \int_0^1 (te^t - t) \, dt = \left(e^t(t-1) - \frac{t^2}{2} \right) \Big|_0^1 = -\frac{1}{2} - (-1) = \frac{1}{2}, \text{ where we use the fact that an antiderivative of } te^t \text{ is } e^t(t-1) \text{ which follows via integration by parts.}$$

$$16.1.18 \text{ This is equal to } \int_0^2 4y \tan^{-1} x^2 \Big|_{x=0}^{x=1} \, dy = \int_0^2 \pi y \, dy = \frac{\pi y^2}{2} \Big|_0^2 = 2\pi.$$

$$16.1.19 \text{ Note that } \int 4(p+q) \ln q \, dp = 4 \left(\frac{p^2}{2} + pq \right) \ln q + C, \text{ so we have } \int_1^e \int_0^1 4(p+q) \ln q \, dp \, dq = \int_1^e (2+4q) \ln q \, dq. \text{ Using integration by parts, we see that an antiderivative of } q \ln q \text{ is } \frac{q^2 \ln q}{2} - \frac{q^2}{4}, \text{ so our integral is equal to } (2q \ln q - 2q + 2q^2 \ln q - q^2) \Big|_1^e = e^2 + 3.$$

$$16.1.20 \text{ This is equal to } \int_0^1 y \sin xy \Big|_{x=0}^{x=\pi} \, dy = \int_0^1 y \sin \pi y \, dy = \left(\frac{\sin \pi y}{\pi^2} - \frac{y \cos \pi y}{\pi} \right) \Big|_0^1 = \frac{1}{\pi}.$$

$$16.1.21 \int_1^2 \int_1^2 \frac{x}{x+y} \, dy \, dx = \int_1^2 (x \ln(x+y)) \Big|_1^2 \, dx = \int_1^2 (x \ln(x+2) - x \ln(x+1)) \, dx = \left(-\frac{1}{2} x^2 \ln(x+1) + \frac{1}{2} x^2 \ln(x+2) + \frac{x}{2} + \frac{1}{2} \ln(x+1) - 2 \ln(x+2) \right) \Big|_1^2 = -2 \ln 3 + 2 \ln 4 + 1 + \frac{\ln 3}{2} - 2 \ln 4 - \left(-\frac{\ln 2}{2} + \frac{\ln 3}{2} + \frac{1}{2} + \frac{\ln 2}{2} - 2 \ln 3 \right) = 1 - \frac{1}{2} = \frac{1}{2}.$$

$$\mathbf{16.1.22} \quad \int_0^2 \int_0^1 x^5 y^2 e^{x^3 y^3} dy dx = \int_0^2 \left(\frac{x^5}{3x^3} e^{x^3 y^3} \right) \Big|_0^1 dx = \frac{1}{3} \int_0^2 (x^2 e^{x^3} - x^2) dx = \frac{1}{3} \left(\frac{1}{3} e^{x^3} - \frac{1}{3} x^3 \right) \Big|_0^2 = \frac{1}{9} (e^8 - 9).$$

$$\mathbf{16.1.23} \quad \int_0^1 \int_1^4 \frac{3y}{\sqrt{x+y^2}} dx dy = \int_0^1 (6y\sqrt{x+y^2}) \Big|_1^4 dy = 6 \int_0^1 (y\sqrt{4+y^2} - y\sqrt{1+y^2}) dy = 6 \left(\frac{1}{3} (4+y^2)^{3/2} - \frac{1}{3} (1+y^2)^{3/2} \right) \Big|_0^1 = 2 (5\sqrt{5} - 2\sqrt{2} - 7).$$

$$\mathbf{16.1.24} \quad \int_0^1 \frac{1}{3} e^{x^3 y} y \Big|_{x=0}^{x=1} dy = \frac{1}{3} \int_0^1 y(e^y - 1) dy = \frac{1}{3} \left(e^y(y-1) - \frac{y^2}{2} \right) \Big|_0^1 = \frac{1}{3} \left(-\frac{1}{2} + 1 \right) = \frac{1}{6}.$$

$$\mathbf{16.1.25} \quad \iint_R (x+2y) dA = \int_1^4 \int_0^3 (x+2y) dx dy = \int_1^4 \left(\frac{x^2}{2} + 2xy \right) \Big|_0^3 dy = \int_1^4 \left(\frac{9}{2} + 6y \right) dy = \left(\frac{9}{2}y + 3y^2 \right) \Big|_1^4 = \frac{117}{2}.$$

$$\mathbf{16.1.26} \quad \iint_R (x^2 + xy) dA = \int_{-1}^1 \int_1^2 (x^2 + xy) dx dy = \int_{-1}^1 \left(\frac{x^3}{3} + \frac{x^2 y}{2} \right) \Big|_1^2 dy = \int_{-1}^1 \left(\frac{7}{3} + \frac{3y}{2} \right) dy = \left(\frac{7}{3}y + \frac{3}{4}y^2 \right) \Big|_{-1}^1 = \frac{14}{3}.$$

$$\mathbf{16.1.27} \quad \text{This is equal to } \int_0^\pi \int_0^1 s^2 t \sin(st^2) dt ds = \int_0^\pi -\frac{1}{2} s \cos(st^2) \Big|_{t=0}^{t=1} ds = -\frac{1}{2} \int_0^\pi (s \cos s - s) ds = -\frac{1}{2} \left(\cos s + s \sin s - \frac{s^2}{2} \right) \Big|_0^\pi = -\frac{1}{2} \left(-1 - \frac{\pi^2}{2} - 1 \right) = 1 + \frac{\pi^2}{4}.$$

$$\mathbf{16.1.28} \quad \text{This is equal to } \int_0^1 \ln(1+xy) \Big|_0^1 dx = \int_0^1 \ln(1+x) dx = ((1+x) \ln(1+x) - (1+x)) \Big|_0^1 = 2 \ln 2 - 2 - (-1) = 2 \ln 2 - 1.$$

$$\mathbf{16.1.29} \quad \iint_R \sqrt{\frac{x}{y}} dA = \int_1^4 \int_0^1 \left(\frac{x}{y} \right)^{1/2} dx dy = \int_1^4 \left(\frac{2}{3} \cdot \frac{x^{3/2}}{y^{1/2}} \right) \Big|_0^1 dy = \int_1^4 \left(\frac{2}{3} y^{1/2} \right) dy = \left(\frac{4}{3} y^{3/2} \right) \Big|_1^4 = \frac{4}{3}.$$

$$\mathbf{16.1.30} \quad \iint_R xy \sin x^2 dA = \int_0^{\sqrt{\pi/2}} \int_0^1 xy \sin x^2 dy dx = \int_0^{\sqrt{\pi/2}} \left(\frac{xy^2}{2} \sin x^2 \right) \Big|_0^1 dx = \frac{1}{2} \int_0^{\sqrt{\pi/2}} (x \sin x^2) dx = -\frac{1}{4} (\cos x^2) \Big|_0^{\sqrt{\pi/2}} = -\frac{1}{4} \left(\cos \left(\frac{\pi}{2} \right) - 1 \right) = -\frac{1}{4} (-1) = \frac{1}{4}.$$

$$\mathbf{16.1.31} \quad \iint_R e^{x+2y} dA = \int_1^{\ln 3} \int_0^{\ln 2} (e^x \cdot e^{2y}) dx dy = \int_1^{\ln 3} (e^x \cdot e^{2y}) \Big|_0^{\ln 2} dy = \int_1^{\ln 3} e^{2y} dy = \left(\frac{1}{2} e^{2y} \right) \Big|_1^{\ln 3} = \frac{1}{2} (9 - e^2).$$

$$\begin{aligned}
 \mathbf{16.1.32} \quad \iint_R (x^2 - y^2)^2 dA &= \int_{-1}^2 \int_0^1 (x^4 - 2x^2y^2 + y^4) dy dx = \int_{-1}^2 (x^4y - 2x^2y^3/3 + y^5/5) \Big|_0^1 dx = \\
 &= \int_{-1}^2 (x^4 - 2x^2/3 + 1/5) dx = (x^5/5 - 2x^3/9 + x/5) \Big|_{-1}^2 = 32/5 - 16/9 + 2/5 - (-1/5 + 2/9 - 1/5) = 26/5.
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{16.1.33} \quad \iint_R (x^5 - y^5)^2 dA &= \int_{-1}^1 \int_0^1 (x^{10} - 2x^5y^5 + y^{10}) dx dy = \int_{-1}^1 \left(\frac{x^{11}}{11} - \frac{1}{3}x^6y^5 + xy^{10} \right) \Big|_0^1 dy = \\
 &= \int_{-1}^1 \left(\frac{1}{11} - \frac{1}{3}y^5 + y^{10} \right) dy = \left(\frac{y}{11} - \frac{y^6}{18} + \frac{y^{11}}{11} \right) \Big|_{-1}^1 = \frac{4}{11}.
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{16.1.34} \quad \int_0^{\pi^2/4} \int_0^1 \cos(x\sqrt{y}) dx dy &= \int_0^{\pi^2/4} \left(\frac{\sin(x\sqrt{y})}{\sqrt{y}} \right) \Big|_0^1 dy = \int_0^{\pi^2/4} \frac{\sin(\sqrt{y})}{\sqrt{y}} dy = \\
 &= (-2 \cos \sqrt{y}) \Big|_0^{\pi^2/4} = 2.
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{16.1.35} \quad \int_0^{\sqrt{\pi/2}} \int_0^1 x^3 y \cos(x^2 y^2) dy dx &= \int_0^{\sqrt{\pi/2}} \left(\frac{x^3}{2x^2} \sin(x^2 y^2) \right) \Big|_0^1 dx = \int_0^{\sqrt{\pi/2}} \frac{1}{2} x \sin(x^2) dx = \\
 &= \left(-\frac{1}{4} \cos(x^2) \right) \Big|_0^{\sqrt{\pi/2}} = \frac{1}{4} (1 - 0) = \frac{1}{4}.
 \end{aligned}$$

$$\mathbf{16.1.36} \quad \int_{-2}^2 \int_0^{\ln 4} e^{-x} dx dy = \int_{-2}^2 (-e^{-x}) \Big|_0^{\ln 4} dy = \int_{-2}^2 \frac{3}{4} dy = \left(\frac{3}{4} y \right) \Big|_{-2}^2 = 3.$$

$$\begin{aligned}
 \mathbf{16.1.37} \quad \int_{-1}^3 \int_0^2 (24 - 3x - 4y) dy dx &= \int_{-1}^3 (24y - 3xy - 2y^2) \Big|_0^2 dx = \int_{-1}^3 (40 - 6x) dx = \\
 &= (40x - 3x^2) \Big|_{-1}^3 = 136.
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{16.1.38} \quad \text{This is given by } \int_0^2 \int_0^1 9xy\sqrt{1-x^2}\sqrt{4-y^2} dx dy &= \int_0^2 (-3)(1-x^2)^{3/2} y\sqrt{4-y^2} \Big|_{x=0}^{x=1} dy = \\
 &= \int_0^2 3y\sqrt{4-y^2} dy = -(4-y^2)^{3/2} \Big|_0^2 = -(0-8) = 8.
 \end{aligned}$$

$$\mathbf{16.1.39} \quad \text{This is given by } \int_0^1 \int_0^3 xy^2\sqrt{1-x^2} dy dx = 9 \int_0^1 x\sqrt{1-x^2} dx = -3(1-x^2)^{3/2} \Big|_0^1 = 3.$$

$$\begin{aligned}
 \mathbf{16.1.40} \quad \text{Integrate first with respect to } x. \quad \iint_R y \cos xy dA &= \int_0^{\pi/3} \int_0^1 y \cos xy dx dy = \int_0^{\pi/3} \sin xy \Big|_0^1 dy = \\
 &= \int_0^{\pi/3} \sin y dy = -\cos y \Big|_0^{\pi/3} = -(1/2 - 1) = 1/2.
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{16.1.41} \quad \text{Integrate first with respect to } x. \quad \iint_R (y+1)e^{x(y+1)} dA &= \int_{-1}^1 \int_0^1 (y+1)e^{x(y+1)} dx dy = \\
 &= \int_{-1}^1 e^{x(y+1)} \Big|_0^1 dy = \int_{-1}^1 (e^{y+1} - 1) dy = (e^{y+1} - y) \Big|_{-1}^1 = (e^2 - 1) - (1 - (-1)) = e^2 - 3.
 \end{aligned}$$

$$\begin{aligned} \mathbf{16.1.42} \quad & \text{Integrate first with respect to } y. \iint_R x \sec^2(xy) \, dA = \int_0^{\pi/3} \int_0^1 x \sec^2(xy) \, dy \, dx = \\ & \int_0^{\pi/3} \left(x \cdot \left(\frac{1}{x} \right) \tan(xy) \right) \Big|_0^1 \, dx = \int_0^{\pi/3} \tan x \, dx = (\ln |\sec x|) \Big|_0^{\pi/3} = \ln 2. \end{aligned}$$

$$\begin{aligned} \mathbf{16.1.43} \quad & \text{Integrate first with respect to } y. \iint_R 6x^5 e^{x^3 y} \, dA = \int_0^2 \int_0^2 6x^5 e^{x^3 y} \, dy \, dx = \int_0^2 6x^2 e^{x^3 y} \Big|_0^2 \, dx = \\ & \int_0^2 (6x^2 e^{2x^3} - 6x^2) \, dx = (e^{2x^3} - 2x^3) \Big|_0^2 = e^{16} - 16 - (1 - 0) = e^{16} - 17. \end{aligned}$$

$$\begin{aligned} \mathbf{16.1.44} \quad & \text{Integrate first with respect to } x. \iint_R y^3 \sin(xy^2) \, dA = \int_0^{\sqrt{\pi/2}} \int_0^2 y^3 \sin(xy^2) \, dx \, dy = \\ & \int_0^{\sqrt{\pi/2}} \left(-\frac{y^3}{y^2} \cos(xy^2) \right) \Big|_0^2 \, dy = \int_0^{\sqrt{\pi/2}} (y - y \cos(2y^2)) \, dy = \left(\frac{y^2}{2} - \frac{1}{4} \sin(2y^2) \right) \Big|_0^{\sqrt{\pi/2}} = \frac{\pi}{4}. \end{aligned}$$

$$\mathbf{16.1.45} \quad \text{Integrate first with respect to } y. \iint_R \frac{x}{(1+xy)^2} \, dA = \int_0^4 \int_1^2 \frac{x}{(1+xy)^2} \, dy \, dx.$$

$$\begin{aligned} \text{Let } u = 1 + xy, \text{ so that } du = x \, dy. \text{ Then we have } & \int_0^4 \left(-\frac{1}{1+xy} \right) \Big|_1^2 \, dx = \int_0^4 \left(\frac{1}{1+x} - \frac{1}{1+2x} \right) \, dx = \\ & \left(\ln|1+x| - \frac{1}{2} \ln|1+2x| \right) \Big|_0^4 = \ln 5 - \frac{1}{2} \ln 9 = \ln \left(\frac{5}{3} \right). \end{aligned}$$

$$\begin{aligned} \mathbf{16.1.46} \quad \bar{f}_{\text{ave}} = \frac{1}{\text{area of } R} \iint_R f(x, y) \, dA &= \frac{1}{4} \int_0^2 \int_0^2 (4 - x - y) \, dy \, dx = \frac{1}{4} \int_0^2 \left(4y - xy - \frac{y^2}{2} \right) \Big|_0^2 \, dx = \\ & \frac{1}{4} \int_0^2 (6 - 2x) \, dx = \frac{1}{4} (6x - x^2) \Big|_0^2 = \frac{1}{4} (8) = 2. \end{aligned}$$

$$\begin{aligned} \mathbf{16.1.47} \quad \bar{f}_{\text{ave}} = \frac{1}{\text{area of } R} \iint_R f(x, y) \, dA &= \frac{1}{6 \ln 2} \int_0^6 \int_0^{\ln 2} e^{-y} \, dy \, dx = \frac{1}{6 \ln 2} \int_0^6 (-e^{-y}) \Big|_0^{\ln 2} \, dx = \\ & \frac{1}{6 \ln 2} \int_0^6 \left(\frac{1}{2} \right) \, dx = \frac{1}{6 \ln 2} \left(\frac{x}{2} \right) \Big|_0^6 = \frac{1}{6 \ln 2} (3) = \frac{1}{2 \ln 2}. \end{aligned}$$

$$\begin{aligned} \mathbf{16.1.48} \quad \bar{f}_{\text{ave}} = \frac{1}{\text{area of } R} \iint_R f(x, y) \, dA &= \frac{1}{\pi^2} \int_0^\pi \int_0^\pi \sin x \sin y \, dy \, dx = \frac{1}{\pi^2} \int_0^\pi (-\sin x \cos y) \Big|_0^\pi \, dx = \\ & \frac{1}{\pi^2} \int_0^\pi 2 \sin x \, dx = \frac{2}{\pi^2} (-\cos x) \Big|_0^\pi = \frac{2}{\pi^2} (2) = \frac{4}{\pi^2}. \end{aligned}$$

$$\begin{aligned} \mathbf{16.1.49} \quad \bar{f}_{\text{ave}} = \frac{1}{\text{area of } R} \iint_R f(x, y) \, dA &= \frac{1}{8} \int_{-2}^2 \int_0^2 (x^2 + y^2) \, dy \, dx = \frac{1}{8} \int_{-2}^2 \left(x^2 y + \frac{1}{3} y^3 \right) \Big|_0^2 \, dx = \\ & \frac{1}{8} \int_{-2}^2 \left(2x^2 + \frac{8}{3} \right) \, dx = \frac{1}{8} \left(\frac{2}{3} x^3 + \frac{8}{3} x \right) \Big|_{-2}^2 = \frac{8}{3}. \end{aligned}$$

$$\begin{aligned} \mathbf{16.1.50} \quad \bar{f}_{\text{ave}} = \frac{1}{\text{area of } R} \iint_R f(x, y) \, dA &= \frac{1}{9} \int_0^3 \int_0^3 ((x-3)^2 + (y-3)^2) \, dy \, dx = \\ & \frac{1}{9} \int_0^3 \left(y(x-3)^2 + \frac{1}{3} (y-3)^3 \right) \Big|_0^3 \, dx = \frac{1}{9} \int_0^3 (3(x-3)^2 + 9) \, dx = \frac{1}{9} ((x-3)^2 + 9x) \Big|_0^3 = 6. \end{aligned}$$

16.1.51

- a. True. The region $R = \{(x, y) \mid 1 \leq x \leq 3, 4 \leq y \leq 6\}$ is a rectangle that has width 2 and length 2, thus is a square.
- b. False. The region for $\int_4^6 \int_1^3 f(x, y) dx dy$ is $R = \{(x, y) \mid 1 \leq x \leq 3, 4 \leq y \leq 6\}$ is not equivalent to the region for $\int_4^6 \int_1^3 f(x, y) dy dx$ which is $R = \{(x, y) \mid 4 \leq x \leq 6, 1 \leq y \leq 3\}$ thus Fubini's Theorem does not apply.
- c. True. The region is $R = \{(x, y) \mid 1 \leq x \leq 3, 4 \leq y \leq 6\}$.

16.1.52

- a. $\iint_R xy e^{-(x^2+y^2)} dA = \int_{-a}^a \int_{-b}^b xy e^{-(x^2+y^2)} dy dx$. Let $f(x, y) = xy e^{-(x^2+y^2)}$. Note that $f(-x, y) = -f(x, y)$, so the integral over the portion of R in the first and fourth quadrants cancels the integral over the portion of R in the second and third quadrants, yielding a value of 0.
- b. Let $f(x, y) = \frac{\sin(x-y)}{x^2+y^2+1}$. Note that $f(-x, -y) = -f(x, y)$, so the integral over the portion of R in the first quadrant cancels the integral over the portion in the third quadrant, and likewise for the second and fourth quadrants, yielding a value of 0.

16.1.53

- a. total population = (total population region 1) + (total population region 2) + ... + (total population region 9) = (population density of region 1) $\times$ (area of region 1) + (population density of region 2) $\times$ (area of region 2) + ... + (population density of region 9) $\times$ (area of region 9) = $(350) \left(\frac{1}{2}\right) + (300) \left(\frac{1}{4}\right) + (150) \left(\frac{3}{4}\right) + (500) \left(\frac{1}{2}\right) + (400) \left(\frac{1}{4}\right) + (250) \left(\frac{3}{4}\right) + (250) (1) + (200) \left(\frac{1}{2}\right) + (150) \left(\frac{3}{2}\right) = 1475$.
- b. The calculation above could be expressed as the sum $\sum_{k=1}^3 f(\bar{x}_k, \bar{y}_k) \Delta A_k$ where $f(\bar{x}_k, \bar{y}_k)$ represents the population density for each sub-region of R with area ΔA_k . This sum can be used as an approximation for a 'continuous' function $f(x, y)$ that represents the population density at each point in R then
- $$\sum_{k=1}^n f(\bar{x}_k, \bar{y}_k) \Delta A_k \approx \iint_R f(x, y) dA.$$

16.1.54 $V \approx \sum_{k=1}^n f(\bar{x}_k, \bar{y}_k) \Delta A_k$. Also, $\Delta A_k = \Delta x_k \Delta y_k = 5 \cdot 6 = 30$. So $\sum_{k=1}^{15} f(\bar{x}_k, \bar{y}_k) \Delta A_k = 30 \cdot (f(\bar{x}_1, \bar{y}_1) + f(\bar{x}_2, \bar{y}_2) + \dots + f(\bar{x}_{15}, \bar{y}_{15})) = 30 \cdot (1 + 1.5 + 2.0 + 2.5 + 3.0 + 1 + 1.5 + 2.0 + 2.5 + 3.0 + 0.75 + 1.25 + 1.75 + 2.25 + 2.75) = 862.5 \text{ m}^3$.

16.1.55 The area of the constant cross section of S is $A = \int_a^b f(x) dx$ for any value of y . The volume of S can be expressed as $\int_c^d \int_a^b f(x) dx dy = \int_c^d A dy = A \cdot y \Big|_c^d = A(d-c)$.

16.1.56

- a. $\int_c^d \int_a^b f(x, y) dx dy = \int_c^d \int_a^b (g(x) \cdot h(y)) dx dy = \int_c^d h(y) \left(\int_a^b g(x) dx \right) dy$, because $h(y)$ is constant relative to x .
- Also, $\int_a^b g(x) dx$ is constant relative to y , so we have $\left(\int_a^b g(x) dx \right) \left(\int_c^d h(y) dy \right)$.

$$\text{b. } \left(\int_a^b g(x) dx \right)^2 = \left(\int_a^b g(x) dx \right) \left(\int_a^b g(x) dx \right) = \int_a^b (g(x))^2 dx.$$

$$\text{c. } \int_0^{2\pi} \int_{10}^{30} (\cos x \cdot e^{-4y^2}) dy dx = \left(\int_0^{2\pi} \cos x dx \right) \left(\int_{10}^{30} e^{-4y^2} dy \right) = (0) \left(\int_{10}^{30} e^{-4y^2} dy \right) = 0.$$

$$\begin{aligned} \text{16.1.57 } \int_0^a \int_0^\pi \sin(x+y) dx dy &= \int_0^a (-\cos(x+y)) \Big|_0^\pi dy = \int_0^a (\cos y - \cos(\pi+y)) dy = \\ (\sin y - \sin(\pi+y)) \Big|_0^a &= (\sin a - \sin(\pi+a)) = 1. \text{ Solving for } a \text{ yields: } (\sin a - \sin(\pi+a)) = 1, \text{ so } 2 \sin a = \\ 1, \text{ so } \sin a &= \frac{1}{2}. \text{ The solutions are } a = \frac{\pi}{6} \text{ and } a = \frac{5\pi}{6}. \end{aligned}$$

$$\begin{aligned} \text{16.1.58 } \bar{f} &= \frac{1}{a^2} \int_0^a \int_0^a (x+y-8) dx dy = \frac{1}{a^2} \int_0^a \left(\frac{x^2}{2} + xy - 8x \right) \Big|_0^a dy = \frac{1}{a^2} \int_0^a \left(\frac{a^2}{2} + ay - 8a \right) dy = \\ \frac{1}{a^2} \left(\frac{a^2}{2} y + \frac{a}{2} y^2 - 8ay \right) \Big|_0^a &= \frac{1}{a^2} \left(\frac{a^3}{2} + \frac{a^3}{2} - 8a^2 \right) = a - 8 = \bar{f}. \end{aligned}$$

Setting $\bar{f} = 0$ and solving for a yields $a - 8 = 0$, thus $a = 8$.

$$\begin{aligned} \text{16.1.59 } \bar{f} &= \frac{1}{a^2} \int_0^a \int_0^a (4 - x^2 - y^2) dy dx = \frac{1}{a^2} \int_0^a \left(4y - x^2 y - \frac{y^3}{3} \right) \Big|_0^a dx = \\ \frac{1}{a^2} \int_0^a \left(4a - ax^2 - \frac{a^3}{3} \right) dx &= \frac{1}{a^2} \left(4ax - \frac{ax^3}{3} - \frac{a^3}{3} x \right) \Big|_0^a = \frac{1}{a^2} \left(4a^2 - \frac{2a^4}{3} \right) = 4 - \frac{2a^2}{3} = \bar{f}. \text{ Setting } \\ \bar{f} = 0 \text{ and solving for } a &\text{ yields } 4 - \frac{2a^2}{3} = 0, \text{ so } a^2 = 6. \text{ Because } a > 0, \text{ we have } a = \sqrt{6}. \end{aligned}$$

$$\begin{aligned} \text{16.1.60 } \text{Consider } \int_0^b \int_0^a (6 - x - 3y) dx dy &= \int_0^b (6x - x^2/2 - 3xy) \Big|_0^a dy = \int_0^b (6a - a^2/2 - 3ay) dy = \\ ((6a - a^2/2)y - 3ay^2/2) \Big|_0^b &= (6ab - a^2b/2 - 3ab^2/2). \end{aligned}$$

Maximize $(6ab - a^2b/2 - 3ab^2/2)$ with the constraint $a = 6 - 3b$. Substituting $6 - 3b$ for a gives $9(2b - b^2)$. This is maximized for $b = 1$. Thus, the desired point is $(3, 1)$ and the value of the integral is 9.

16.1.61

$$\begin{aligned} \text{a. } m &= \iint_R \rho(x, y) dA = \int_0^\pi \int_0^{\pi/2} (1 + \sin x) dx dy = \int_0^\pi (x - \cos x) \Big|_0^{\pi/2} dy = \\ \int_0^\pi \left(\frac{\pi}{2} + 1 \right) dy &= \left(\frac{\pi}{2} + 1 \right) y \Big|_0^\pi = \frac{\pi^2}{2} + \pi. \end{aligned}$$

$$\begin{aligned} \text{b. } m &= \iint_R \rho(x, y) dA = \int_0^{\pi/2} \int_0^\pi (1 + \sin y) dy dx = \int_0^{\pi/2} (y - \cos y) \Big|_0^\pi dx = \\ \int_0^{\pi/2} (\pi + 2) dx &= (\pi + 2) x \Big|_0^{\pi/2} = \frac{\pi^2}{2} + \pi. \end{aligned}$$

$$\begin{aligned} \text{c. } m &= \iint_R \rho(x, y) dA = \int_0^\pi \int_0^{\pi/2} (1 + \sin x \sin y) dx dy = \\ \int_0^\pi (x - \cos x \sin y) \Big|_0^{\pi/2} dy &= \int_0^\pi \left(\frac{\pi}{2} + \sin y \right) dy = \left(\frac{\pi}{2} y - \cos y \right) \Big|_0^\pi = \frac{\pi^2}{2} + 2. \end{aligned}$$

16.1.62 Let the base of the shed be in the xy -plane with corners at $(0, 0)$, $(16, 0)$, $(16, 10)$, and $(0, 10)$. Let the peak of the roof be over the point $(0, 0)$. With these assumptions the height of the roof $h(x, y)$ over any point (x, y) in the base is the lesser of these two functions $f(x, y) = 12 - \frac{x}{4}$ and $g(x, y) = 12 - \frac{2y}{5}$. Divide the base of the shed into 40 squares each of size 2 by 2 (that is, $\Delta x = \Delta y = 2$), and produce the Riemann sum for $h(x, y)$ over the base, evaluating $h(x, y)$ at the midpoint of each square. Thus $\bar{x}_1 = 1$, $\bar{x}_2 = 3$, $\bar{x}_3 = 5$, $\bar{x}_4 = 7$, $\bar{x}_5 = 9$, $\bar{x}_6 = 11$, $\bar{x}_7 = 13$, and $\bar{x}_8 = 15$. Likewise $\bar{y}_1 = 1$, $\bar{y}_2 = 3$, $\bar{y}_3 = 5$, $\bar{y}_4 = 7$, and $\bar{y}_5 = 9$.

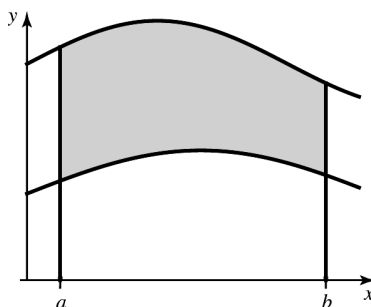
The volume will thus be approximated by the Riemann sum $\sum_{j=1}^8 \sum_{i=1}^5 h(\bar{x}_i, \bar{y}_j) \Delta x_i \Delta y_j = 373.7 \times 4 = 1494.8$.

$$\mathbf{16.1.63} \quad \iint_R \frac{\partial^2 f}{\partial x \partial y} dA = \int_0^b \int_0^a \left(\frac{\partial^2 f}{\partial x \partial y} \right) dx dy = \int_0^b \left(\frac{\partial f}{\partial y} \right) \Big|_0^a dy = \int_0^b \left(\frac{\partial f}{\partial y}(a, y) - \frac{\partial f}{\partial y}(0, y) \right) dy =$$

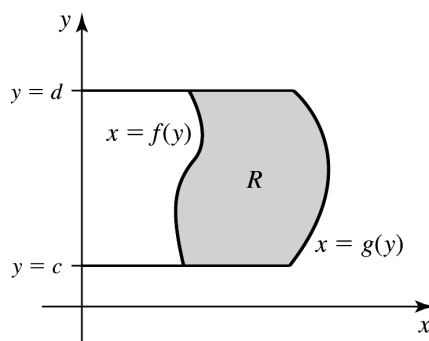
$$(f(a, y) - f(0, y)) \Big|_0^b = f(a, b) - f(0, b) - f(a, 0) + f(0, 0).$$

16.2 Double Integrals over General Regions

16.2.1 A region is bounded below by $y = f(x)$ and above by $y = g(x)$, on the left by $x = a$ and on the right by $x = b$. $R = \{(x, y) \mid a \leq x \leq b, f(x) \leq y \leq g(x)\}$.



16.2.2 A region is bounded below by $y = c$ and above by $y = d$, on the left by $x = f(y)$ and on the right by $x = g(y)$. $R = \{(x, y) \mid f(y) \leq x \leq g(y), c \leq y \leq d\}$.

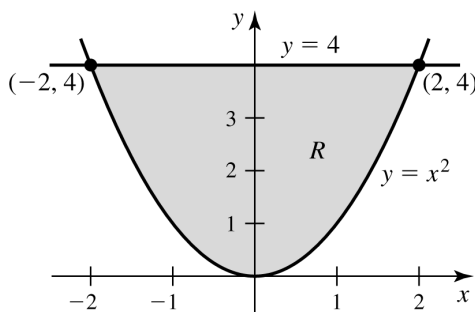


16.2.3 Integrate first with respect to x , then with respect to y . $\iint_R f(x, y) dA = \int_0^1 \int_{y-1}^{1-y} xy dx dy.$

16.2.4 Integrate first with respect to x , then with respect to y . $A = \iint_R (1) dA = \int_0^{17} \int_{\frac{y-3}{2}}^{\frac{y+4}{3}} (1) dx dy.$

16.2.5 $\int_0^1 \int_{x^2}^{\sqrt{x}} f(x, y) dy dx.$

16.2.6



16.2.7 This is a right triangle with base 2 and height 4, so the area is 4.

16.2.8 This is circular cylinder of radius 4 and height 10, so its volume is $\pi r^2 h = \pi \cdot 16 \cdot 10 = 160\pi$.

$$\mathbf{16.2.9} \quad \iint_R f(x, y) \, dA = \int_0^2 \int_{x^2}^{4x} f(x, y) \, dy \, dx.$$

$$\mathbf{16.2.10} \quad \iint_R f(x, y) \, dA = \int_{-3}^4 \int_{2x^2}^{2x+24} f(x, y) \, dy \, dx.$$

$$\mathbf{16.2.11} \quad \int_0^1 3y^2 \Big|_x^1 dx = \int_0^1 (3 - 3x^2) dx = (3x - x^3) \Big|_0^1 = 2.$$

$$\mathbf{16.2.12} \quad \int_0^1 5xy^3 \Big|_0^{2x} dx = \int_0^1 40x^4 dx = 8x^5 \Big|_0^1 = 8.$$

$$\mathbf{16.2.13} \quad \int_0^2 \int_{x^2}^{2x} xy \, dy \, dx = \int_0^2 \left(\frac{xy^2}{2} \right) \Big|_{x^2}^{2x} dx = \frac{1}{2} \int_0^2 (4x^3 - x^5) dx = \frac{1}{2} \left(x^4 - \frac{x^6}{6} \right) \Big|_0^2 = \frac{8}{3}.$$

$$\mathbf{16.2.14} \quad \int_{-\pi/4}^{\pi/4} \int_{\sin x}^{\cos x} dy \, dx = \int_{-\pi/4}^{\pi/4} (y) \Big|_{\sin x}^{\cos x} dx = \int_{-\pi/4}^{\pi/4} (\cos x - \sin x) dx = (\sin x + \cos x) \Big|_{-\pi/4}^{\pi/4} = \sqrt{2}.$$

$$\mathbf{16.2.15} \quad \int_{-2}^2 \int_{x^2}^{8-x^2} x \, dy \, dx = \int_{-2}^2 (xy) \Big|_{x^2}^{8-x^2} dx = \int_{-2}^2 (8x - 2x^3) dx = \left(4x^2 - \frac{x^4}{2} \right) \Big|_{-2}^2 = 0.$$

$$\mathbf{16.2.16} \quad \int_0^{\ln 2} \int_{e^x}^2 dy \, dx = \int_0^{\ln 2} (y) \Big|_{e^x}^2 dx = \int_0^{\ln 2} (2 - e^x) dx = (2x - e^x) \Big|_0^{\ln 2} = 2 \ln 2 - 1.$$

$$\mathbf{16.2.17} \quad \int_0^1 \int_0^x 2e^{x^2} dy \, dx = \int_0^1 2xe^{x^2} dx = e^{x^2} \Big|_0^1 = e - 1.$$

$$\mathbf{16.2.18} \quad \int_0^{\sqrt[3]{\pi/2}} \int_0^x y \cos x^3 dy \, dx = \int_0^{\sqrt[3]{\pi/2}} (y^2/2) \cos x^3 \Big|_0^x dx = \frac{1}{2} \int_0^{\sqrt[3]{\pi/2}} x^2 \cos x^3 dx = \frac{1}{6} (\sin x^3) \Big|_0^{\sqrt[3]{\pi/2}} = \frac{1}{6}.$$

$$\mathbf{16.2.19} \quad \int_0^{\ln 2} \int_{e^y}^2 \left(\frac{y}{x} \right) dx \, dy = \int_0^{\ln 2} (y \ln |x|) \Big|_{e^y}^2 dy = \int_0^{\ln 2} (y \ln 2 - y^2) dy = \left(\frac{\ln 2}{2} y^2 - \frac{y^3}{3} \right) \Big|_0^{\ln 2} = \frac{(\ln 2)^3}{6}.$$

$$16.2.20 \quad \int_0^4 \int_y^{2y} (xy) \, dx \, dy = \int_0^4 \left(\frac{x^2 y}{2} \right) \Big|_y^{2y} dy = \int_0^4 \left(\frac{3}{2} y^3 \right) dy = \left(\frac{3}{8} y^4 \right) \Big|_0^4 = 96.$$

$$16.2.21 \quad \int_0^{\pi/2} \int_y^{\pi/2} 6 \sin(2x - 3y) \, dx \, dy = \int_0^{\pi/2} (-3 \cos(2x - 3y)) \Big|_y^{\pi/2} dy = \\ -3 \int_0^{\pi/2} (\cos(\pi - 3y) - \cos(-y)) \, dy = (\sin(\pi - 3y) - 3 \sin(-y)) \Big|_0^{\pi/2} = -1 - (-3) = 2.$$

$$16.2.22 \quad \int_0^{\pi/2} \int_0^{\cos y} e^{\sin y} \, dx \, dy = \int_0^{\pi/2} (xe^{\sin y}) \Big|_0^{\cos y} dy = \int_0^{\pi/2} \cos y e^{\sin y} \, dy = e^{\sin y} \Big|_0^{\pi/2} = e - 1.$$

$$16.2.23 \quad \int_0^{\pi/2} y \cos y \, dy = y \sin y \Big|_0^{\pi/2} - \int_0^{\pi/2} \sin y \, dy = \pi/2 + \left(\cos y \Big|_0^{\pi/2} \right) = \frac{\pi}{2} - 1, \text{ where the first integral is computed via integration by parts with } u = y \text{ and } dv = \cos y.$$

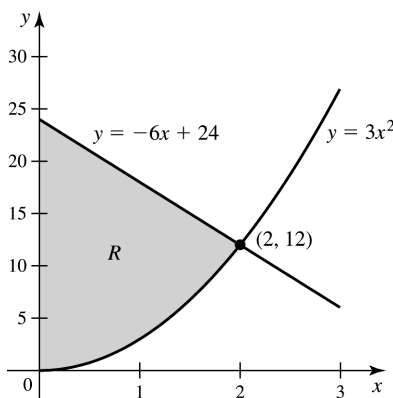
$$16.2.24 \quad \int_0^1 2x \left(\frac{\pi}{4} - \tan^{-1} x \right) dx = \frac{x^2 \pi}{4} \Big|_0^1 - \int_0^1 2x \tan^{-1} x \, dx. \text{ Using Integration by Parts, (with } u = \tan^{-1} x \text{ and } dv = 2x \, dx, \text{ we can deduce that } \int 2x \tan^{-1} x \, dx = -x + \tan^{-1} x + x^2 \tan^{-1} x + C, \text{ so our integral is equal to } \frac{\pi}{4} - (-x + \tan^{-1} x + x^2 \tan^{-1} x) \Big|_0^1 = 1 - \frac{\pi}{4}.$$

$$16.2.25 \quad R = \{(x, y) \mid -\sqrt{16 - y^2} \leq x \leq \sqrt{16 - y^2}, 0 \leq y \leq 4\}. \text{ We have } \int_0^4 \int_{-\sqrt{16 - y^2}}^{\sqrt{16 - y^2}} 2xy \, dx \, dy = \\ \int_0^4 (x^2 y) \Big|_{-\sqrt{16 - y^2}}^{\sqrt{16 - y^2}} dy = \int_0^4 0 \, dy = 0.$$

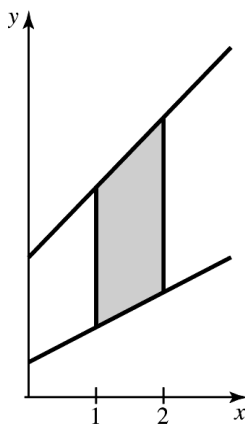
$$16.2.26 \quad \int_0^1 \int_0^x 2e^x \, dy \, dx = \int_0^1 2e^x y \Big|_{y=0}^{y=x} dx = \int_0^1 2xe^x \, dx = 2e^x(x - 1) \Big|_0^1 = 0 - (-2) = 2, \text{ where the last integral is computed via integration by parts with } u = x \text{ and } dv = e^x \, dx.$$

$$16.2.27 \quad \int_{\pi/2}^{\pi} \int_0^{y^2} \cos \frac{x}{y} \, dx \, dy = \int_{\pi/2}^{\pi} y \sin \frac{x}{y} \Big|_{x=0}^{x=y^2} dy = \int_{\pi/2}^{\pi} y \sin y \, dy = (\sin y - y \cos y) \Big|_{\pi/2}^{\pi} = \pi - 1.$$

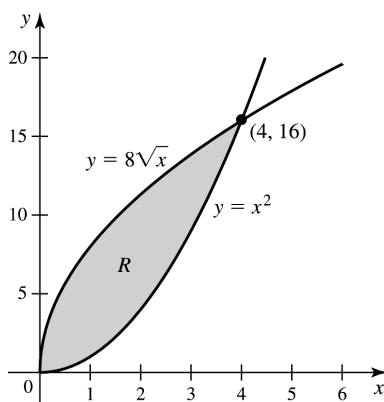
$$16.2.28 \quad \iint_R f(x, y) \, dA = \int_0^2 \int_{3x^2}^{-6x+24} f(x, y) \, dy \, dx.$$



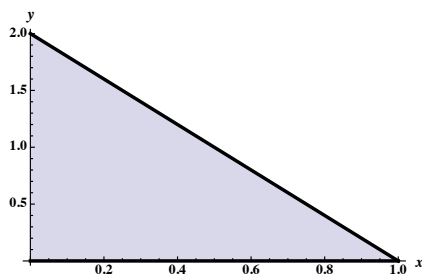
$$16.2.29 \quad \iint_R f(x, y) \, dA = \int_1^2 \int_{x+1}^{2x+4} f(x, y) \, dy \, dx.$$



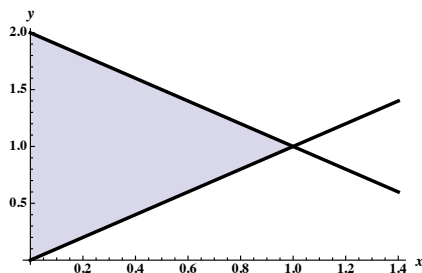
$$16.2.30 \quad \iint_R f(x, y) \, dA = \int_0^4 \int_{x^2}^{8\sqrt{x}} f(x, y) \, dy \, dx \quad (\text{or}) \quad = \int_0^{16} \int_{y^2/16}^{\sqrt{y}} f(x, y) \, dx \, dy.$$



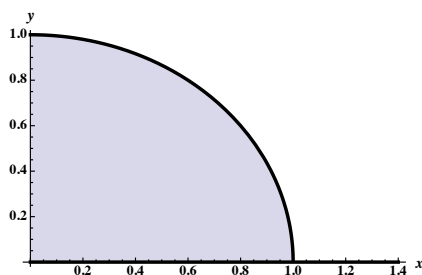
$$16.2.31 \quad \iint_R f(x, y) \, dA = \int_0^1 \int_0^{-2x+2} f(x, y) \, dy \, dx.$$



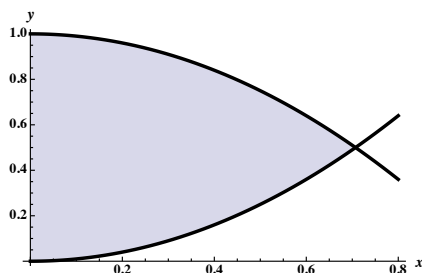
$$16.2.32 \quad \iint_R f(x, y) \, dA = \int_0^1 \int_x^{-x+2} f(x, y) \, dy \, dx.$$



$$16.2.33 \quad \iint_R f(x, y) \, dA = \int_0^1 \int_0^{\sqrt{1-x^2}} f(x, y) \, dy \, dx.$$



$$16.2.34 \quad \iint_R f(x, y) \, dA = \int_0^{1/\sqrt{2}} \int_{x^2}^{1-x^2} f(x, y) \, dy \, dx.$$



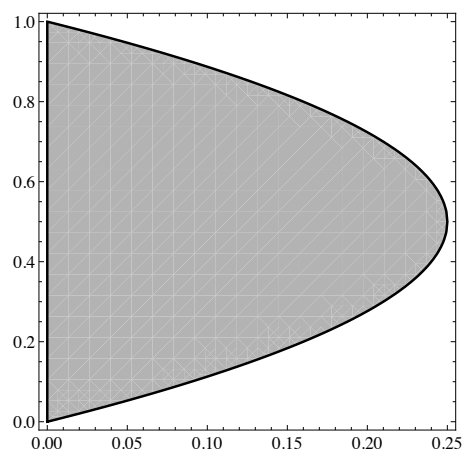
$$16.2.35 \quad \iint_R f(x, y) \, dA = \int_0^{18} \int_{y/2}^{(y+9)/3} f(x, y) \, dx \, dy.$$

$$16.2.36 \quad \iint_R f(x, y) \, dA = \int_{-4}^5 \int_{(5-y)/3}^{\sqrt{25-y^2}} f(x, y) \, dx \, dy.$$

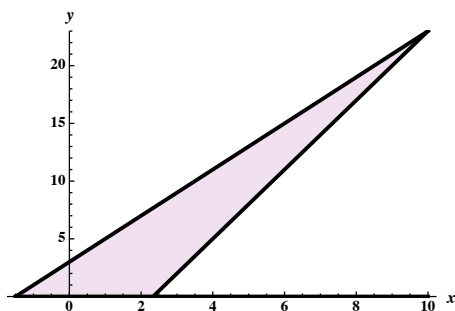
$$16.2.37 \quad \iint_R f(x, y) \, dA = \int_1^4 \int_0^{4-y} f(x, y) \, dx \, dy = \int_0^3 \int_1^{4-x} f(x, y) \, dy \, dx.$$



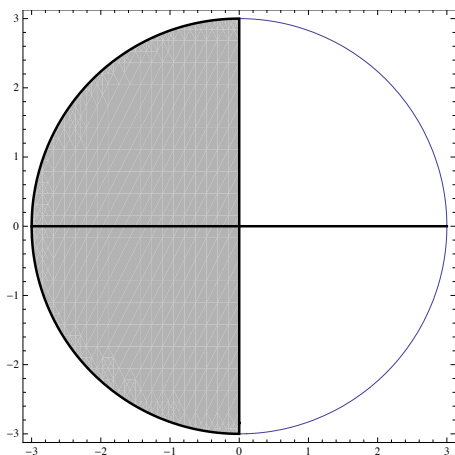
$$16.2.38 \quad \iint_R f(x, y) \, dA = \int_0^1 \int_0^{y(1-y)} f(x, y) \, dx \, dy.$$



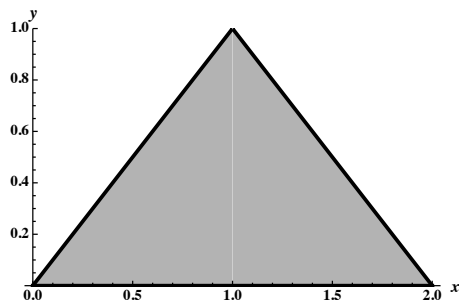
$$16.2.39 \quad \iint_R f(x, y) \, dA = \int_0^{23} \int_{(y-3)/2}^{(y+7)/3} f(x, y) \, dx \, dy.$$



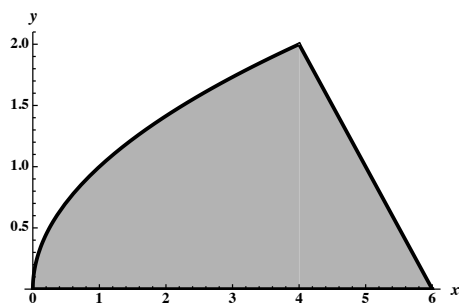
$$16.2.40 \quad \iint_R f(x, y) \, dA = \int_{-3}^3 \int_{-\sqrt{9-y^2}}^0 f(x, y) \, dx \, dy.$$



$$16.2.41 \quad \iint_R f(x, y) \, dA = \int_0^1 \int_y^{2-y} f(x, y) \, dx \, dy.$$

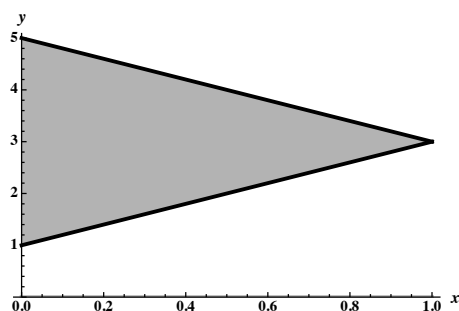


$$16.2.42 \quad \iint_R f(x, y) \, dA = \int_0^2 \int_{y^2}^{6-y} f(x, y) \, dx \, dy.$$



$$16.2.43 \quad \iint_R xy \, dA = \int_0^1 \int_{2x+1}^{-2x+5} xy \, dy \, dx = \int_0^1 \left(\frac{xy^2}{2} \right) \Big|_{2x+1}^{-2x+5} dx = \int_0^1 (12x - 12x^2) \, dx =$$

$$(6x^2 - 4x^3) \Big|_0^1 = 2.$$

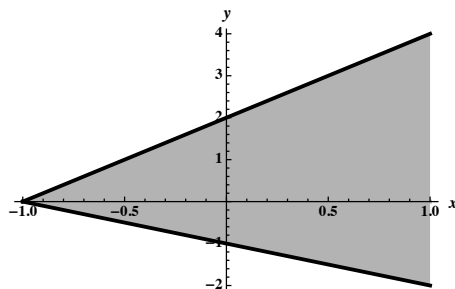


$$16.2.44 \quad \iint_R (x + y) \, dA = \int_0^2 \int_{x^2}^{8-x^2} (x + y) \, dy \, dx = \int_0^2 \left(xy + \frac{y^2}{2} \right) \Big|_{x^2}^{8-x^2} dx =$$

$$\int_0^2 (32 + 8x - 8x^2 - 2x^3) \, dx = \left(32x + 4x^2 - \frac{8}{3}x^3 - \frac{x^4}{2} \right) \Big|_0^2 = \frac{152}{3}.$$

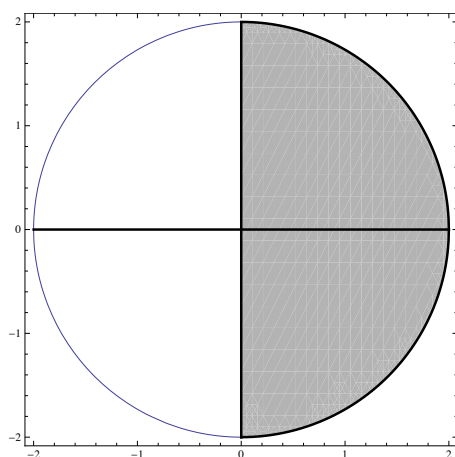
$$16.2.45 \quad \iint_R y^2 \, dA = \int_{-1}^1 \int_{-x-1}^{2x+2} y^2 \, dy \, dx = \int_{-1}^1 \left(\frac{y^3}{3} \right) \Big|_{-x-1}^{2x+2} dx = \frac{1}{3} \int_{-1}^1 (8(x+1)^3 + (x+1)^3) \, dx =$$

$$3 \int_{-1}^1 (x+1)^3 \, dx = \frac{3}{4} (x+1)^4 \Big|_{-1}^1 = 12.$$

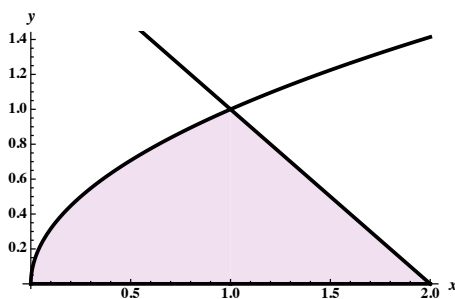


$$\mathbf{16.2.46} \quad \iint_R x^2 y \, dA = \int_{-4}^4 \int_0^{\sqrt{16-y^2}} x^2 y \, dx \, dy = \int_{-4}^4 \left(\frac{x^3 y}{3} \right) \Big|_0^{\sqrt{16-y^2}} dy = \frac{1}{3} \int_{-4}^4 y (16-y^2)^{3/2} \, dy. \quad \text{Let}$$

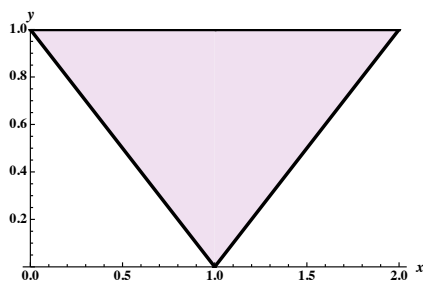
$u = 16 - y^2$ so that $du = -2y \, dy$, Then the integral is equal to $\frac{1}{6} \int_0^0 u^{3/2} \, du = 0$.



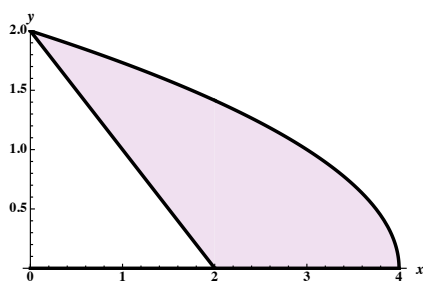
$$\mathbf{16.2.47} \quad \iint_R 12y \, dA = \int_0^1 \int_{y^2}^{2-y} 12y \, dx \, dy = \int_0^1 (12xy) \Big|_{y^2}^{2-y} dy = 12 \int_0^1 (2y - y^2 - y^3) \, dy = 12(y^2 - y^3/3 - y^4/4) \Big|_0^1 = 12(5/12) = 5.$$



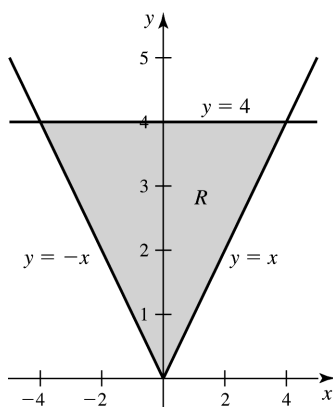
$$\mathbf{16.2.48} \quad \iint_R y^2 \, dA = \int_0^1 \int_{1-y}^{y+1} y^2 \, dx \, dy = \int_0^1 (xy^2) \Big|_{1-y}^{y+1} dy = \int_0^1 (y^3 + y^2 - y^2 + y^3) \, dy = \int_0^1 2y^3 \, dy = y^4/2 \Big|_0^1 = 1/2.$$



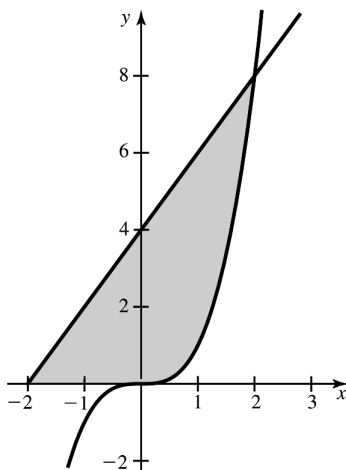
$$\begin{aligned}
 \mathbf{16.2.49} \quad \iint_R 3xy \, dA &= \int_0^2 \int_{2-y}^{4-y^2} 3xy \, dx \, dy = \int_0^2 (3x^2 y/2) \Big|_{2-y}^{4-y^2} dy = \int_0^2 (3(4-y^2)^2 y/2 - 3(2-y)^2 y/2) dy = \\
 &= \frac{3}{2} \int_0^2 (16y - 8y^3 + y^5 - (4y - 4y^2 + y^3)) dy = \frac{3}{2} \int_0^2 (y^5 - 9y^3 + 4y^2 + 12y) dy = \frac{3}{2} (y^6/6 - 9y^4/4 + 4y^3/3 + 6y^2) \Big|_0^2 = \\
 &= \frac{3}{2} (32/3 - 36 + 32/3 + 24) = 14.
 \end{aligned}$$



$$\begin{aligned}
 \mathbf{16.2.50} \quad \iint_R (x+y) \, dA &= \int_0^4 \int_{-y}^y (x+y) \, dx \, dy = \int_0^4 \left(\frac{x^2}{2} + xy \right) \Big|_{-y}^y dy = \int_0^4 (2y^2) \, dy = \left(\frac{2}{3} y^3 \right) \Big|_0^4 = \\
 &= \frac{128}{3}.
 \end{aligned}$$



$$\begin{aligned}
 \mathbf{16.2.51} \quad \iint_R 3x^2 \, dA &= \int_0^8 \int_{\sqrt[3]{y-4}}^{\sqrt[3]{y}} 3x^2 \, dx \, dy = \int_0^8 (x^3) \Big|_{\sqrt[3]{y-4}}^{\sqrt[3]{y}} dy = \int_0^8 (y - (1/8)(y^3 - 12y^2 + 48y - 64)) dy = \\
 &= \int_0^8 (-y^3/8 + 3y^2/2 - 5y + 8) dy = (-y^4/32 + y^3/2 - 5y^2/2 + 8y) \Big|_0^8 = 32.
 \end{aligned}$$



$$\mathbf{16.2.52} \quad \iint_R 8xy \, dA = \int_0^{\pi/4} \int_0^{\sec x} 8xy \, dy \, dx = \int_0^{\pi/4} 4x \sec^2 x \, dx = 4(\ln \cos x + x \tan x) \Big|_0^{\pi/4} = 4(\ln \sqrt{2}/2 + \pi/4) - 0 = \ln(1/4) + \pi = -2 \ln 2 + \pi$$

$$\mathbf{16.2.53} \quad \iint_R (x+y) \, dA = \int_{1/2}^2 \int_{1/x}^{5/2-x} (x+y) \, dy \, dx = \int_{1/2}^2 \left(xy + \frac{1}{2}y^2 \right) \Big|_{\frac{1}{x}}^{\frac{5}{2}-x} dx = \int_{1/2}^2 \left(\frac{17}{8} - \frac{x^2}{2} - \frac{1}{2x^2} \right) dx = \left(\frac{17}{8}x - \frac{x^3}{6} + \frac{1}{2x} \right) \Big|_{1/2}^2 = \frac{9}{8}.$$

16.2.54

$$\begin{aligned} \int_0^1 \int_0^{y^2} \frac{y}{1+x+y^2} \, dx \, dy &= \int_0^1 y \ln(1+x+y^2) \Big|_{x=0}^{x=y^2} dy \\ &= \int_0^1 (y \ln(1+2y^2) - y \ln(1+y^2)) \, dy \\ &= \int_0^1 y \ln(1+2y^2) \, dy - \int_0^1 y \ln(1+y^2) \, dy \end{aligned}$$

For the first integral we let $u = 1 + 2y^2$ and for the second we let $u = 1 + y^2$. We have

$$\begin{aligned} &\frac{1}{4} (2y^2 + 1) (\ln(2y^2 + 1) - 1) \Big|_0^1 - \left(-\frac{y^2}{2} + \frac{1}{2}y^2 \ln(y^2 + 1) + \frac{1}{2} \ln(y^2 + 1) \right) \Big|_0^1 \\ &= \frac{3}{4}(\ln 3 - 1) + \frac{1}{4} - \left(\ln 2 - \frac{1}{2} \right) = \frac{3}{4} \ln 3 - \ln 2. \end{aligned}$$

$$\mathbf{16.2.55} \quad \iint_R x \sec^2 y \, dA = \int_0^{\sqrt{\pi}/2} \int_0^{x^2} x \sec^2 y \, dy \, dx = \int_0^{\sqrt{\pi}/2} x \tan y \Big|_0^{x^2} dx = \int_0^{\sqrt{\pi}/2} x \tan x^2 \, dx = \left(\frac{1}{2} \ln |\sec x^2| \right) \Big|_0^{\sqrt{\pi}/2} = \frac{1}{2} \ln(\sqrt{2}) = \frac{1}{4} \ln 2.$$

$$\begin{aligned} \mathbf{16.2.56} \quad \iint_R \frac{8xy}{1+x^2+y^2} \, dA &= \int_0^2 \int_0^x \frac{8xy}{1+x^2+y^2} \, dy \, dx = \int_0^2 4x \ln(1+x^2+y^2) \Big|_0^x dx = \\ &4 \int_0^2 (x \ln(1+2x^2) - x \ln(1+x^2)) \, dx. \text{ Let } u = 1+2x^2 \text{ and } v = 1+x^2. \text{ Substituting gives } \int_1^9 \ln u \, du - \\ &2 \int_1^5 \ln v \, dv = (u \ln u - u) \Big|_{u=1}^{u=9} - 2(v \ln v - v) \Big|_{v=1}^{v=5} = (9 \ln 9 - 8) - (10 \ln 5 - 8) = 9 \ln 9 - 10 \ln 5. \end{aligned}$$

$$16.2.57 \quad \int_0^2 \int_{x^2}^{2x} f(x, y) \, dy \, dx = \int_0^4 \int_{y/2}^{\sqrt{y}} f(x, y) \, dx \, dy.$$

$$16.2.58 \quad \int_0^3 \int_0^{6-2x} f(x, y) \, dy \, dx = \int_0^6 \int_0^{\frac{6-y}{2}} f(x, y) \, dx \, dy.$$

$$16.2.59 \quad \int_{1/2}^1 \int_0^{-\ln y} f(x, y) \, dx \, dy = \int_0^{\ln 2} \int_{1/2}^{e^{-x}} f(x, y) \, dy \, dx.$$

$$16.2.60 \quad \int_0^1 \int_1^{e^y} f(x, y) \, dx \, dy = \int_1^e \int_{\ln x}^1 f(x, y) \, dy \, dx.$$

$$16.2.61 \quad \int_0^1 \int_0^{\cos^{-1} y} f(x, y) \, dx \, dy = \int_0^{\pi/2} \int_0^{\cos x} f(x, y) \, dy \, dx.$$

$$16.2.62 \quad \int_1^e \int_0^{\ln x} f(x, y) \, dy \, dx = \int_0^1 \int_{e^y}^e f(x, y) \, dx \, dy.$$

$$16.2.63 \quad \int_0^1 \int_y^1 e^{x^2} \, dx \, dy = \int_0^1 \int_0^x e^{x^2} \, dy \, dx = \int_0^1 y e^{x^2} \Big|_0^x \, dx = \int_0^1 (x e^{x^2}) \, dx = \left(\frac{1}{2} e^{x^2} \right) \Big|_0^1 = \frac{1}{2} (e - 1).$$

$$16.2.64 \quad \int_0^\pi \int_x^\pi \sin y^2 \, dy \, dx = \int_0^\pi \int_0^y \sin y^2 \, dx \, dy = \int_0^\pi (x \sin y^2) \Big|_0^y \, dy = \int_0^\pi y \sin y^2 \, dy = \left(-\frac{1}{2} \cos y^2 \right) \Big|_0^\pi = \frac{1}{2} - \frac{1}{2} \cos \pi^2.$$

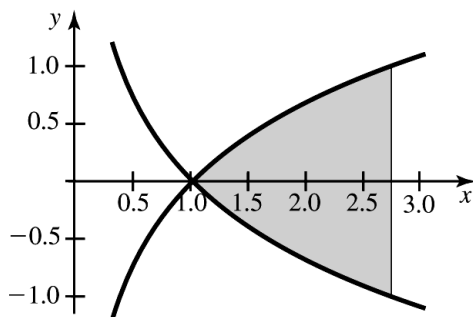
$$16.2.65 \quad \int_0^{1/2} \int_{y^2}^{1/4} y \cos(16\pi x^2) \, dx \, dy = \int_0^{1/4} \int_0^{\sqrt{x}} y \cos(16\pi x^2) \, dy \, dx = \int_0^{1/4} \left(\frac{1}{2} y^2 \cos(16\pi x^2) \right) \Big|_0^{\sqrt{x}} \, dx = \int_0^{1/4} \frac{1}{2} x \cos(16\pi x^2) \, dx. \text{ Let } u = 16\pi x^2 \text{ so that } du = 32\pi \, dx. \text{ We have } \int_0^\pi \frac{\cos u}{64\pi} \, du = \left(\frac{\sin u}{64\pi} \right) \Big|_{u=0}^{u=\pi} = 0.$$

$$16.2.66 \quad \int_0^4 \int_{\sqrt{x}}^2 \frac{x}{y^5 + 1} \, dy \, dx = \int_0^2 \int_0^{y^2} \frac{x}{y^5 + 1} \, dx \, dy = \int_0^2 \left(\frac{x^2}{2(y^5 + 1)} \right) \Big|_0^{y^2} \, dy = \int_0^2 \frac{y^4}{2(y^5 + 1)} \, dy. \text{ Let } u = y^5 + 1 \text{ so that } du = 5y^4 \, dy. \text{ Substituting gives } \int_1^{33} \frac{1}{10u} \, du = \frac{1}{10} (\ln|u|) \Big|_{u=1}^{u=33} = \frac{\ln 33}{10}.$$

$$16.2.67 \quad \int_0^{\sqrt[3]{\pi}} \int_y^{\sqrt[3]{\pi}} x^4 \cos(x^2 y) \, dx \, dy = \int_0^{\sqrt[3]{\pi}} \int_0^x x^4 \cos(x^2 y) \, dy \, dx = \int_0^{\sqrt[3]{\pi}} (x^2 \sin(x^2 y)) \Big|_0^x \, dx = \int_0^{\sqrt[3]{\pi}} x^2 \sin(x^3) \, dx = -\frac{1}{3} (\cos x^3) \Big|_{u=0}^{u=\sqrt[3]{\pi}} = \frac{2}{3}.$$

$$16.2.68 \quad \int_0^2 \int_0^{4-x^2} \frac{x e^{2y}}{4-y} \, dy \, dx = \int_0^4 \int_0^{\sqrt{4-y}} \frac{x e^{2y}}{4-y} \, dx \, dy = \int_0^4 \left(\frac{x^2 e^{2y}}{2(4-y)} \right) \Big|_0^{\sqrt{4-y}} \, dy = \int_0^4 \frac{(4-y) e^{2y}}{2(4-y)} \, dy = \int_0^4 \frac{e^{2y}}{2} \, dy = \frac{1}{4} (e^{2y}) \Big|_0^4 = \frac{1}{4} (e^8 - 1).$$

$$16.2.69 \quad \int_0^1 \int_{e^y}^e f(x, y) \, dx \, dy + \int_{-1}^0 \int_{e^{-y}}^e f(x, y) \, dx \, dy = \int_1^e \int_{-\ln x}^{\ln x} f(x, y) \, dy \, dx.$$



$$16.2.70 \quad \int_{-4}^0 \int_0^{\sqrt{16-x^2}} f(x, y) \, dy \, dx + \int_0^4 \int_0^{4-x} f(x, y) \, dy \, dx = \int_0^4 \int_{-\sqrt{16-y^2}}^{4-y} f(x, y) \, dx \, dy.$$

$$16.2.71 \quad V = \int_0^1 \int_0^x (2 - y^2) \, dy \, dx = \int_0^1 \left(2y - \frac{1}{3}y^3 \right) \Big|_0^x \, dx = \frac{1}{3} \int_0^1 (6x - x^3) \, dx = \frac{1}{12} (12x^2 - x^4) \Big|_0^1 = \frac{11}{12}.$$

$$16.2.72 \quad 2 \int_0^\pi \int_0^y \sin^2 x \, dx \, dy = \int_0^\pi \int_0^y (1 - \cos 2x) \, dx \, dy = \int_0^\pi \left(y - \frac{\sin 2y}{2} \right) \, dy = \left(\frac{y^2}{2} + \frac{\cos 2y}{4} \right) \Big|_0^\pi = \frac{\pi^2}{2}.$$

$$16.2.73 \quad V = \int_0^4 \int_0^{2-\frac{x}{2}} (8 - 2x - 4y) \, dy \, dx = \int_0^4 (8y - 2xy - 2y^2) \Big|_0^{2-\frac{x}{2}} \, dx = \int_0^4 \left(8 - 4x - \frac{1}{2}x^2 \right) \, dx = \left(8x - 2x^2 - \frac{1}{6}x^3 \right) \Big|_0^4 = \frac{32}{3}, \text{ or } V = \int_0^2 \int_0^{4-2y} (8 - 2x - 4y) \, dx \, dy = \frac{32}{3}.$$

$$16.2.74 \quad \int_0^1 \int_0^{1-x^2} (1 - y - x^2) \, dy \, dx = \int_0^1 (y - y^2/2 - x^2y) \Big|_0^{1-x^2} \, dx = \int_0^1 (1 - x^2 - (1/2)(1 - 2x^2 + x^4) - x^2 + x^4) \, dx = \frac{1}{2} \int_0^1 (x^4 - 2x^2 + 1) \, dx = \frac{1}{2} (x^5/5 - 2x^3/3 + x) \Big|_0^1 = \frac{4}{15}.$$

$$16.2.75 \quad V = \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (12 + x + y) \, dy \, dx = \int_{-1}^1 \left(12y + xy + \frac{1}{2}y^2 \right) \Big|_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \, dx = \int_{-1}^1 \left(24\sqrt{1-x^2} + 2x\sqrt{1-x^2} \right) \, dx = \left(12x\sqrt{1-x^2} + 12\sin^{-1} x - \frac{2}{3}(1-x^2)^{3/2} \right) \Big|_{-1}^1 = 12(\sin^{-1}(1) - \sin^{-1}(-1)) = 12\pi.$$

$$16.2.76 \quad V = \iint_R (e^{x-y} - (-e^{x-y})) \, dA = \int_0^1 \int_0^y 2e^{x-y} \, dx \, dy = \int_0^1 (2e^{x-y}) \Big|_0^y \, dy = \int_0^1 2(1 - e^{-y}) \, dy = 2(y + e^{-y}) \Big|_0^1 = \frac{2}{e}.$$

$$16.2.77 \quad V = \iint_R (8 - 2x - 3y) \, dA = \int_0^1 \int_0^{2-x} (8 - 2x - 3y) \, dy \, dx = \int_0^1 \left((8 - 2x)y - \frac{3}{2}y^2 \right) \Big|_0^{2-x} \, dx = \int_0^1 \left(10 - 6x + \frac{1}{2}x^2 \right) \, dx = \left(10x - 3x^2 + \frac{1}{6}x^3 \right) \Big|_0^1 = \frac{43}{6}.$$

$$16.2.78 \quad V = \int_0^1 \int_0^{1-x^2} (2 - y - 1) \, dy \, dx = \int_0^1 (y - y^2/2) \Big|_0^{1-x^2} \, dx = \frac{1}{2} \int_0^1 (2 - 2x^2 - (1 - 2x^2 + x^4)) \, dx = \frac{1}{2} \int_0^1 (-x^4 + 1) \, dx = \frac{1}{2} (-x^5/5 + x) \Big|_0^1 = \frac{2}{5}.$$

$$\begin{aligned}
 \mathbf{16.2.79} \quad V &= \int_0^1 \int_0^{1-x} ((2 - x^2 - y^2) - (x^2 + y^2)) dy dx = 2 \int_0^1 \int_0^{1-x} (1 - x^2 - y^2) dy dx = 2 \int_0^1 (y - x^2 y - y^3/3) \Big|_0^{1-x} dx \\
 &= \frac{2}{3} \int_0^1 ((3 - 3x) - (3x^2 - 3x^3) - (1 - 3x + 3x^2 - x^3)) dx = \frac{2}{3} \int_0^1 (4x^3 - 6x^2 + 2) dx = \frac{2}{3} (x^4 - 2x^3 + 2x) \Big|_0^1 = \frac{2}{3}.
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{16.2.80} \quad V &= \iint_R ((y+1) - (x^2+1)) dA = \int_{-1}^1 \int_{x^2}^1 (y - x^2) dy dx = \int_{-1}^1 \left(\frac{y^2}{2} - x^2 y \right) \Big|_{x^2}^1 dx = \\
 &= \int_{-1}^1 \left(\left(\frac{1}{2} - x^2 \right) - \left(\frac{x^4}{2} - x^4 \right) \right) dx = \int_{-1}^1 \left(\frac{1}{2} - x^2 + \frac{x^4}{2} \right) dx = \left(\frac{x}{2} - \frac{x^3}{3} + \frac{x^5}{10} \right) \Big|_{-1}^1 = \frac{8}{15}.
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{16.2.81} \quad V &= \iint_R (4 - x - y) dA = \int_{-1}^1 \int_{-1}^1 (4 - x - y) dy dx = \int_{-1}^1 \left((4-x)y - \frac{1}{2}y^2 \right) \Big|_{-1}^1 dx = \\
 &= \int_{-1}^1 2(4-x) dx = 2 \left(4x - \frac{1}{2}x^2 \right) \Big|_{-1}^1 = 16.
 \end{aligned}$$

$$\mathbf{16.2.82} \quad V = \iint_R (z_{\text{top}} - z_{\text{bottom}}) dA \text{ where } R \text{ is enclosed by the intersection of the surfaces } z = x^2 + y^2$$

and $z = 1 - 2y$. Setting $x^2 + y^2 = 1 - 2y$ gives the circle $x^2 + (y+1)^2 = 2$.

$$\begin{aligned}
 \text{So } V &= \int_{-\sqrt{2}}^{\sqrt{2}} \int_{-1-\sqrt{2-x^2}}^{-1+\sqrt{2-x^2}} ((1-2y) - (x^2+y^2)) dy dx = \int_{-\sqrt{2}}^{\sqrt{2}} \left(y - y^2 - x^2 y - \frac{1}{3}y^3 \right) \Big|_{-1-\sqrt{2-x^2}}^{-1+\sqrt{2-x^2}} dx = \\
 &= \int_{-\sqrt{2}}^{\sqrt{2}} \left(\frac{8}{3}\sqrt{2-x^2} - \frac{4}{3}x^2\sqrt{2-x^2} \right) dx = \left(\frac{5}{3}x\sqrt{2-x^2} - \frac{1}{3}x^3\sqrt{2-x^2} + 2\sin^{-1}\left(\frac{x}{\sqrt{2}}\right) \right) \Big|_{-\sqrt{2}}^{\sqrt{2}} = 2\pi.
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{16.2.83} \quad V &= \iint_R (a(2-x) - a(x-2)) dA = 2a \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (2-x) dy dx = \\
 &= 2a \int_{-1}^1 \left[(2-x)y \right] \Big|_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dx = 4a \int_{-1}^1 (2-x)\sqrt{1-x^2} dx = 4a \int_{-1}^1 \left(2\sqrt{1-x^2} - x\sqrt{1-x^2} \right) dx = \\
 &= \left[4a \left(x\sqrt{1-x^2} + \sin^{-1}x \right) + \frac{4a}{3}(1-x^2)^{3/2} \right] \Big|_{-1}^1 = 4a(\sin^{-1}(1) - \sin^{-1}(-1)) = 4a\pi.
 \end{aligned}$$

16.2.84 Using symmetry, this is given by

$$\begin{aligned}
 &2 \int_0^2 \int_{-\sqrt{12-3y^2}}^{\sqrt{12-3y^2}} (3x^2 + y^2 + 1) dx dy = 2 \int_0^2 (x^3 + xy^2 + x) \Big|_{-\sqrt{12-3y^2}}^{\sqrt{12-3y^2}} dy = \\
 &2 \int_0^2 \left(26\sqrt{12-3y^2} - 4y^2\sqrt{12-3y^2} \right) dy = 2 \left(\sqrt{3} \left(22\sin^{-1}\left(\frac{y}{2}\right) - \frac{1}{2}y\sqrt{4-y^2}(y^2-15) \right) \right) \Big|_0^2 = 22\sqrt{3}\pi,
 \end{aligned}$$

where the last integral is found either via a table of integrals or a computer algebra system.

$$\mathbf{16.2.85} \quad A = \iint_R 1 dA = \int_{-2}^2 \int_{x^2}^4 1 dy dx = \int_{-2}^2 (y) \Big|_{x^2}^4 dx = \int_{-2}^2 (4 - x^2) dx = \left(4x - \frac{1}{3}x^3 \right) \Big|_{-2}^2 = \frac{32}{3}.$$

$$\begin{aligned}
 \mathbf{16.2.86} \quad A &= \iint_R 1 dA = \int_{-1}^2 \int_{x^2}^{x+2} 1 dy dx = \int_{-1}^2 (y) \Big|_{x^2}^{x+2} dx = \int_{-1}^2 (x+2-x^2) dx = \\
 &= \left(\frac{1}{2}x^2 + 2x - \frac{1}{3}x^3 \right) \Big|_{-1}^2 = \frac{9}{2}.
 \end{aligned}$$

$$16.2.87 \quad A = \iint_R 1 \, dA = \int_0^{\ln 2} \int_0^{e^x} 1 \, dy \, dx = \int_0^{\ln 2} (y) \Big|_0^{e^x} dx = \int_0^{\ln 2} e^x \, dx = (e^x) \Big|_0^{\ln 2} = 1.$$

$$16.2.88 \quad A = \iint_R 1 \, dA = \int_0^\pi \int_{1-\sin x}^{1+\sin x} 1 \, dy \, dx = \int_0^\pi (y) \Big|_{1-\sin x}^{1+\sin x} dx = \int_0^\pi 2 \sin x \, dx = (-2 \cos x) \Big|_0^\pi = 4.$$

$$16.2.89 \quad A = \iint_R 1 \, dA = \int_0^2 \int_{6-x}^{5x+6} 1 \, dy \, dx + \int_2^6 \int_{x^2}^{5x+6} 1 \, dy \, dx = \int_0^2 (y) \Big|_{6-x}^{5x+6} dx + \int_2^6 (y) \Big|_{x^2}^{5x+6} dx = \int_0^2 6x \, dx + \int_2^6 (5x+6-x^2) \, dx = (3x^2) \Big|_0^2 + \left(\frac{5}{2}x^2 + 6x - \frac{1}{3}x^3 \right) \Big|_2^6 = \frac{140}{3}.$$

$$16.2.90 \quad A = \iint_R 1 \, dA = \int_0^4 \int_x^{2x+1} 1 \, dy \, dx = \int_0^4 (y) \Big|_x^{2x+1} dx = \int_0^4 (x+1) \, dx = \left(\frac{1}{2}x^2 + x \right) \Big|_0^4 = 12.$$

16.2.91

- False. a and b must be constants or functions of y .
- False. a and d must be constants.
- False. Variable limits of integration are not allowed in the outermost integral.

16.2.92

$$a. \quad \int_0^4 \int_0^4 (4-x-y) \, dx \, dy = \int_0^4 \left(4x - \frac{x^2}{2} - xy \right) \Big|_{x=0}^{x=4} dy = \int_0^4 (8-4y) \, dy = (8y-2y^2) \Big|_0^4 = 32-32=0.$$

- The integrand is positive provided that $4-x-y \geq 0$ or $y \leq 4-x$. It follows that

$$\begin{aligned} \int_0^4 \int_0^4 |4-x-y| \, dx \, dy &= 2 \int_0^4 \int_0^{4-x} (4-x-y) \, dy \, dx = 2 \int_0^4 \left(4y - xy - \frac{y^2}{2} \right) \Big|_{y=0}^{y=4-x} dx = \\ &= 2 \int_0^4 \left(16-4x - (4x-x^2) - \frac{(4-x)^2}{2} \right) dx = 2 \int_0^4 \left(8-4x + \frac{1}{2}x^2 \right) dx = 2 \left(8x-2x^2 + \frac{x^3}{6} \right) \Big|_0^4 = \\ &= \frac{64}{3}. \end{aligned}$$

$$16.2.93 \quad V = \iint_R f(x, y) \, dA \quad (\text{orient } y\text{-axis vertically}) = \iint_R (y_{\text{top}} - y_{\text{bottom}}) \, dz \, dx = \int_0^5 \int_0^2 \left[(z+1) - \left(\frac{-z-1}{2} \right) \right] dz \, dx = \int_0^5 \int_0^2 \left(\frac{3}{2}z + \frac{3}{2} \right) dz \, dx = \int_0^5 \left(\frac{3}{4}z^2 + \frac{3}{2}z \right) \Big|_0^2 dx = \int_0^5 6 \, dx = (6x) \Big|_0^5 = 30.$$

16.2.94

$$a. \quad A = \int_{-1}^0 \int_{-1-x}^{1+x} 1 \, dy \, dx + \int_0^1 \int_{x-1}^{1-x} 1 \, dy \, dx = \int_{-1}^0 (2+2x) \, dx + \int_0^1 (2-2x) \, dx = (2x+x^2) \Big|_{-1}^0 + (2x-x^2) \Big|_0^1 = 1+1=2.$$

$$\begin{aligned} b. \quad A &= \int_{-1}^0 \int_{-1-x}^{1+x} (12-3x-4y) \, dy \, dx + \int_0^1 \int_{x-1}^{1-x} (12-3x-4y) \, dy \, dx = \\ &= \int_{-1}^0 \left((12-3x)y - 2y^2 \right) \Big|_{-1-x}^{1+x} dx + \int_0^1 \left((12-3x)y - 2y^2 \right) \Big|_{x-1}^{1-x} dx = \int_{-1}^0 (24+18x-6x^2) \, dx + \\ &= \int_0^1 (24-30x+6x^2) \, dx = (24x+9x^2-2x^3) \Big|_{-1}^0 + (24x-15x^2+2x^3) \Big|_0^1 = 13+11=24. \end{aligned}$$

c. $V = \int_{-1}^0 \int_{-1-x}^{1+x} \sqrt{1-x^2} dy dx + \int_0^1 \int_{x-1}^{1-x} \sqrt{1-x^2} dy dx = 2 \int_0^1 \int_{x-1}^{1-x} \sqrt{1-x^2} dy dx$, where the last equality is due to symmetry. Then we have $2 \int_0^1 \left(y \sqrt{1-x^2} \right) \Big|_{x-1}^{1-x} dx = 2 \int_0^1 (2-2x) \sqrt{1-x^2} dx = 4 \int_0^1 \left(\sqrt{1-x^2} - x\sqrt{1-x^2} \right) dx = 4 \left(\frac{1}{2}x\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}x + \frac{1}{3}(1-x^2)^{3/2} \right) \Big|_0^1 = 4 \left(0 + \frac{\pi}{4} + 0 - 0 - 0 - \frac{1}{3} \right) = \pi - \frac{4}{3}$.

d. Use symmetry.

$$V = 4 \int_0^1 \int_0^{1-x} 6(1-x-y) dy dx = 24 \int_0^1 \left((1-x)y - \frac{1}{2}y^2 \right) \Big|_0^{1-x} dx = 24 \int_0^1 \frac{1}{2}(1-x)^2 dx = 12 \left(-\frac{1}{3}(1-x)^3 \right) \Big|_0^1 = 4.$$

16.2.95

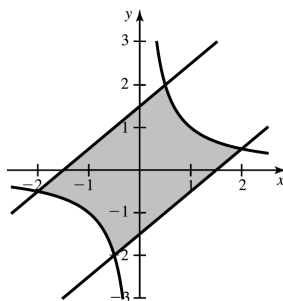
$$\begin{aligned} \bar{f} &= \frac{1}{\text{Area of } R} \iint_R f(x, y) dA = \frac{1}{\frac{1}{2}a^2} \int_0^a \int_0^{a-x} (a-x-y) dy dx \\ &= \frac{2}{a^2} \int_0^a \left((a-x)y - \frac{1}{2}y^2 \right) \Big|_0^{a-x} dx = \frac{2}{a^2} \int_1^9 \frac{1}{2}(a-x)^2 dx = \frac{1}{a^2} \left(-\frac{1}{3}(a-x)^3 \right) \Big|_0^a = \frac{1}{a^2} \left(\frac{a^3}{3} \right) = \frac{a}{3}. \end{aligned}$$

16.2.96 $\bar{f} = \frac{1}{\text{Area of } R} \iint_R f(x, y) dA = \frac{1}{\pi a^2} \int_{-a}^a \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} (a^2 - x^2 - y^2) dy dx$, which by symmetry is equal to

$$\begin{aligned} \frac{4}{\pi a^2} \int_0^a \int_0^{\sqrt{a^2-x^2}} (a^2 - x^2 - y^2) dy dx &= \frac{4}{\pi a^2} \int_0^a \left((a^2 - x^2)y - \frac{1}{3}y^3 \right) \Big|_0^{\sqrt{a^2-x^2}} dx \\ &= \frac{4}{\pi a^2} \int_0^a \frac{2}{3}(a^2 - x^2)^{3/2} dx = \frac{8}{3\pi a^2} \left(-\frac{x}{8}(2x^2 - 5a^2)\sqrt{a^2 - x^2} + \frac{3a^4}{8}\sin^{-1}\left(\frac{x}{a}\right) \right) \Big|_0^a = \frac{a^2}{3}. \end{aligned}$$

16.2.97

a.



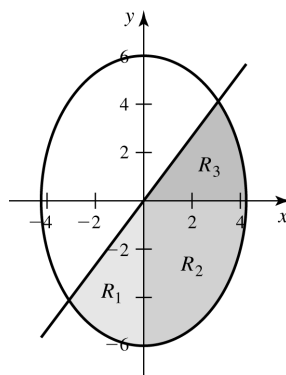
b.

$$\begin{aligned}
A &= \iint_R 1 \, dA = \iint_{R_1} 1 \, dA + \iint_{R_2} 1 \, dA + \iint_{R_3} 1 \, dA \\
&= \int_{-2}^{-1/2} \int_{1/x}^{x+3/2} 1 \, dy \, dx + \int_{-1/2}^{1/2} \int_{x-3/2}^{x+3/2} 1 \, dy \, dx + \int_{1/2}^2 \int_{x-3/2}^{1/x} 1 \, dy \, dx \\
&= \int_{-2}^{-1/2} \left(x + \frac{3}{2} - \frac{1}{x} \right) dx + \int_{-1/2}^{1/2} (3) \, dx + \int_{1/2}^2 \left(\frac{1}{x} - x + \frac{3}{2} \right) dx \\
&= \left(\frac{1}{2}x^2 + \frac{3}{2}x - \ln |x| \right) \Big|_{-2}^{-1/2} + (3x) \Big|_{-1/2}^{1/2} + \left(\ln |x| - \frac{1}{2}x^2 + \frac{3}{2}x \right) \Big|_{1/2}^2 \\
&= \frac{3}{8} + 2 \ln 2 + 3 + \frac{3}{8} + 2 \ln 2 = \frac{15}{4} + 4 \ln 2.
\end{aligned}$$

$$\begin{aligned}
\text{c. } \iint_R xy \, dA &= \int_{-2}^{-1/2} \int_{1/x}^{x+3/2} xy \, dy \, dx + \int_{-1/2}^{1/2} \int_{x-3/2}^{x+3/2} xy \, dy \, dx + \int_{1/2}^2 \int_{x-3/2}^{1/x} xy \, dy \, dx = \\
&\int_{-2}^{-1/2} \left(\frac{1}{2}xy^2 \right) \Big|_{1/x}^{x+3/2} dx + \int_{-1/2}^{1/2} \left(\frac{1}{2}xy^2 \right) \Big|_{x-3/2}^{x+3/2} dx + \int_{1/2}^2 \left(\frac{1}{2}xy^2 \right) \Big|_{x-3/2}^{1/x} dx = \\
&\frac{1}{2} \int_{-2}^{-1/2} \left(x^3 + 3x^2 + \frac{9}{4}x - \frac{1}{x} \right) dx + \frac{1}{2} \int_{-1/2}^{1/2} (6x^2) \, dx + \frac{1}{2} \int_{1/2}^2 \left(\frac{1}{x} - \frac{9}{4}x + 3x^2 - x^3 \right) dx = \\
&\frac{1}{2} \left(\frac{1}{4}x^4 + x^3 + \frac{9}{8}x^2 - \ln |x| \right) \Big|_{-2}^{-1/2} + (x^3) \Big|_{-1/2}^{1/2} + \frac{1}{2} \left(\ln |x| - \frac{9}{8}x^2 + x^3 - \frac{1}{4}x^4 \right) \Big|_{1/2}^2 = \\
&\frac{1}{2} \left(-\frac{21}{64} + 2 \ln 2 \right) + \frac{1}{4} + \frac{1}{2} \left(-\frac{21}{64} + 2 \ln 2 \right) = -\frac{5}{64} + 2 \ln 2.
\end{aligned}$$

16.2.98

a.



b. $A = \iint_R 1 \, dA = \iint_{R_1} 1 \, dA + \iint_{R_2} 1 \, dA + \iint_{R_3} 1 \, dA$. Note: By symmetry we see that the area of $(R_1 + R_3)$ is equal to the area of R_2 . Thus

$$\begin{aligned}
A &= 2 \iint_{R_2} 1 \, dA = 2 \int_0^{3\sqrt{2}} \int_{-\sqrt{36-2x^2}}^0 1 \, dy \, dx = 2\sqrt{2} \int_0^{3\sqrt{2}} \sqrt{18-x^2} \, dx \\
&= 2\sqrt{2} \left(\frac{1}{2}x\sqrt{18-x^2} + 9 \sin^{-1} \left(\frac{x}{3\sqrt{2}} \right) \right) \Big|_0^{3\sqrt{2}} = 9\sqrt{2}\pi.
\end{aligned}$$

c. Again by symmetry $R_1 + R_3$ will be equivalent to R_2 but with opposite sign. $xy > 0$ in Quadrant I and III and $xy < 0$ in Quadrant II and IV. By symmetry about the origin we can deduce equivalent components of surface above the xy -plane over regions R_1 and R_3 as below the xy -plane in R_2 . Thus

$$\iint_R xy \, dA = 0.$$

$$\begin{aligned} 16.2.99 \quad \int_1^\infty \int_0^{e^{-x}} xy \, dy \, dx &= \int_1^\infty \left(\frac{1}{2} xy^2 \right) \Big|_0^{e^{-x}} dx = \int_1^\infty \frac{1}{2} x e^{-2x} dx = \lim_{b \rightarrow \infty} \int_1^b \frac{1}{2} x e^{-2x} dx = \\ \lim_{b \rightarrow \infty} \frac{1}{2} \left(e^{-2x} \left(-\frac{1}{2} x - \frac{1}{4} \right) \right) \Big|_1^b &= \frac{1}{4} \lim_{b \rightarrow \infty} \left(e^{-2b} \left(-b - \frac{1}{2} \right) + e^{-2} \left(\frac{3}{2} \right) \right) = \frac{1}{4} \left(0 + \frac{3}{2} e^{-2} \right) = \frac{3}{8e^2}. \end{aligned}$$

$$\begin{aligned} 16.2.100 \quad \int_1^\infty \int_0^{1/x^2} \frac{2y}{x} \, dy \, dx &= \int_1^\infty \left(\frac{y^2}{x} \right) \Big|_0^{1/x^2} dx = \int_1^\infty \frac{1}{x^5} dx = \lim_{b \rightarrow \infty} \int_1^b \frac{1}{x^5} dx = \\ \lim_{b \rightarrow \infty} \left(-\frac{1}{4x^4} \right) \Big|_1^b &= \lim_{b \rightarrow \infty} \left(-\frac{1}{4b^4} + \frac{1}{4} \right) = 0 + \frac{1}{4} = \frac{1}{4}. \end{aligned}$$

$$\begin{aligned} 16.2.101 \quad \int_0^\infty \int_0^\infty e^{-x-y} \, dy \, dx &= \int_0^\infty \left(\lim_{b \rightarrow \infty} \int_0^b e^{-x-y} \, dy \right) dx = \int_0^\infty \left(\lim_{b \rightarrow \infty} (-e^{-x-y}) \Big|_0^b \right) dx = \\ \int_0^\infty \left(\lim_{b \rightarrow \infty} (e^{-x} - e^{-x-b}) \right) dx &= \int_0^\infty e^{-x} dx = \lim_{b \rightarrow \infty} \int_0^b -e^{-x} dx = \lim_{b \rightarrow \infty} (-e^{-x}) \Big|_0^b = \\ \lim_{b \rightarrow \infty} (1 - e^{-b}) &= 1. \end{aligned}$$

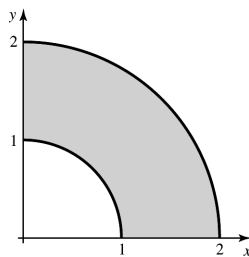
16.2.102 First compute $\int_{-\infty}^\infty \frac{1}{u^2 + 1} du$. This is equal to

$$\begin{aligned} \int_{-\infty}^0 \frac{1}{u^2 + 1} du + \int_0^\infty \frac{1}{u^2 + 1} du &= \lim_{a \rightarrow -\infty} \int_a^0 \frac{1}{u^2 + 1} du + \lim_{b \rightarrow \infty} \int_0^b \frac{1}{u^2 + 1} du = \lim_{a \rightarrow -\infty} \left(\tan^{-1} u \Big|_{u=a}^{u=0} \right) + \\ \lim_{b \rightarrow \infty} \left(\tan^{-1} u \Big|_{u=0}^{u=b} \right) &= \lim_{a \rightarrow -\infty} -\tan^{-1} a + \lim_{b \rightarrow \infty} \tan^{-1} b = \frac{\pi}{2} + \frac{\pi}{2} = \pi \end{aligned}$$

Now use this result to compute the inner integral: $\int_{-\infty}^\infty \frac{1}{(x^2 + 1)(y^2 + 1)} dy = \frac{1}{x^2 + 1} \int_{-\infty}^\infty \frac{1}{y^2 + 1} dy = \frac{\pi}{x^2 + 1}$, so the integral is $\int_{-\infty}^\infty \int_{-\infty}^\infty \frac{1}{(x^2 + 1)(y^2 + 1)} dy dx = \int_{-\infty}^\infty \frac{\pi}{x^2 + 1} dx = \pi^2$.

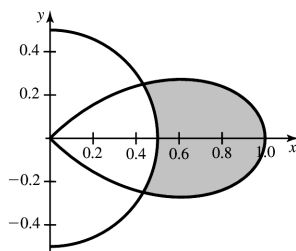
16.3 Double Integrals in Polar Coordinates

16.3.1 It is called a polar rectangle because it is analogous to a cartesian rectangle; they both have each variable constrained by constants.



$$16.3.2 \quad \iint_R f(x, y) \, dA = \int_\alpha^\beta \int_a^b f(r \cos \theta, r \sin \theta) r \, dr \, d\theta.$$

$$16.3.3 \quad R = \left\{ (r, \theta) : \frac{1}{2} \leq r \leq \cos 2\theta, -\frac{\pi}{6} \leq \theta \leq \frac{\pi}{6} \right\}$$



16.3.4 Geometrically $dx dy$ represents the length times width of a rectangular subsection of the region of integration. For equal partitions with respect to x and y each subsection has equivalent area. In polar coordinates each partition is a portion of a circular sector. For equal partitions with respect to r and θ the areas of the subsections are proportional to the distance from the origin, or r .

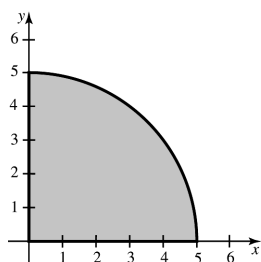
16.3.5 For $R = \left\{ (r, \theta) : g(\theta) \leq r \leq h(\theta), \alpha \leq \theta \leq \beta \right\}$, the area of R given by $\iint_R (1) dy dx$ converts to polar coordinates $\int_{\alpha}^{\beta} \int_{g(\theta)}^{h(\theta)} r dr d\theta$.

16.3.6 For $R = \left\{ (r, \theta) : g(\theta) \leq r \leq h(\theta), \alpha \leq \theta \leq \beta \right\}$ the average value of a function $f(x, y)$ is given by

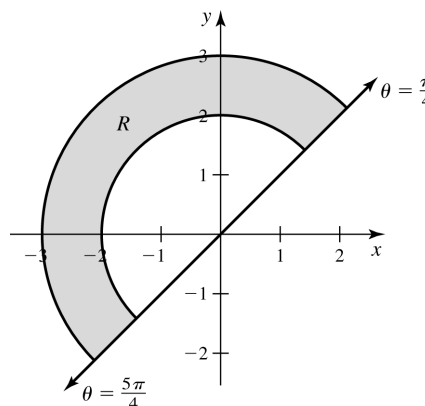
$$\bar{f} = \frac{1}{\text{area of } R} \iint_R f(x, y) dy dx$$

which converts to polar coordinates $\frac{1}{\text{area of } R} \int_{\alpha}^{\beta} \int_{g(\theta)}^{h(\theta)} f(r \cos \theta, r \sin \theta) r dr d\theta$.

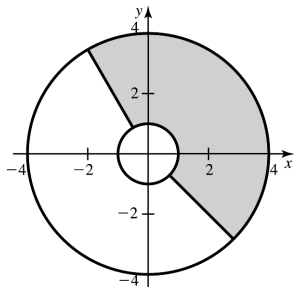
16.3.7



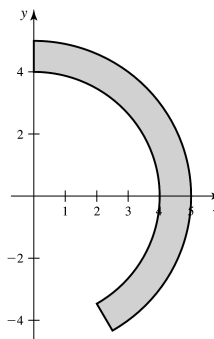
16.3.8



16.3.9



16.3.10



16.3.11 Setting $z \geq 0$ gives $x^2 + y^2 \leq 16$, and thus $0 \leq r \leq 4$ and $0 \leq \theta \leq 2\pi$. It follows that $V = \int_0^{2\pi} \int_0^4 (4 - \sqrt{r^2}) r dr d\theta = \int_0^{2\pi} \int_0^4 (4r - r^2) dr d\theta = \int_0^{2\pi} \left(2r^2 - \frac{1}{3}r^3 \right) \Big|_0^4 d\theta = \int_0^{2\pi} \frac{32}{3} d\theta = \frac{64\pi}{3}$.

16.3.12 Setting $z \geq 0$ gives $x^2 + y^2 \leq 4$, and thus $0 \leq r \leq 2$ and $0 \leq \theta \leq 2\pi$. It follows that $V = \int_0^{2\pi} \int_0^2 (16 - 4r^2) r dr d\theta = \int_0^{2\pi} \int_0^2 (16r - 4r^3) dr d\theta = \int_0^{2\pi} (8r^2 - r^4) \Big|_0^2 d\theta = \int_0^{2\pi} 16 d\theta = 32\pi$.

16.3.13 Setting $z \geq 0$, gives $x^2 + y^2 \leq 16$, and thus $0 \leq r \leq 4$ and $0 \leq \theta \leq 2\pi$. It follows that $V = \int_0^{2\pi} \int_0^4 (e^{-r^2/8} - e^{-2}) r dr d\theta = \int_0^{2\pi} \int_0^4 (r e^{-r^2/8} - r e^{-2}) dr d\theta = \int_0^{2\pi} \left(-4e^{-r^2/8} - \frac{1}{2}r^2 e^{-2} \right) \Big|_0^4 d\theta = \int_0^{2\pi} (-12e^{-2} + 4) d\theta = 8\pi(1 - 3e^{-2})$.

16.3.14 Setting $z \geq 0$ gives $x^2 + y^2 \leq 9$, and thus $0 \leq r \leq 3$ and $0 \leq \theta \leq 2\pi$. It follows that $V = \int_0^{2\pi} \int_0^3 \left(\frac{20}{1+r^2} - 2 \right) r dr d\theta = \int_0^{2\pi} \int_0^3 \left(\frac{20r}{1+r^2} - 2r \right) dr d\theta = \int_0^{2\pi} (10 \ln(1+r^2) - r^2) \Big|_0^3 d\theta = \int_0^{2\pi} (10 \ln 10 - 9) d\theta = 2\pi(10 \ln 10 - 9)$.

16.3.15 $V = \iint_R f(r \cos \theta, r \sin \theta) r dr d\theta = \int_0^{2\pi} \int_0^1 (4 - r^2) r dr d\theta = \int_0^{2\pi} \left(2r^2 - \frac{1}{4}r^4 \right) \Big|_0^1 d\theta = \int_0^{2\pi} \left(\frac{7}{4} \right) d\theta = \frac{7\pi}{2}$.

16.3.16 $V = \iint_R f(r \cos \theta, r \sin \theta) r dr d\theta = \int_0^{2\pi} \int_0^2 (4 - r^2) r dr d\theta = \int_0^{2\pi} \left(2r^2 - \frac{1}{4}r^4 \right) \Big|_0^2 d\theta = \int_0^{2\pi} 4 d\theta = 8\pi$.

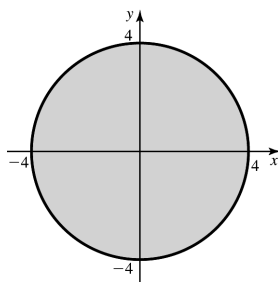
16.3.17 $V = \iint_R f(r \cos \theta, r \sin \theta) r dr d\theta = \int_0^{2\pi} \int_1^2 (4 - r^2) r dr d\theta = \int_0^{2\pi} \left(2r^2 - \frac{1}{4}r^4 \right) \Big|_1^2 d\theta = \int_0^{2\pi} \frac{9}{4} d\theta = \frac{9\pi}{2}$.

16.3.18 $V = \iint_R f(r \cos \theta, r \sin \theta) r dr d\theta = \int_{-\pi/2}^{\pi/2} \int_1^2 (4 - r^2) r dr d\theta = \int_{-\pi/2}^{\pi/2} \left(2r^2 - \frac{1}{4}r^4 \right) \Big|_1^2 d\theta = \int_{-\pi/2}^{\pi/2} \frac{9}{4} d\theta = \frac{9\pi}{4}$.

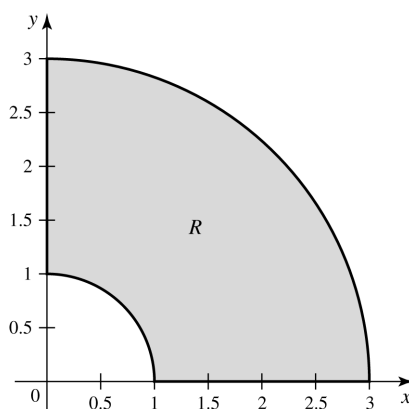
$$\begin{aligned}
 \mathbf{16.3.19} \quad V &= \iint_R (5 - \sqrt{1+x^2+y^2}) \, dA = \int_0^{2\pi} \int_{\sqrt{3}}^{2\sqrt{2}} (5 - \sqrt{1+r^2}) \, r \, dr \, d\theta = \\
 &= \int_0^{2\pi} \left(\frac{5}{2}r^2 - \frac{1}{3}(1+r^2)^{3/2} \right) \Big|_{\sqrt{3}}^{2\sqrt{2}} d\theta = \int_0^{2\pi} (20 - 9 - (15/2 - 8/3)) \, d\theta = \int_0^{2\pi} (37/6) \, d\theta = 37\pi/3.
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{16.3.20} \quad V &= \iint_R (5 - \sqrt{1+x^2+y^2}) \, dA = \int_{-\pi/2}^{\pi} \int_{\sqrt{3}}^{\sqrt{15}} (5 - \sqrt{1+r^2}) \, r \, dr \, d\theta = \\
 &= \int_{-\pi/2}^{\pi} \left(\frac{5}{2}r^2 - \frac{1}{3}(1+r^2)^{3/2} \right) \Big|_{\sqrt{3}}^{\sqrt{15}} d\theta = \int_{-\pi/2}^{\pi} (75/2 - 64/3 - (15/2 - 8/3)) \, d\theta = \int_{-\pi/2}^{\pi} (34/3) \, d\theta = \\
 &= \left(\frac{34}{3} \cdot \frac{3\pi}{2} \right) = 17\pi.
 \end{aligned}$$

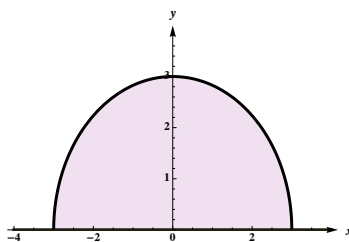
$$\mathbf{16.3.21} \quad \iint_R (x^2 + y^2) \, dA = \int_0^{2\pi} \int_0^4 (r^2) \, r \, dr \, d\theta = \int_0^{2\pi} \int_0^4 r^3 \, dr \, d\theta = \int_0^{2\pi} \left(\frac{1}{4}r^4 \right) \Big|_0^4 d\theta = \int_0^{2\pi} 64 \, d\theta = 128\pi.$$



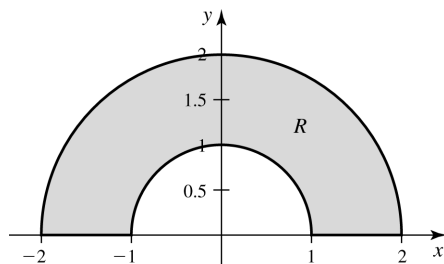
$$\begin{aligned}
 \mathbf{16.3.22} \quad \iint_R (2xy) \, dA &= \int_0^{\pi/2} \int_1^3 2(r \cos \theta)(r \sin \theta) \, r \, dr \, d\theta = \int_0^{\pi/2} \int_1^3 2r^3 \sin \theta \cos \theta \, dr \, d\theta = \\
 &= \int_0^{\pi/2} \sin \theta \cos \theta \left(\frac{1}{2}r^4 \right) \Big|_1^3 d\theta = \int_0^{\pi/2} 40 \sin \theta \cos \theta \, d\theta = (20 \sin^2 \theta) \Big|_{\theta=0}^{\theta=\pi/2} = 20.
 \end{aligned}$$



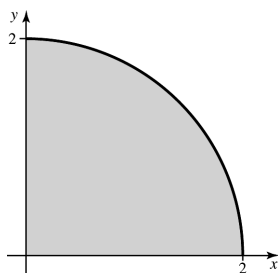
$$\begin{aligned}
 \mathbf{16.3.23} \quad \iint_R (2xy) \, dA &= \int_0^{\pi} \int_0^3 2(r \cos \theta)(r \sin \theta) \, r \, dr \, d\theta = \int_0^{\pi} \int_0^3 2r^3 \sin \theta \cos \theta \, dr \, d\theta = \\
 &= \int_0^{\pi} \cos \theta \sin \theta \left(\frac{1}{2}r^4 \right) \Big|_0^3 d\theta = (81/2) \int_0^{\pi} \cos \theta \sin \theta \, d\theta = \left(\frac{81}{4} \sin^2 \theta \right) \Big|_{\theta=0}^{\theta=\pi} = 0.
 \end{aligned}$$



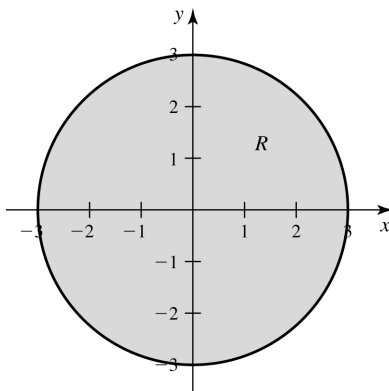
$$\begin{aligned} \mathbf{16.3.24} \quad \iint_R \frac{1}{1+x^2+y^2} dA &= \int_0^\pi \int_1^2 \left(\frac{1}{1+r^2} \right) r dr d\theta = \int_0^\pi \int_1^2 \frac{r}{1+r^2} dr d\theta = \\ &= \int_0^\pi \left(\frac{1}{2} \ln(1+r^2) \right) \Big|_1^2 d\theta = \int_0^\pi \frac{1}{2} (\ln 5 - \ln 2) d\theta = \frac{\pi}{2} \ln \left(\frac{5}{2} \right). \end{aligned}$$



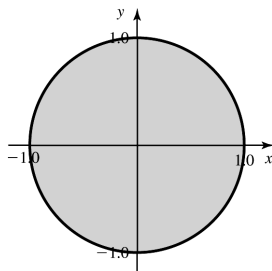
$$\begin{aligned} \mathbf{16.3.25} \quad \iint_R \frac{1}{\sqrt{16-x^2-y^2}} dA &= \int_0^{\pi/2} \int_0^2 \frac{1}{\sqrt{16-r^2}} r dr d\theta = \int_0^{\pi/2} \int_0^2 \frac{r}{\sqrt{16-r^2}} dr d\theta = \\ &= \int_0^{\pi/2} \left(-\sqrt{16-r^2} \right) \Big|_0^2 d\theta = \int_0^{\pi/2} (-2\sqrt{3} + 4) d\theta = \pi(2 - \sqrt{3}). \end{aligned}$$



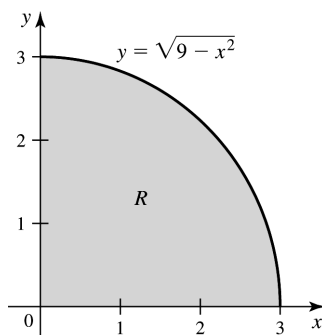
$$\begin{aligned} \mathbf{16.3.26} \quad \iint_R e^{-x^2-y^2} dA &= \int_0^{2\pi} \int_0^3 e^{-r^2} r dr d\theta = \int_0^{2\pi} \left(-\frac{1}{2} e^{-r^2} \right) \Big|_0^3 d\theta = \int_0^{2\pi} \left(-\frac{1}{2} e^{-9} + \frac{1}{2} \right) d\theta \\ &= \pi(1 - e^{-9}). \end{aligned}$$



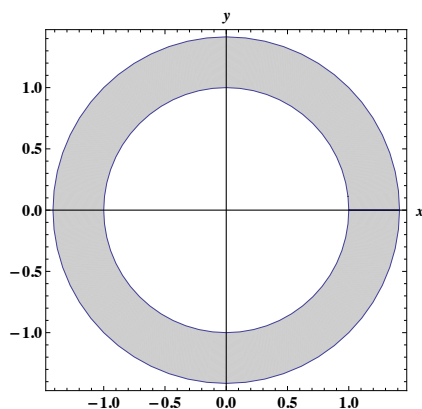
$$\begin{aligned}
 \mathbf{16.3.27} \quad \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (x^2 + y^2)^{3/2} dy dx &= \int_0^{2\pi} \int_0^1 (r^2)^{3/2} r dr d\theta = \int_0^{2\pi} \left(\frac{1}{5} r^5 \right) \Big|_0^1 d\theta = \int_0^{2\pi} \left(\frac{1}{5} \right) d\theta = \\
 \left(\frac{1}{5} \theta \right) \Big|_{\theta=0}^{\theta=2\pi} &= \frac{2\pi}{5}.
 \end{aligned}$$



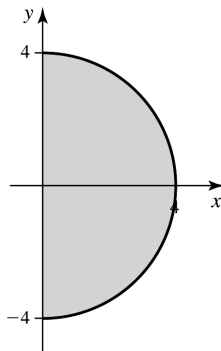
$$\begin{aligned}
 \mathbf{16.3.28} \quad \int_0^3 \int_0^{\sqrt{9-x^2}} \sqrt{x^2 + y^2} dy dx &= \int_0^{\pi/2} \int_0^3 \sqrt{r^2} r dr d\theta = \int_0^{\pi/2} \left(\frac{1}{3} r^3 \right) \Big|_0^3 d\theta = \int_0^{\pi/2} (9) d\theta = \\
 (9\theta) \Big|_{\theta=0}^{\theta=\pi/2} &= \frac{9\pi}{2}.
 \end{aligned}$$



$$\mathbf{16.3.29} \quad \text{Let } x = \cos \theta \text{ and } y = \sin \theta. \text{ Then we have } \int_0^{2\pi} \int_1^2 r^2 dr d\theta = \int_0^{2\pi} \left(\frac{r^3}{3} \right) \Big|_1^2 d\theta = \frac{7}{3} \cdot 2\pi = \frac{14\pi}{3}.$$



$$\begin{aligned}
 \mathbf{16.3.30} \quad \int_{-4}^4 \int_0^{\sqrt{16-y^2}} (16 - x^2 - y^2) dx dy &= \int_{-\pi/2}^{\pi/2} \int_0^4 (16 - r^2) r dr d\theta = \int_{-\pi/2}^{\pi/2} \left(8r^2 - \frac{1}{4} r^4 \right) \Big|_0^4 d\theta = \\
 \int_{-\pi/2}^{\pi/2} 64 d\theta &= (64\theta) \Big|_{\theta=-\pi/2}^{\theta=\pi/2} = 64\pi.
 \end{aligned}$$



$$\begin{aligned}
 \mathbf{16.3.31} \quad V &= \iint_R (9 - x^2 - y^2) \, dA = \int_{-3}^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} (9 - x^2 - y^2) \, dy \, dx = \\
 &= \int_{-3}^3 \left((9 - x^2)y - \frac{1}{3}y^3 \right) \Big|_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} dx = \int_{-3}^3 \left(12\sqrt{9-x^2} - \frac{4}{3}x^2\sqrt{9-x^2} \right) dx = \\
 &= \left(\frac{15}{2}x\sqrt{9-x^2} - \frac{1}{3}x^2\sqrt{9-x^2} + \frac{81}{2}\sin^{-1}\left(\frac{x}{3}\right) \right) \Big|_{-3}^3 = \frac{81\pi}{2}.
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{16.3.32} \quad \text{Let } x = r \cos \theta \text{ and } y = r \sin \theta. \quad V &= \iint_R (2 - x^2 - y^2 - 1) \, dA = \int_0^{2\pi} \int_0^1 (1 - r^2)r \, dr \, d\theta = \int_0^{2\pi} \int_0^1 (r - \\
 r^3) \, dr \, d\theta &= \int_0^{2\pi} \left(r^2/2 - r^4/4 \right) \Big|_0^1 d\theta = (1/4) \cdot 2\pi = \frac{\pi}{2}.
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{16.3.33} \quad \text{Let } x = r \cos \theta \text{ and } y = r \sin \theta. \quad V &= \iint_R ((2 - x^2 - y^2) - (x^2 + y^2)) \, dA = \int_0^{2\pi} \int_0^1 ((2 - r^2) - \\
 (r^2))r \, dr \, d\theta &= \int_0^{2\pi} \int_0^1 (2r - 2r^3) \, dr \, d\theta = \int_0^{2\pi} (r^2 - r^4/2) \Big|_0^1 d\theta = \frac{1}{2} \cdot 2\pi = \pi.
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{16.3.34} \quad \text{Let } x = r \cos \theta \text{ and } y = r \sin \theta. \quad V &= \iint_R ((27 - x^2 - 2y^2) - (2x^2 + y^2)) \, dA = \int_0^{2\pi} \int_0^3 (27 - 3r^2)r \, dr \, d\theta = \\
 \int_0^{2\pi} \int_0^3 (27r - 3r^3) \, dr \, d\theta &= \int_0^{2\pi} (27r^2/2 - 3r^4/4) \Big|_0^3 d\theta = ((243/2) - (243/4)) \cdot 2\pi = \frac{243\pi}{2}.
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{16.3.35} \quad \text{Let } x = r \cos \theta \text{ and } y = r \sin \theta. \quad V &= \iint_R ((4 - x - y) - (x^2 + y^2 - x - y)) \, dA = \int_0^{2\pi} \int_0^2 (4 - r^2)r \, dr \, d\theta = \\
 \int_0^{2\pi} \left(2r^2 - \frac{r^4}{4} \right) \Big|_0^2 d\theta &= 8\pi.
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{16.3.36} \quad \text{Let } x = r \cos \theta \text{ and } y = r \sin \theta. \quad V &= \iint_R ((3 - x) - (x - 3)) \, dA = \int_0^{2\pi} \int_0^2 (6 - 2r \cos \theta)r \, dr \, d\theta = \\
 \int_0^{2\pi} \left(3r^2 - \frac{2r^3}{3} \cos \theta \right) \Big|_0^2 d\theta &= \int_0^{2\pi} \left(12 - \frac{16}{3} \cos \theta \right) d\theta = \left(12\theta - \frac{16}{3} \sin \theta \right) \Big|_0^{2\pi} = 24\pi - 0 = 24\pi.
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{16.3.37} \quad \text{Let } x = r \cos \theta \text{ and } y = r \sin \theta. \quad V &= \iint_R (18 - x^2 - 3y^2 - x^2 + y^2) \, dA = \int_0^{2\pi} \int_0^3 (18 - 2r^2)r \, dr \, d\theta = \\
 \int_0^{2\pi} \int_0^3 (18r - 2r^3) \, dr \, d\theta &= \int_0^{2\pi} (9r^2 - r^4/2) \Big|_0^3 d\theta = \frac{81}{2} \cdot 2\pi = 81\pi.
 \end{aligned}$$

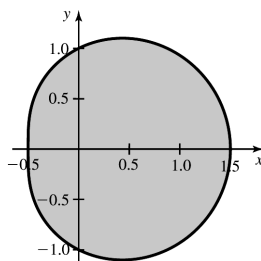
16.3.38 Let $x = r \cos \theta$ and $y = r \sin \theta$. $V = \iint_R ((8 - x^2 + y^2) - (x^2 + 3y^2)) dA = \int_0^{2\pi} \int_1^2 (8r - 2r^3) dr d\theta =$
 $\int_0^{2\pi} \left(4r^2 - \frac{1}{2}r^4 \right) \Big|_1^2 d\theta = 2\pi \left(16 - 8 - 4 + \frac{1}{2} \right) = 9\pi.$

16.3.39 Let $x = r \cos \theta$ and $y = r \sin \theta$. $V = \iint_R (\sqrt{8 - (x^2 + y^2)} - \sqrt{x^2 + y^2}) dA = \int_0^{2\pi} \int_1^2 (\sqrt{8 - r^2} -$
 $r)r dr d\theta = \int_0^{2\pi} \int_1^2 r\sqrt{8 - r^2} dr d\theta - \int_0^{2\pi} \int_1^2 r^2 dr d\theta = \int_0^{2\pi} \left. \frac{-1}{3}(8 - r^2)^{3/2} \right|_1^2 d\theta - \int_0^{2\pi} \left. \frac{r^3}{3} \right|_1^2 d\theta =$
 $\frac{-2\pi}{3} (8 - 7^{3/2}) - \frac{2\pi}{3} (7) = 2\pi \left(\frac{7\sqrt{7}}{3} - 5 \right).$

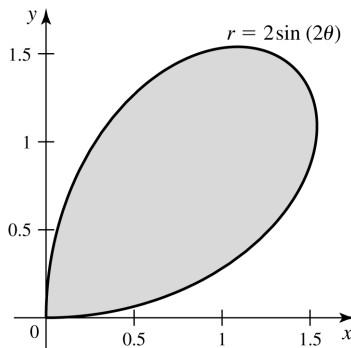
16.3.40 Let $x = r \cos \theta$ and $y = r \sin \theta$.

$V = \iint_R (2 - \sqrt{x^2 + y^2} - \sqrt{1 + x^2 + y^2}) dA = \int_0^{2\pi} \int_0^{3/4} (2 - r - \sqrt{1 + r^2})r dr d\theta =$
 $\int_0^{2\pi} \left(r^2 - \frac{r^3}{3} - \frac{1}{3}(1 + r^2)^{3/2} \right) \Big|_0^{3/4} d\theta = \int_0^{2\pi} \frac{1}{3} \left(\frac{27}{16} - \frac{27}{64} - \frac{125}{64} \right) + \frac{1}{3} d\theta = \frac{2\pi}{3} \left(1 - \frac{33}{48} \right) = \frac{5\pi}{24}.$

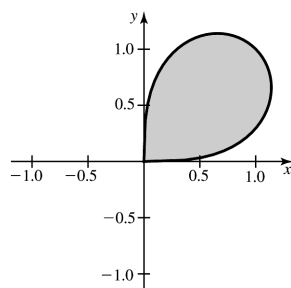
16.3.41 $\iint_R g(r, \theta) dA = \int_0^{2\pi} \int_0^{1 + \frac{1}{2} \cos \theta} g(r, \theta) r dr d\theta.$



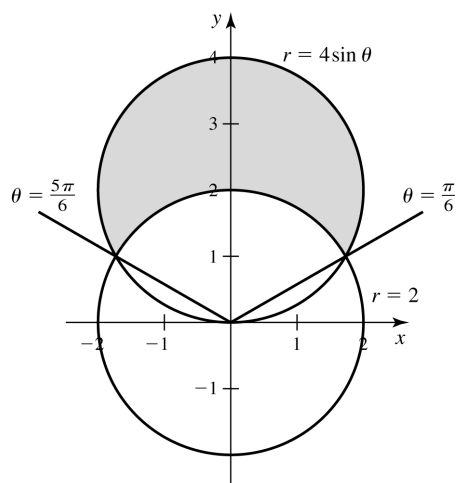
16.3.42 $\iint_R g(r, \theta) dA = \int_0^{\pi/2} \int_0^{2 \sin(2\theta)} g(r, \theta) r dr d\theta.$



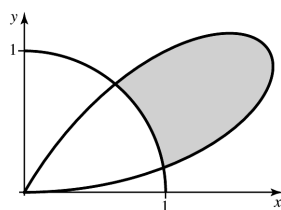
16.3.43 $\iint_R g(r, \theta) dA = \int_0^{\pi/2} \int_0^{\sqrt{2 \sin(2\theta)}} g(r, \theta) r dr d\theta.$



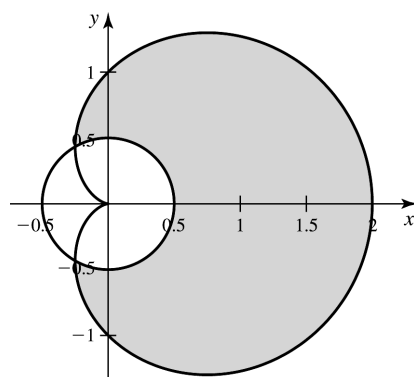
$$16.3.44 \quad \iint_R g(r, \theta) \, dA = \int_{\pi/6}^{5\pi/6} \int_2^{4 \sin \theta} g(r, \theta) \, r \, dr \, d\theta.$$



$$16.3.45 \quad \iint_R g(r, \theta) \, dA = \int_{\pi/18}^{5\pi/18} \int_1^{2 \sin(3\theta)} g(r, \theta) \, r \, dr \, d\theta.$$



$$16.3.46 \quad \iint_R g(r, \theta) \, dA = \int_{-2\pi/3}^{2\pi/3} \int_{1/2}^{1+\cos \theta} g(r, \theta) \, r \, dr \, d\theta.$$



$$16.3.47 \quad A = \iint_R 1 \, dA = \int_0^\pi \int_1^2 r \, dr \, d\theta = \int_0^\pi \left(\frac{1}{2} r^2 \right) \Big|_1^2 d\theta = \int_0^\pi \left(\frac{3}{2} \right) d\theta = \left(\frac{3}{2} \theta \right) \Big|_{\theta=0}^{\theta=\pi} = \frac{3\pi}{2}.$$

16.3.48

$$\begin{aligned} A &= \iint_R 1 \, dA = \int_0^{2\pi} \int_0^{2(1-\sin \theta)} r \, dr \, d\theta = \int_0^{2\pi} \left(\frac{1}{2} r^2 \right) \Big|_0^{2(1-\sin \theta)} d\theta \\ &= \frac{1}{2} \int_0^{2\pi} 4(1 - 2\sin \theta + \sin^2 \theta) \, d\theta = 2 \int_0^{2\pi} \left(\frac{3}{2} - 2\sin \theta - \frac{1}{2} \cos 2\theta \right) d\theta \\ &= 2 \left(\frac{3}{2} \theta + 2 \cos \theta - \frac{1}{4} \sin 2\theta \right) \Big|_{\theta=0}^{\theta=2\pi} = 6\pi. \end{aligned}$$

$$16.3.49 \quad A = \iint_R 1 \, dA = \int_0^\pi \int_0^{2\cos(3\theta)} r \, dr \, d\theta = \int_0^\pi \left(\frac{1}{2} r^2 \right) \Big|_0^{2\cos(3\theta)} d\theta = \int_0^\pi (2\cos^2(3\theta)) \, d\theta = \int_0^\pi (1 + \cos(6\theta)) \, d\theta = \left(\theta + \frac{1}{6} \sin(6\theta) \right) \Big|_{\theta=0}^{\theta=\pi} = \pi.$$

$$\begin{aligned} 16.3.50 \quad A &= \iint_R 1 \, dA = \int_{-\pi/2}^{\pi/2} \int_0^{1-\cos \theta} r \, dr \, d\theta + \int_{\pi/2}^{3\pi/2} \int_0^1 r \, dr \, d\theta. \text{ By symmetry this is equal to} \\ 2 \left(\int_0^{\pi/2} \int_0^{1-\cos \theta} r \, dr \, d\theta + \int_{\pi/2}^\pi \int_0^1 r \, dr \, d\theta \right) &= \int_0^{\pi/2} (r^2) \Big|_0^{1-\cos \theta} d\theta + \int_{\pi/2}^\pi (r^2) \Big|_0^1 d\theta = \int_0^{\pi/2} (1 - \cos \theta)^2 \, d\theta + \int_{\pi/2}^\pi (1) \, d\theta \\ &= \int_0^{\pi/2} \left(\frac{3}{2} - 2\cos \theta + \frac{1}{2} \cos 2\theta \right) d\theta + \int_{\pi/2}^\pi (1) \, d\theta = \left(\frac{3}{2} \theta - 2\sin \theta + \frac{1}{4} \sin 2\theta \right) \Big|_{\theta=0}^{\theta=\pi/2} + (\theta) \Big|_{\theta=\pi/2}^{\theta=\pi} \\ &= \left(\frac{3\pi}{4} - 2 \right) + \left(\frac{\pi}{2} \right) = \frac{5\pi}{4} - 2. \end{aligned}$$

$$\begin{aligned} 16.3.51 \quad A &= \iint_R 1 \, dA = \int_{-3\pi/4}^{\pi/4} \int_0^{1+\sin \theta} r \, dr \, d\theta + \int_{\pi/4}^{5\pi/4} \int_0^{1+\cos \theta} r \, dr \, d\theta. \text{ By symmetry this is equal to} \\ 2 \int_{\pi/4}^{5\pi/4} \int_0^{1+\cos \theta} r \, dr \, d\theta &= \int_{\pi/4}^{5\pi/4} (r^2) \Big|_0^{1+\cos \theta} d\theta = \int_{\pi/4}^{5\pi/4} (1 + 2\cos \theta + \cos^2 \theta) \, d\theta = \int_{\pi/4}^{5\pi/4} \left(\frac{3}{2} + 2\cos \theta + \frac{1}{2} \cos 2\theta \right) d\theta \\ &= \left(\frac{3}{2} \theta + 2\sin \theta + \frac{1}{4} \sin 2\theta \right) \Big|_{\theta=\pi/4}^{\theta=5\pi/4} = \frac{3\pi}{2} - 2\sqrt{2}. \end{aligned}$$

$$16.3.52 \quad A = \iint_R 1 \, dA = \int_0^\pi \int_0^{2\theta} r \, dr \, d\theta = \int_0^\pi \left(\frac{1}{2} r^2 \right) \Big|_0^{2\theta} d\theta = \int_0^\pi (2\theta^2) \, d\theta = \left(\frac{2}{3} \theta^3 \right) \Big|_{\theta=0}^{\theta=\pi} = \frac{2\pi^3}{3}.$$

$$\begin{aligned} 16.3.53 \quad \bar{f} &= \frac{1}{\text{area of } R} \iint_R f(r, \theta) \, dA = \frac{1}{\pi a^2} \int_0^{2\pi} \int_0^a (r) \, r \, dr \, d\theta = \frac{1}{\pi a^2} \int_0^{2\pi} \left(\frac{1}{3} r^3 \right) \Big|_0^a d\theta = \frac{1}{\pi a^2} \int_0^{2\pi} \left(\frac{1}{3} a^3 \right) d\theta \\ &= \frac{a}{3\pi} (\theta) \Big|_{\theta=0}^{\theta=2\pi} = \frac{2a}{3}. \end{aligned}$$

$$\begin{aligned} 16.3.54 \quad \bar{f} &= \frac{1}{\text{area of } R} \iint_R f(r, \theta) \, dA = \frac{1}{12\pi} \int_0^{2\pi} \int_2^4 \frac{1}{r^2} r \, dr \, d\theta = \frac{1}{12\pi} \int_0^{2\pi} \int_2^4 \frac{1}{r} \, dr \, d\theta = \frac{1}{12\pi} \int_0^{2\pi} (\ln |r|) \Big|_2^4 d\theta \\ &= \frac{1}{12\pi} \int_0^{2\pi} (\ln 4 - \ln 2) \, d\theta = \frac{\ln 2}{6}. \end{aligned}$$

16.3.55

a. False. $\iint_R (x^2 + y^2) dA = \int_0^{2\pi} \int_0^1 (r^2) r dr d\theta.$

b. True. The distance from every point on a hemisphere with radius 2 to the origin is 2.

c. True. $\int_0^1 \int_0^{\sqrt{1-y^2}} e^{x^2+y^2} dx dy$ converts to $\int_0^{\pi/2} \int_0^r r e^{r^2} dr d\theta.$

16.3.56 For $r = 2a \cos \theta$: $A = \iint_R 1 dA = \int_0^\pi \int_0^{2a \cos \theta} r dr d\theta = \int_0^\pi \left(\frac{1}{2} r^2 \right) \Big|_0^{2a \cos \theta} d\theta = \int_0^\pi 2a^2 \cos^2 \theta d\theta.$

$$= \int_0^\pi a^2 (1 + \cos 2\theta) d\theta = a^2 \left(\theta + \frac{1}{2} \sin 2\theta \right) \Big|_{\theta=0}^{\theta=\pi} = \pi a^2.$$

For $r = 2a \sin \theta$: $A = \iint_R 1 dA = \int_0^\pi \int_0^{2a \sin \theta} r dr d\theta = \int_0^\pi \left(\frac{1}{2} r^2 \right) \Big|_0^{2a \sin \theta} d\theta = \int_0^\pi 2a^2 \sin^2 \theta d\theta$

$$= \int_0^\pi a^2 (1 - \cos 2\theta) d\theta = a^2 \left(\theta - \frac{1}{2} \sin 2\theta \right) \Big|_{\theta=0}^{\theta=\pi} = \pi a^2.$$

16.3.57 Paraboloid: $V = \iint_R (4 - (x^2 + y^2)) dA = \int_0^{2\pi} \int_0^2 (4 - r^2) r dr d\theta = \int_0^{2\pi} \int_0^2 (4r - r^3) dr d\theta =$

$$\int_0^{2\pi} \left(2r^2 - \frac{1}{4} r^4 \right) \Big|_0^2 d\theta = \int_0^{2\pi} 4 d\theta = 8\pi.$$

Cone: $V = \iint_R (4 - \sqrt{x^2 + y^2}) dA = \int_0^{2\pi} \int_0^4 (4r - r^2) dr d\theta = \int_0^{2\pi} (2r^2 - r^3/3) \Big|_0^4 d\theta = \int_0^{2\pi} \frac{32}{3} d\theta = \frac{64\pi}{3}.$

Hyperboloid: $V = \iint_R (5 - \sqrt{1 + x^2 + y^2}) dA = \int_0^{2\pi} \int_0^{\sqrt{24}} (5 - \sqrt{1 + r^2}) r dr d\theta =$

$$\int_0^{2\pi} \int_0^{\sqrt{24}} (5r - r\sqrt{1 + r^2}) dr d\theta = \int_0^{2\pi} \left(5r^2/2 - \frac{1}{3} (1 + r^2)^{3/2} \right) \Big|_0^{\sqrt{24}} d\theta = \int_0^{2\pi} \frac{56}{3} d\theta = \frac{112\pi}{3}.$$

The bowl that holds the most water is the hyperboloid.

16.3.58 Paraboloid: $V = \int_0^{2\pi} \int_0^{\sqrt{h}} (r^2) r dr d\theta = \int_0^{2\pi} \int_0^{\sqrt{h}} r^3 dr d\theta = \int_0^{2\pi} \left(\frac{1}{4} r^4 \right) \Big|_0^{\sqrt{h}} d\theta = \int_0^{2\pi} \frac{h^2}{4} d\theta =$

$$\frac{\pi h^2}{2}.$$
 Setting this equal to $\frac{176\pi}{3}$ and solving gives $h = \sqrt{\frac{352}{3}} \approx 10.83$ units.

Cone: $V = \int_0^{2\pi} \int_0^h r \cdot r dr d\theta = \int_0^{2\pi} \int_0^h r^2 dr d\theta = \int_0^{2\pi} \left(\frac{1}{3} r^3 \right) \Big|_0^h d\theta = \int_0^{2\pi} \frac{h^3}{3} d\theta = \frac{2\pi h^3}{3}.$ Setting this equal to $\frac{176\pi}{3}$ and solving gives $h = \sqrt[3]{88} \approx 4.45$ units.

16.3.59

a. $z \geq 0$ implies that $x^2 - y^2 \geq 0$, which in turn implies that $x^2 \geq y^2$. Thus $x \geq |y|$ or $x \leq -|y|$ so

$$R = \left\{ (r, \theta) \mid -\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4} \text{ or } \frac{3\pi}{4} \leq \theta \leq \frac{5\pi}{4} \right\}.$$

b. $V = \iint_R (x^2 - y^2) dA = \int_{-\pi/4}^{\pi/4} \int_0^a (r^2 \cos^2 \theta - r^2 \sin^2 \theta) r dr d\theta =$

$$\int_{-\pi/4}^{\pi/4} \int_0^a (\cos^2 \theta - \sin^2 \theta) r^3 dr d\theta = \int_{-\pi/4}^{\pi/4} (\cos^2 \theta - \sin^2 \theta) \left(\frac{1}{4} r^4 \right) \Big|_0^a d\theta = \left(\frac{1}{4} a^4 \right) \int_{-\pi/4}^{\pi/4} \cos 2\theta d\theta = \left(\frac{1}{4} a^4 \right) \left(\frac{1}{2} \sin 2\theta \right) \Big|_{\theta=-\pi/4}^{\theta=\pi/4} = \frac{a^4}{4}.$$

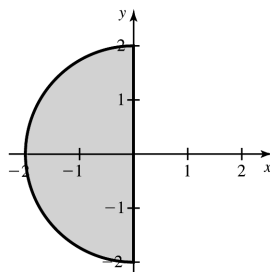
16.3.60 $2 \int_0^{2\pi} \int_0^a \sqrt{a^2 - r^2} r dr d\theta$ represents the volume. Let $u = a^2 - r^2$ so that $du = -2r dr$. Then we have $2 \cdot \frac{-1}{2} \int_0^{2\pi} \int_{a^2}^0 \sqrt{u} du d\theta = 2\pi \int_0^{a^2} u^{1/2} du = \frac{4}{3} \pi u^{3/2} \Big|_0^{a^2} = \frac{4}{3} \pi a^3$.

16.3.61 The volume is given by $\iint_R \sqrt{x^2 + y^2} dA = \int_{-\pi/2}^{\pi/2} \int_0^{2\cos\theta} r^2 dr d\theta = \int_{-\pi/2}^{\pi/2} \frac{8\cos^3\theta}{3} d\theta = \frac{8}{3} \int_{-\pi/2}^{\pi/2} \cos\theta(1 - \sin^2\theta) d\theta = \frac{8}{3} \left(\sin\theta \Big|_{-\pi/2}^{\pi/2} - \frac{\sin^3\theta}{3} \Big|_{-\pi/2}^{\pi/2} \right) = \frac{8}{3} \left(2 - \frac{2}{3} \right) = \frac{32}{9}$.

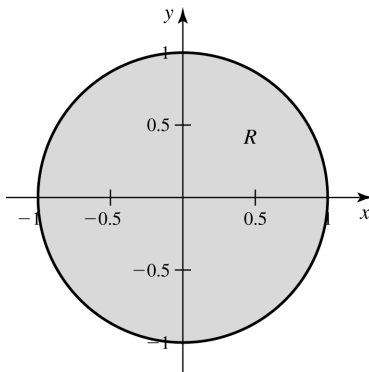
16.3.62 The volume is given by

$$\begin{aligned} \iint_R 2(x^2 + y^2) dA &= \int_0^\pi \int_0^{2\sin\theta} 2r^3 dr d\theta = 2 \int_0^\pi \frac{r^4}{4} \Big|_0^{2\sin\theta} d\theta = 8 \int_0^\pi \sin^4\theta d\theta \\ &= 8 \int_0^\pi \left(\frac{1 - \cos 2\theta}{2} \right)^2 d\theta = 2 \int_0^\pi (1 - 2\cos 2\theta + \cos^2 2\theta) d\theta \\ &= 2 \int_0^\pi \left(\frac{3}{2} - 2\cos 2\theta + \frac{1}{2} \cos 4\theta \right) d\theta = \left(3\theta - 2\sin 2\theta + \frac{1}{4} \sin 4\theta \right) \Big|_0^\pi = 3\pi. \end{aligned}$$

16.3.63 $\iint_R \frac{1}{4 + \sqrt{x^2 + y^2}} dA = \int_{\pi/2}^{3\pi/2} \int_0^2 \frac{1}{4 + r} r dr d\theta = \int_{\pi/2}^{3\pi/2} \int_0^2 \left(1 - \frac{4}{r + 4} \right) dr d\theta = \int_{\pi/2}^{3\pi/2} (r - 4 \ln|r + 4|) \Big|_0^2 d\theta = 2 \int_{\pi/2}^{3\pi/2} \left(1 - 2 \ln \left(\frac{3}{2} \right) \right) d\theta = 2 \left(\theta \left(1 - 2 \ln \left(\frac{3}{2} \right) \right) \right) \Big|_{\theta=\pi/2}^{\theta=3\pi/2} = 2\pi \left(1 - 2 \ln \left(\frac{3}{2} \right) \right).$



16.3.64 $\iint_R \frac{x - y}{x^2 + y^2 + 1} dA = \int_0^{2\pi} \int_0^1 \frac{r \cos\theta - r \sin\theta}{r^2 + 1} r dr d\theta = \int_0^{2\pi} \int_0^1 \frac{r^2 (\cos\theta - \sin\theta)}{r^2 + 1} dr d\theta = \left(\int_0^{2\pi} (\cos\theta - \sin\theta) \left(\int_0^1 \left(1 - \frac{1}{r^2 + 1} \right) dr \right) d\theta \right) = \int_0^{2\pi} (\cos\theta - \sin\theta) \left((r - \tan^{-1}r) \Big|_0^1 \right) d\theta = \int_0^{2\pi} (\cos\theta - \sin\theta) (1 - \tan^{-1}(1)) d\theta = \left(1 - \frac{\pi}{4} \right) \int_0^{2\pi} (\cos\theta - \sin\theta) d\theta = \left(1 - \frac{\pi}{4} \right) (\sin\theta + \cos\theta) \Big|_{\theta=0}^{2\pi} = \left(1 - \frac{\pi}{4} \right) \cdot 0 = 0$



$$\begin{aligned} \mathbf{16.3.65} \quad & \int_0^{\pi/2} \int_1^\infty \frac{\cos \theta}{r^3} r \, dr \, d\theta = \lim_{b \rightarrow \infty} \int_0^{\pi/2} \int_1^b \frac{\cos \theta}{r^2} \, dr \, d\theta = \lim_{b \rightarrow \infty} \int_0^{\pi/2} \left(-\frac{\cos \theta}{r} \right) \Big|_1^b \, d\theta = \\ & \lim_{b \rightarrow \infty} \int_0^{\pi/2} \cos \theta \left(1 - \frac{1}{b} \right) \, d\theta = \lim_{b \rightarrow \infty} \left(\sin \theta \left(1 - \frac{1}{b} \right) \right) \Big|_{\theta=0}^{\theta=\pi/2} = \lim_{b \rightarrow \infty} \left(1 - \frac{1}{b} \right) = 1. \end{aligned}$$

$$\begin{aligned} \mathbf{16.3.66} \quad & \iint_R \frac{dA}{(x^2 + y^2)^{5/2}} = \int_0^{2\pi} \int_1^\infty \frac{1}{(r^2)^{5/2}} r \, dr \, d\theta = \lim_{b \rightarrow \infty} \int_0^{2\pi} \int_1^b \frac{1}{r^4} \, dr \, d\theta = \lim_{b \rightarrow \infty} \int_0^{2\pi} \left(-\frac{1}{3r^3} \right) \Big|_1^b \, d\theta = \\ & \lim_{b \rightarrow \infty} \int_0^{2\pi} \frac{1}{3} \left(1 - \frac{1}{b^3} \right) \, d\theta = \lim_{b \rightarrow \infty} \left(\frac{1}{3} \theta \left(1 - \frac{1}{b^3} \right) \right) \Big|_{\theta=0}^{\theta=2\pi} = \lim_{b \rightarrow \infty} \frac{2\pi}{3} \left(1 - \frac{1}{b^3} \right) = \frac{2\pi}{3}. \end{aligned}$$

$$\begin{aligned} \mathbf{16.3.67} \quad & \iint_R e^{-x^2-y^2} \, dA = \int_0^{\pi/2} \int_0^\infty e^{-r^2} r \, dr \, d\theta = \int_0^{\pi/2} \left(\lim_{b \rightarrow \infty} \int_0^b e^{-r^2} r \, dr \right) \, d\theta = \\ & \int_0^{\pi/2} \left(\lim_{b \rightarrow \infty} \left(-\frac{1}{2} e^{-r^2} \right) \Big|_0^b \right) \, d\theta = \int_0^{\pi/2} \left(\lim_{b \rightarrow \infty} \frac{1}{2} (1 - e^{-b^2}) \right) \, d\theta = \int_0^{\pi/2} \left(\frac{1}{2} \right) \, d\theta = \frac{\pi}{4}. \end{aligned}$$

$$\begin{aligned} \mathbf{16.3.68} \quad & \iint_R \frac{1}{(1+x^2+y^2)^2} \, dA = \int_0^{\pi/2} \int_0^\infty \frac{1}{(1+r^2)^2} r \, dr \, d\theta = \int_0^{\pi/2} \left(\lim_{b \rightarrow \infty} \int_0^b \frac{r}{(1+r^2)^2} \, dr \right) \, d\theta = \\ & \int_0^{\pi/2} \left(\lim_{b \rightarrow \infty} \left(-\frac{1}{2(1+r^2)} \right) \Big|_0^b \right) \, d\theta = \int_0^{\pi/2} \left(\lim_{b \rightarrow \infty} \frac{1}{2} \left(1 - \frac{1}{1+b^2} \right) \right) \, d\theta = \int_0^{\pi/2} \left(\frac{1}{2} \right) \, d\theta = \frac{\pi}{4}. \end{aligned}$$

16.3.69

$$\begin{aligned} \text{a. } V &= \iint_R \sqrt{16-x^2-y^2} \, dA = \int_0^{\pi/4} \int_0^4 \sqrt{16-r^2} r \, dr \, d\theta = \int_0^{\pi/4} \left(-\frac{1}{3} (16-r^2)^{3/2} \right) \Big|_0^4 \, d\theta = \\ & \int_0^{\pi/4} \frac{64}{3} \, d\theta = \frac{16\pi}{3}. \text{ Because this slice is } \frac{1}{8} \text{ of the hemispherical cake, the formula for the volume of a} \\ & \text{sphere can be used to confirm that the volume of the slice is } V = \frac{1}{8} \cdot \frac{1}{2} \cdot \frac{4}{3} \pi \cdot 4^3 = \frac{16\pi}{3}. \end{aligned}$$

$$\begin{aligned} \text{b. } V &= \iint_R (\sqrt{16-x^2-y^2}-a) \, dA = \int_0^{2\pi} \int_0^{\sqrt{16-a^2}} (\sqrt{16-r^2}-a) r \, dr \, d\theta = 2\pi \left(\frac{64}{3} - 8a + \frac{a^3}{6} \right). \text{ Setting} \\ & \text{this equal to } \frac{16\pi}{3} \text{ and using a root finder gives } a \approx 2.78. \end{aligned}$$

16.3.70 The mass $m = \iint_R \rho(r, \theta) \, dA$ where $\rho(r, \theta)$ estimates the density at each point on the plate.

Approximate $m = \iint_R \rho(r, \theta) \, dA \approx \sum_{i=1}^m \sum_{j=1}^n \rho(r_i, \theta_j) \cdot r_i \Delta r \Delta \theta$. Let $\Delta r = 1$, $\Delta \theta = \frac{\pi}{4}$, then sum the upper

right (clockwise outermost) corner for each subsection of region. We have $(2.0 + 2.1 + 2.2 + 2.3) \cdot (1) \cdot (1) \cdot \left(\frac{\pi}{4}\right) + (2.5 + 2.7 + 2.9 + 3.1) \cdot (2) \cdot (1) \cdot \left(\frac{\pi}{4}\right) + (3.2 + 3.4 + 3.5 + 3.6) \cdot (2) \cdot (1) \cdot \left(\frac{\pi}{4}\right) \approx 56.6$ grams.

$$\begin{aligned} 16.3.71 \quad m &= \iint_R \rho(r, \theta) \, dA = \int_0^\pi \int_1^4 (4 + r \sin \theta) \, r \, dr \, d\theta = \int_0^\pi \left(2r^2 + \frac{1}{3}r^3 \sin \theta \right) \Big|_1^4 \, d\theta = \\ &= \int_0^\pi (30 + 21 \sin \theta) \, d\theta = (30\theta - 21 \cos \theta) \Big|_{\theta=0}^{\theta=\pi} = 30\pi + 42 \end{aligned}$$

$$16.3.72 \quad R = \{(r, \theta) \mid 0 \leq r \leq g(\theta), \alpha \leq \theta \leq \beta\}, \quad A = \iint_R 1 \, dA = \int_\alpha^\beta \int_0^{g(\theta)} r \, dr \, d\theta = \int_\alpha^\beta \left(\frac{1}{2}r^2 \right) \Big|_0^{g(\theta)} d\theta.$$

$$\text{Thus } A = \frac{1}{2} \int_\alpha^\beta (g(\theta))^2 \, d\theta.$$

16.3.73

a. Because e^{-x^2} is an even function, $\int_{-\infty}^0 e^{-x^2} \, dx = \int_0^\infty e^{-x^2} \, dx$. Because both of these are convergent, $\int_{-\infty}^\infty e^{-x^2} \, dx$ is convergent too.

$$\begin{aligned} \text{b. } \int_{-\infty}^\infty \int_{-\infty}^\infty e^{-x^2-y^2} \, dx \, dy &= \int_0^{2\pi} \int_0^\infty e^{-r^2} \, r \, dr \, d\theta = \\ &= \int_0^{2\pi} \left(\lim_{b \rightarrow \infty} \int_0^b e^{-r^2} \, r \, dr \right) d\theta = \int_0^{2\pi} \left(\lim_{b \rightarrow \infty} \left(-\frac{1}{2}e^{-r^2} \right) \Big|_0^b \right) d\theta = \int_0^{2\pi} \left(\lim_{b \rightarrow \infty} \frac{1}{2} (1 - e^{-b^2}) \right) d\theta \\ &= \int_0^{2\pi} \frac{1}{2} \, d\theta = \pi. \end{aligned}$$

Thus $\int_{-\infty}^\infty \int_{-\infty}^\infty e^{-x^2-y^2} \, dx \, dy = \pi$. $\int_{-\infty}^\infty \int_{-\infty}^\infty e^{-x^2-y^2} \, dx \, dy = \int_{-\infty}^\infty \int_{-\infty}^\infty (e^{-x^2} \cdot e^{-y^2}) \, dx \, dy = \left(\int_{-\infty}^\infty e^{-x^2} \, dx \right) \cdot \left(\int_{-\infty}^\infty e^{-y^2} \, dy \right) = \left(\int_{-\infty}^\infty e^{-x^2} \, dx \right)^2 = \pi$. Thus $\int_{-\infty}^\infty e^{-x^2} \, dx = \sqrt{\pi}$. Note that the possibility $\int_{-\infty}^\infty e^{-x^2} \, dx = -\sqrt{\pi}$ is rejected because the integrand is everywhere positive and the integral of a positive function is positive.

$$\text{c. i. } \int_0^\infty e^{-x^2} \, dx = \frac{1}{2} \int_{-\infty}^\infty e^{-x^2} \, dx = \frac{\sqrt{\pi}}{2}.$$

$$\begin{aligned} \text{ii. Let } u &= -x^2. \text{ Then } -\frac{du}{2} = x \, dx, \text{ and the integral becomes } \frac{1}{2} \int_{-\infty}^0 e^u \, du = \left(\frac{1}{2} \right) \lim_{b \rightarrow -\infty} \int_b^0 e^u \, du = \\ &= \left(\frac{1}{2} \right) \lim_{b \rightarrow -\infty} (e^u) \Big|_{u=b}^{u=0} = \left(\frac{1}{2} \right) \lim_{b \rightarrow -\infty} (1 - e^b) = \frac{1}{2} (1 - 0) = \frac{1}{2}. \end{aligned}$$

$$\begin{aligned} \text{iii. } \int_0^\infty x^2 e^{-x^2} \, dx &= \lim_{b \rightarrow \infty} \int_0^b x^2 e^{-x^2} \, dx \text{ By parts: let } u = x, \, dv = x e^{-x^2}, \text{ then } du = dx, \, v = -\frac{1}{2} e^{-x^2}. \\ \text{We have } \lim_{b \rightarrow \infty} \left(-\frac{1}{2} x e^{-x^2} \Big|_{u=0}^b + \frac{1}{2} \int_0^b e^{-x^2} \, dx \right) &= 0 + \frac{1}{2} \cdot \frac{\sqrt{\pi}}{2} = \frac{\sqrt{\pi}}{4} \text{ from (i).} \end{aligned}$$

16.3.74

$$\begin{aligned} \text{a. } \iint_R \frac{k}{(x^2 + y^2)^p} \, dA &= \int_0^{2\pi} \int_1^\infty \frac{k}{(r^2)^p} \, r \, dr \, d\theta = \int_0^{2\pi} \left(\lim_{b \rightarrow \infty} \int_1^b \frac{k \, r}{r^{2p}} \, dr \right) d\theta = \\ &= \int_0^{2\pi} \left(\lim_{b \rightarrow \infty} \int_1^b \frac{k}{r^{2p-1}} \, dr \right) d\theta \text{ which converges if } 2p - 1 > 1 \text{ or } p > 1. \end{aligned}$$

b.
$$\iint_R \frac{k}{(x^2 + y^2)^p} dA = \int_0^{2\pi} \int_0^1 \frac{k}{(r^2)^p} r dr d\theta = \int_0^{2\pi} \left(\lim_{b \rightarrow 0} \int_b^1 \frac{k r}{r^{2p}} dr \right) d\theta = \int_0^{2\pi} \left(\lim_{b \rightarrow 0} \int_b^1 \frac{k}{r^{2p-1}} dr \right) d\theta$$

which converges if $2p - 1 \leq 1$ or $p < 1$.

16.3.75

a.

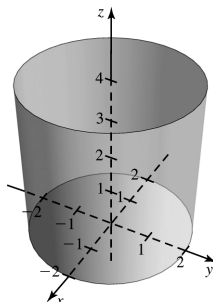
$$\begin{aligned} \int_0^1 \int_0^1 \frac{1}{(1+x^2+y^2)^2} dy dx &= \int_0^{\pi/4} \int_0^{\sec \theta} \frac{1}{(1+r^2)^2} r dr d\theta + \int_{\pi/4}^{\pi/2} \int_0^{\csc \theta} \frac{1}{(1+r^2)^2} r dr d\theta \\ &= \frac{1}{2} \int_0^{\pi/4} \frac{\sec^2 \theta}{1+\sec^2 \theta} d\theta + \frac{1}{2} \int_{\pi/4}^{\pi/2} \frac{\csc^2 \theta}{1+\csc^2 \theta} d\theta \\ &= \frac{1}{2} \int_0^{\pi/4} \frac{\sec^2 \theta}{2+\tan^2 \theta} d\theta + \frac{1}{2} \int_{\pi/4}^{\pi/2} \frac{\csc^2 \theta}{2+\cot^2 \theta} d\theta = \frac{1}{2} \int_0^1 \frac{du}{2+u^2} + \frac{1}{2} \int_0^1 \frac{du}{2+u^2} \\ &= \int_0^1 \frac{du}{2+u^2} = \frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{u}{\sqrt{2}} \right) \Big|_{u=0}^{u=1} = \frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{1}{\sqrt{2}} \right). \end{aligned}$$

b.

$$\begin{aligned} \int_0^1 \int_0^a \frac{1}{(1+x^2+y^2)^2} dy dx &= \int_0^{\tan^{-1}(a)} \int_0^{\sec \theta} \frac{1}{(1+r^2)^2} r dr d\theta + \int_{\tan^{-1}(a)}^{\pi/2} \int_0^{\csc \theta} \frac{1}{(1+r^2)^2} r dr d\theta \\ &= \frac{1}{2} \int_0^{\tan^{-1}(a)} \frac{\sec^2 \theta}{1+\sec^2 \theta} d\theta + \frac{1}{2} \int_{\tan^{-1}(a)}^{\pi/2} \frac{a^2 \csc^2 \theta}{1+a^2 \csc^2 \theta} d\theta \\ &= \frac{1}{2} \int_0^{\tan^{-1}(a)} \frac{\sec^2 \theta}{2+\tan^2 \theta} d\theta + \frac{1}{2} \int_{\tan^{-1}(a)}^{\pi/2} \frac{\csc^2 \theta}{\left(\frac{1+a^2}{a^2}\right) + \cot^2 \theta} d\theta \\ &= \frac{1}{2} \int_0^1 \frac{du}{2+u^2} + \frac{1}{2} \int_0^{1/a} \frac{du}{\left(\frac{1+a^2}{a^2}\right) + u^2} \\ &= \frac{1}{2\sqrt{2}} \tan^{-1} \left(\frac{u}{\sqrt{2}} \right) \Big|_{u=0}^{u=1} + \frac{a}{2\sqrt{1+a^2}} \tan^{-1} \left(\frac{a-u}{\sqrt{1+a^2}} \right) \Big|_{u=0}^{u=1/a} \\ &= \frac{1}{2\sqrt{2}} \tan^{-1} \left(\frac{a}{\sqrt{2}} \right) + \frac{a}{2\sqrt{1+a^2}} \tan^{-1} \left(\frac{1}{\sqrt{1+a^2}} \right). \end{aligned}$$

c.
$$\lim_{a \rightarrow \infty} \int_0^1 \int_0^a \frac{1}{(1+x^2+y^2)^2} dy dx = \lim_{a \rightarrow \infty} \left[\frac{1}{2\sqrt{2}} \tan^{-1} \left(\frac{a}{\sqrt{2}} \right) + \frac{a}{2\sqrt{1+a^2}} \tan^{-1} \left(\frac{1}{\sqrt{1+a^2}} \right) \right] =$$

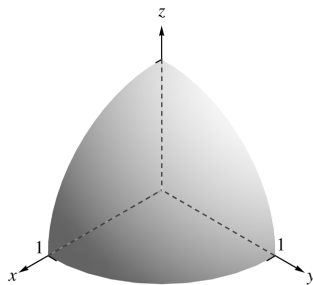
$$\frac{1}{2\sqrt{2}} \left(\frac{\pi}{2} \right) + \frac{1}{2} (1)(0) = \frac{\pi\sqrt{2}}{8}.$$

16.4 Triple Integrals**16.4.1**

$$16.4.2 \quad \iiint_D f(x, y, z) \, dV = \int_0^4 \int_0^6 \int_0^3 f(x, y, z) \, dx \, dy \, dz.$$

$$16.4.3 \quad \iiint_D f(x, y, z) \, dV = \int_{-9}^9 \int_{-\sqrt{81-x^2}}^{\sqrt{81-x^2}} \int_{-\sqrt{81-x^2-y^2}}^{\sqrt{81-x^2-y^2}} f(x, y, z) \, dz \, dy \, dx.$$

16.4.4



$$16.4.5 \quad \int_0^1 \int_0^{\sqrt{1-z^2}} \int_0^{\sqrt{1-x^2-z^2}} f(x, y, z) \, dy \, dx \, dz.$$

$$16.4.6 \quad \bar{f} = \frac{1}{\text{volume of } D} \iiint_D f(x, y, z) \, dV = \frac{1}{\text{volume of } D} \int_{-3}^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_0^{9-x^2-y^2} xyz \, dz \, dy \, dx.$$

$$16.4.7 \quad \int_{-2}^2 \int_3^6 \int_0^2 dx \, dy \, dz = \int_{-2}^2 \int_3^6 (x) \Big|_0^2 dy \, dz = \int_{-2}^2 \int_3^6 2 \, dy \, dz = \int_{-2}^2 (2y) \Big|_3^6 dz = \int_{-2}^2 6 \, dz = (6z) \Big|_{-2}^2 = 24.$$

$$16.4.8 \quad \int_{-1}^1 \int_{-1}^2 \int_0^1 6xyz \, dy \, dx \, dz = \int_{-1}^1 \int_{-1}^2 (3xy^2z) \Big|_0^1 dx \, dz = \int_{-1}^1 \int_{-1}^2 3xz \, dx \, dz = \int_{-1}^1 \left(\frac{3}{2} x^2 z \right) \Big|_{-1}^2 dz = \int_{-1}^1 \frac{9}{2} z \, dz = \left(\frac{9}{4} z^2 \right) \Big|_{-1}^1 = 0.$$

$$16.4.9 \quad \int_{-2}^2 \int_1^2 \int_1^e \frac{xy^2}{z} dz \, dx \, dy = \int_{-2}^2 \int_1^2 xy^2 \left(xy^2 \ln |z| \Big|_1^e \right) dx \, dy = \int_{-2}^2 \int_1^2 xy^2 dx \, dy = \int_{-2}^2 \left(\frac{1}{2} x^2 y^2 \right) \Big|_1^2 dy = \int_{-2}^2 \frac{3}{2} y^2 dy = \left(\frac{1}{2} y^3 \right) \Big|_{-2}^2 = 8.$$

$$16.4.10 \quad \int_0^{\ln 4} \int_0^{\ln 3} \int_0^{\ln 2} e^{-x+y+z} dx \, dy \, dz = \int_0^{\ln 4} \int_0^{\ln 3} -e^{-x+y+z} \Big|_0^{\ln 2} dy \, dz = \int_0^{\ln 4} \int_0^{\ln 3} ((-1/2)(e^{y+z}) + e^{y+z}) dy \, dz = \frac{1}{2} \int_0^{\ln 4} \int_0^{\ln 3} e^{y+z} dy \, dz = \frac{1}{2} \int_0^{\ln 4} e^{y+z} \Big|_0^{\ln 3} dz = \frac{1}{2} \int_0^{\ln 4} (3e^z - e^z) dz = \int_0^{\ln 4} e^z dz = 4 - 1 = 3.$$

$$16.4.11 \quad \int_0^{\pi/2} \int_0^1 \int_0^{\pi/2} \sin \pi x \cdot \cos y \cdot \sin 2z \, dy \, dx \, dz = \int_0^{\pi/2} \int_0^1 \sin \pi x \sin 2z \left((\sin y) \Big|_0^{\pi/2} \right) dx \, dz = \int_0^{\pi/2} \int_0^1 \sin \pi x \sin 2z (1 - 0) dx \, dz = \int_0^{\pi/2} \int_0^1 \sin \pi x \sin 2z dx \, dz = \int_0^{\pi/2} \sin 2z \left(\left(-\frac{1}{\pi} \cos \pi x \right) \Big|_0^1 \right) dz = \int_0^{\pi/2} \sin 2z \left(\frac{1}{\pi} + \frac{1}{\pi} \right) dz = \frac{2}{\pi} \int_0^{\pi/2} \sin 2z dz = \frac{2}{\pi} \left(-\frac{1}{2} \cos 2z \right) \Big|_0^{\pi/2} = \frac{2}{\pi} \left(\frac{1}{2} + \frac{1}{2} \right) = \frac{2}{\pi}.$$

$$\begin{aligned}
 \mathbf{16.4.12} \quad & \int_0^2 \int_1^2 \int_0^1 yz e^x dx dz dy = \int_0^2 \int_1^2 yz \left(e^x \right) \Big|_0^1 dz dy = \int_0^2 \int_1^2 yz (e - 1) dz dy = \\
 & (e - 1) \int_0^2 y \left(\frac{1}{2} z^2 \right) \Big|_1^2 dy = (e - 1) \int_0^2 \frac{3}{2} y dy = \frac{3(e - 1)}{2} \left(\frac{1}{2} y^2 \right) \Big|_0^2 = 3(e - 1).
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{16.4.13} \quad & \iiint_D (xy + xz + yz) dV = \int_{-1}^1 \int_{-2}^2 \int_{-3}^3 (xy + xz + yz) dz dy dx = \\
 & \int_{-1}^1 \int_{-2}^2 \left(xyz + \frac{1}{2} xz^2 + \frac{1}{2} yz^2 \right) \Big|_{-3}^3 dy dx = \int_{-1}^1 \int_{-2}^2 (6xy) dy dx = \int_{-1}^1 (3xy^2) \Big|_{-2}^2 dx = \int_{-1}^1 (0) dx = 0.
 \end{aligned}$$

16.4.14

$$\begin{aligned}
 \iiint_D xyz e^{-x^2-y^2} dV &= \int_0^{\sqrt{\ln 2}} \int_0^{\sqrt{\ln 4}} \int_0^1 xyz e^{-x^2-y^2} dz dy dx = \int_0^{\sqrt{\ln 2}} \int_0^{\sqrt{\ln 4}} \frac{1}{2} xyz^2 e^{-x^2-y^2} \Big|_0^1 dy dx \\
 &= \frac{1}{2} \int_0^{\sqrt{\ln 2}} \int_0^{\sqrt{\ln 4}} xy e^{-x^2-y^2} dy dx = \frac{1}{2} \int_0^{\sqrt{\ln 2}} \int_0^{\sqrt{\ln 4}} xy e^{-x^2} e^{-y^2} dy dx \\
 &= \frac{1}{2} \int_0^{\sqrt{\ln 2}} x e^{-x^2} \left(-\frac{1}{2} e^{-y^2} \right) \Big|_0^{\sqrt{\ln 4}} dx = \frac{1}{2} \int_0^{\sqrt{\ln 2}} x e^{-x^2} \left(\frac{1}{2} \left(1 - \frac{1}{4} \right) \right) dx \\
 &= \frac{3}{16} \int_0^{\sqrt{\ln 2}} x e^{-x^2} dx = \frac{3}{16} \left(\left(-\frac{1}{2} e^{-x^2} \right) \Big|_0^{\sqrt{\ln 2}} \right) = \frac{3}{16} \left(\frac{1}{2} \left(1 - \frac{1}{2} \right) \right) = \frac{3}{64}.
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{16.4.15} \quad & V = \iiint_D 1 dV = \int_0^6 \int_0^{4-2x/3} \int_0^{2-x/3-y/2} 1 dz dy dx = \int_0^6 \int_0^{4-2x/3} (z) \Big|_0^{2-x/3-y/2} dy dx = \\
 & \int_0^6 \int_0^{4-2x/3} \left(2 - \frac{1}{3}x - \frac{1}{2}y \right) dy dx = \int_0^6 \left(2y - \frac{1}{3}xy - \frac{1}{4}y^2 \right) \Big|_0^{4-2x/3} dx = \\
 & \int_0^6 \left(\frac{1}{9}x^2 - \frac{4}{3}x + 4 \right) dx = \left(\frac{1}{27}x^3 - \frac{2}{3}x^2 + 4x \right) \Big|_0^6 = 8.
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{16.4.16} \quad & V = \iiint_D 1 dV = \int_0^\pi \int_x^\pi \int_0^{\sin y} 1 dz dy dx = \int_0^\pi \int_x^\pi (z) \Big|_0^{\sin y} dy dx = \int_0^\pi \int_x^\pi (\sin y) dy dx = \\
 & \int_0^\pi (-\cos y) \Big|_x^\pi dx = \int_0^\pi (1 + \cos x) dx = (x + \sin x) \Big|_0^\pi = \pi.
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{16.4.17} \quad & V = \iiint_D 1 dV = \int_{-2}^2 \int_{-\sqrt{4-x^2}}^0 \int_0^{-y} 1 dz dy dx = \int_{-2}^2 \int_{-\sqrt{4-x^2}}^0 (z) \Big|_0^{-y} dy dx = \\
 & \int_{-2}^2 \int_{-\sqrt{4-x^2}}^0 (-y) dy dx = \int_{-2}^2 \left(-\frac{1}{2} y^2 \right) \Big|_{-\sqrt{4-x^2}}^0 dx = \frac{1}{2} \int_{-2}^2 (4 - x^2) dx = \frac{1}{2} \left(4x - \frac{1}{3} x^3 \right) \Big|_{-2}^2 = \frac{16}{3}.
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{16.4.18} \quad & V = \iiint_D 1 dV = \int_0^{1/2} \int_0^8 \int_0^{2-4x} 1 dz dy dx = \int_0^{1/2} \int_0^8 (z) \Big|_0^{2-4x} dy dx = \\
 & \int_0^{1/2} \int_0^8 (2 - 4x) dy dx = \int_0^{1/2} ((2 - 4x)y) \Big|_0^8 dx = \int_0^{1/2} (16 - 32x) dx = (16x - 16x^2) \Big|_0^{1/2} = 4.
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{16.4.19} \quad & V = \iiint_D 1 dV = \int_0^{\ln 2} \int_0^1 \int_1^{e^y} 1 dz dx dy = \int_0^{\ln 2} \int_0^1 (z) \Big|_1^{e^y} dx dy = \int_0^{\ln 2} \int_0^1 (e^y - 1) dx dy = \\
 & \int_0^{\ln 2} (e^y - 1)x \Big|_0^1 dy = \int_0^{\ln 2} (e^y - 1) dy = (e^y - y) \Big|_0^{\ln 2} = 1 - \ln 2.
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{16.4.20} \quad V &= \iiint_D 1 \, dV = \int_{-\sqrt{3}}^{\sqrt{3}} \int_{x^2}^3 \int_0^{3-y} 1 \, dz \, dy \, dx = \int_{-\sqrt{3}}^{\sqrt{3}} \int_{x^2}^3 (z) \Big|_0^{3-y} dy \, dx = \int_{-\sqrt{3}}^{\sqrt{3}} \int_{x^2}^3 (3-y) \, dy \, dx = \\
 &= \int_{-\sqrt{3}}^{\sqrt{3}} \left(3y - \frac{1}{2}y^2 \right) \Big|_{x^2}^3 dx = \int_{-\sqrt{3}}^{\sqrt{3}} \left(\frac{9}{2} - 3x^2 + \frac{1}{2}x^4 \right) dx = \left(\frac{9}{2}x - x^3 + \frac{1}{10}x^5 \right) \Big|_{-\sqrt{3}}^{\sqrt{3}} = \frac{24\sqrt{3}}{5}.
 \end{aligned}$$

16.4.21

$$\begin{aligned}
 V &= \iiint_D 1 \, dV = \int_{-3}^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_{\sqrt{1+x^2+y^2}}^{\sqrt{19-x^2-y^2}} 1 \, dz \, dy \, dx \\
 &= \int_{-3}^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} (z) \Big|_{\sqrt{1+x^2+y^2}}^{\sqrt{19-x^2-y^2}} dy \, dx \\
 &= \int_{-3}^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \left(\sqrt{19-x^2-y^2} - \sqrt{1+x^2+y^2} \right) dy \, dx.
 \end{aligned}$$

Converting to polar coordinates gives

$$\begin{aligned}
 &\int_0^{2\pi} \int_0^3 \left(\sqrt{19-r^2} - \sqrt{1+r^2} \right) r \, dr \, d\theta \\
 &= \int_0^{2\pi} \int_0^3 \left(r\sqrt{19-r^2} - r\sqrt{1+r^2} \right) dr \, d\theta \\
 &= \int_0^{2\pi} \left(-\frac{1}{3} \right) \left((19-r^2)^{3/2} + (1+r^2)^{3/2} \right) \Big|_0^3 d\theta = \left(-\frac{1}{3} \right) \int_0^{2\pi} \left(20\sqrt{10} - 19\sqrt{19} - 1 \right) d\theta \\
 &= \frac{1 + 19\sqrt{19} - 20\sqrt{10}}{3} \int_0^{2\pi} d\theta = \frac{2\pi}{3} \left(1 + 19\sqrt{19} - 20\sqrt{10} \right).
 \end{aligned}$$

16.4.22

$$\begin{aligned}
 V &= \iiint_D 1 \, dV = \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{8-x^2-y^2}} 1 \, dz \, dy \, dx = \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (z) \Big|_{\sqrt{x^2+y^2}}^{\sqrt{8-x^2-y^2}} dy \, dx \\
 &= \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \left(\sqrt{8-x^2-y^2} - \sqrt{x^2+y^2} \right) dy \, dx.
 \end{aligned}$$

Converting to polar coordinates gives

$$\begin{aligned}
 \int_0^{2\pi} \int_0^2 \left(\sqrt{8-r^2} - \sqrt{r^2} \right) r \, dr \, d\theta &= \int_0^{2\pi} \int_0^2 \left(r\sqrt{8-r^2} - r^2 \right) dr \, d\theta = \int_0^{2\pi} \left(-\frac{1}{3} (8-r^2)^{3/2} - \frac{1}{3} r^3 \right) \Big|_0^2 d\theta \\
 &= \int_0^{2\pi} \left(\frac{-16 + 16\sqrt{2}}{3} \right) d\theta = \frac{32\pi}{3} (\sqrt{2} - 1).
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{16.4.23} \quad V &= \iiint_D 1 \, dV = \iint_R (9 - 3(x^2 + z^2)) \, dA = \int_0^{2\pi} \int_0^{\sqrt{3}} (9 - 3r^2) r \, dr \, d\theta = \int_0^{2\pi} \left(\frac{9r^2}{2} - \frac{3r^4}{4} \right) \Big|_0^{\sqrt{3}} d\theta = \\
 &= 2\pi \left(\frac{27}{2} - \frac{27}{4} \right) = \frac{27\pi}{2}.
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{16.4.24} \quad V &= \iiint_D 1 \, dV = \int_0^1 \int_0^2 \int_{z^2}^{2-z} 1 \, dx \, dy \, dz = \int_0^1 \int_0^2 (2 - z - z^2) \, dy \, dz = 2 \int_0^1 (2 - z - z^2) \, dz = \\
 &= 2 \left(2z - \frac{z^2}{2} - \frac{z^3}{3} \right) \Big|_0^1 = 2 \left(2 - \frac{1}{2} - \frac{1}{3} \right) = \frac{7}{3}.
 \end{aligned}$$

$$\begin{aligned}
16.4.25 \quad V &= \iiint_D 1 \, dV = \int_{-1}^1 \int_{-\sqrt{4-4z^2}}^{\sqrt{4-4z^2}} \int_{x-3}^{3-x} 1 \, dy \, dx \, dz = \int_{-1}^1 \int_{-\sqrt{4-4z^2}}^{\sqrt{4-4z^2}} y \Big|_{x-3}^{3-x} dx \, dz = \\
&\int_{-1}^1 \int_{-\sqrt{4-4z^2}}^{\sqrt{4-4z^2}} (6-2x) \, dx \, dz = \int_{-1}^1 (6x-x^2) \Big|_{-\sqrt{4-4z^2}}^{\sqrt{4-4z^2}} dz = \int_{-1}^1 12\sqrt{4-4z^2} \, dx = 24 \int_{-1}^1 \sqrt{1-z^2} \, dx = \\
&24 \cdot \frac{\pi}{2} = 12\pi.
\end{aligned}$$

$$\begin{aligned}
16.4.26 \quad V &= \int_0^1 \int_0^2 \int_0^{e^{-z}} 1 \, dy \, dx \, dz = \int_0^1 \int_0^2 y \Big|_0^{e^{-z}} dx \, dz \\
&= \int_0^1 \int_0^2 e^{-z} \, dx \, dz = \int_0^1 2e^{-z} \, dz = (-2e^{-z}) \Big|_0^1 = 2 - \frac{2}{e}.
\end{aligned}$$

$$\begin{aligned}
16.4.27 \quad V &= \int_0^1 \int_0^{1-z^2} \int_0^{1-z} 1 \, dy \, dx \, dz = \int_0^1 \int_0^{1-z^2} y \Big|_0^{1-z} dx \, dz = \int_0^1 \int_0^{1-z^2} (1-z) \, dx \, dz = \\
&\int_0^1 (1-z)x \Big|_0^{1-z^2} = \int_0^1 (1-z)(1-z^2) \, dz = \int_0^1 (1-z-z^2+z^3) \, dz = \left(z - \frac{z^2}{2} - \frac{z^3}{3} + \frac{z^4}{4} \right) \Big|_0^1 = \frac{5}{12}.
\end{aligned}$$

16.4.28

$$\begin{aligned}
V &= \int_0^1 \int_0^{2-z-z^2} \int_{z^2}^{2-x-z} 1 \, dy \, dx \, dz = \int_0^1 \int_0^{2-z-z^2} y \Big|_{z^2}^{2-x-z} dx \, dz = \int_0^1 \int_0^{2-z-z^2} (2-x-z-z^2) \, dx \, dz \\
&= \int_0^1 \left(2x - \frac{x^2}{2} - zx - z^2x \right) \Big|_0^{2-z-z^2} dz = \int_0^1 \left(\frac{z^4}{2} + z^3 - \frac{3z^2}{2} - 2z + 2 \right) dz \\
&= \left(\frac{z^5}{10} + \frac{z^4}{4} - \frac{3z^3}{2} - z^2 + 2z \right) \Big|_0^1 = \frac{17}{20}.
\end{aligned}$$

$$16.4.29 \quad V = \int_0^2 \int_0^4 \int_z^{z+1} 1 \, dy \, dz \, dx = \int_0^2 \int_0^4 y \Big|_z^{z+1} dz \, dx = \int_0^2 \int_0^4 1 \, dz \, dx = \int_0^2 4 \, dx = 8.$$

$$16.4.30 \quad \int_0^1 \int_0^x \int_0^{2\sqrt{1-x^2}} f(x, y, z) \, dz \, dy \, dx.$$

$$16.4.31 \quad \int_0^1 \int_y^1 \int_0^{2\sqrt{1-x^2}} f(x, y, z) \, dz \, dx \, dy.$$

$$16.4.32 \quad \int_0^2 \int_0^{\frac{1}{2}\sqrt{4-z^2}} \int_0^x f(x, y, z) \, dy \, dx \, dz.$$

$$16.4.33 \quad \int_0^1 \int_0^{2\sqrt{1-x^2}} \int_0^x f(x, y, z) \, dy \, dz \, dx.$$

$$16.4.34 \quad \int_0^2 \int_0^{\frac{1}{2}\sqrt{4-z^2}} \int_y^{\frac{1}{2}\sqrt{4-z^2}} f(x, y, z) \, dx \, dy \, dz.$$

$$16.4.35 \quad \int_0^1 \int_0^{2\sqrt{1-y^2}} \int_{y^2}^{\frac{1}{2}\sqrt{4-z^2}} f(x, y, z) \, dx \, dz \, dy.$$

16.4.36

$$\text{i. } \int_0^1 \int_x^1 \int_0^{1-y^2} dz \, dy \, dx.$$

$$\text{ii. } \int_0^1 \int_0^y \int_0^{1-y^2} dz \, dx \, dy.$$

$$\text{iii. } \int_0^1 \int_0^{1-x^2} \int_x^{\sqrt{1-z}} dy \, dz \, dx.$$

$$\text{iv. } \int_0^1 \int_0^{\sqrt{1-z}} \int_x^{\sqrt{1-z}} dy \, dx \, dz.$$

$$\text{v. } \int_0^1 \int_0^{\sqrt{1-z}} \int_0^y dx \, dy \, dz.$$

$$\text{vi. } \int_0^1 \int_0^{1-y^2} \int_0^y dx \, dz \, dy.$$

It is a good exercise to see that each of these evaluate to $\frac{1}{4}$.

16.4.37

$$\text{i. } \int_0^1 \int_0^2 \int_0^{1-y} dz \, dx \, dy.$$

$$\text{ii. } \int_0^2 \int_0^1 \int_0^{1-z} dy \, dz \, dx.$$

$$\text{iii. } \int_0^1 \int_0^2 \int_0^{1-z} dy \, dx \, dz.$$

$$\text{iv. } \int_0^1 \int_0^{1-z} \int_0^2 dx \, dy \, dz.$$

$$\text{v. } \int_0^1 \int_0^{1-y} \int_0^2 dx \, dz \, dy.$$

$$\begin{aligned} \text{16.4.38 } \int_0^\pi \int_0^\pi \int_0^{\sin x} \sin y \, dz \, dx \, dy &= \int_0^\pi \int_0^\pi \sin x \cdot \sin y \, dx \, dy = \int_0^\pi (-\cos x \cdot \sin y) \Big|_0^\pi dy = \\ &= \int_0^\pi 2 \sin y \, dy = (-2 \cos y) \Big|_0^\pi = 4. \end{aligned}$$

$$\begin{aligned} \text{16.4.39 } \int_0^2 \int_0^4 \int_{y^2}^4 \sqrt{x} \, dz \, dx \, dy &= \int_0^2 \int_0^4 z \sqrt{x} \Big|_{y^2}^4 dx \, dy = \int_0^2 \int_0^4 (4 - y^2) \sqrt{x} \, dx \, dy = \\ &= \int_0^2 (4 - y^2) \left(\frac{2}{3} x^{3/2} \right) \Big|_0^4 dy = \frac{16}{3} \int_0^2 (4 - y^2) \, dy = \frac{16}{3} \left(4y - \frac{1}{3} y^3 \right) \Big|_0^2 = \frac{256}{9}. \end{aligned}$$

$$\begin{aligned} \text{16.4.40 } \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} 2xz \, dz \, dy \, dx &= \int_0^1 \int_0^{\sqrt{1-x^2}} (xz^2) \Big|_0^{\sqrt{1-x^2-y^2}} dy \, dx = \\ &= \int_0^1 \int_0^{\sqrt{1-x^2}} (x - x^3 - xy^2) \, dy \, dx. \text{ Switching the order of integration yields} \\ &= \int_0^1 \int_0^{\sqrt{1-y^2}} (x - x^3 - xy^2) \, dx \, dy = \int_0^1 \left(\frac{1}{2} x^2 - \frac{1}{4} x^4 - \frac{1}{2} x^2 y^2 \right) \Big|_0^{\sqrt{1-y^2}} dy = \int_0^1 \left(\frac{1}{4} - \frac{1}{2} y^2 + \frac{1}{4} y^4 \right) dy = \\ &= \left(\frac{1}{4} y - \frac{1}{6} y^3 + \frac{1}{20} y^5 \right) \Big|_0^1 = \frac{2}{15}. \end{aligned}$$

$$\begin{aligned}
 16.4.41 \quad \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2}} dz \, dy \, dx &= \int_0^1 \int_0^{\sqrt{1-x^2}} (z) \Big|_0^{\sqrt{1-x^2}} dy \, dx = \int_0^1 \int_0^{\sqrt{1-x^2}} \sqrt{1-x^2} \, dy \, dx = \\
 \int_0^1 y \sqrt{1-x^2} \Big|_0^{\sqrt{1-x^2}} dx &= \int_0^1 (1-x^2) \, dx = \left(x - \frac{1}{3}x^3 \right) \Big|_0^1 = \frac{2}{3}.
 \end{aligned}$$

16.4.42

$$\begin{aligned}
 \int_1^6 \int_0^{4-2y/3} \int_0^{12-2y-3z} \frac{1}{y} \, dx \, dz \, dy &= \int_1^6 \int_0^{4-2y/3} \left(\frac{x}{y} \right) \Big|_0^{12-2y-3z} dz \, dy \\
 &= \int_1^6 \int_0^{4-2y/3} \frac{12-2y-3z}{y} \, dz \, dy \\
 &= \int_1^6 \frac{1}{y} \left((12-2y)z - \frac{3}{2}z^2 \right) \Big|_0^{4-2y/3} dy \\
 &= \int_1^6 \frac{1}{y} \left(24 - 8y + \frac{2}{3}y^2 \right) dy = \int_1^6 \left(\frac{24}{y} - 8 + \frac{2}{3}y \right) dy \\
 &= \left(24 \ln |y| - 8y + \frac{1}{3}y^2 \right) \Big|_1^6 = 24 \ln 6 + \frac{85}{3}.
 \end{aligned}$$

$$16.4.43 \quad \int_0^3 \int_0^{\sqrt{9-z^2}} \int_0^{\sqrt{1+x^2+z^2}} dy \, dx \, dz = \int_0^3 \int_0^{\sqrt{9-z^2}} \sqrt{1+x^2+z^2} \, dz \, dy. \text{ Use polar coordinates.}$$

$$\int_0^{\pi/2} \int_0^3 \sqrt{1+r^2} \, r \, dr \, d\theta = \int_0^{\pi/2} \left(\frac{1}{3} (1+r^2)^{3/2} \right) \Big|_0^3 d\theta = \int_0^{\pi/2} \frac{1}{3} (10^{3/2} - 1) \, d\theta = \frac{\pi}{6} (10\sqrt{10} - 1).$$

16.4.44

$$\begin{aligned}
 \int_0^1 \int_y^{2-y} \int_0^{2-x-y} 15xy \, dz \, dx \, dy &= 15 \int_0^1 \int_y^{2-y} (xyz) \Big|_0^{2-x-y} dx \, dy \\
 &= 15 \int_0^1 \int_y^{2-y} (2xy - x^2y - xy^2) \, dx \, dy \\
 &= 15 \int_0^1 \left(x^2y - \frac{1}{3}x^3y - \frac{1}{2}x^2y^2 \right) \Big|_y^{2-y} dy \\
 &= 15 \int_0^1 \left((2-y)^2y - \frac{1}{3}(2-y)^3y - \frac{1}{2}(2-y)^2y^2 - y^3 + \frac{1}{3}y^4 + \frac{1}{2}y^4 \right) dy \\
 &= 15 \int_0^1 \left(\frac{4}{3}y - 2y^2 + \frac{2}{3}y^4 \right) dy = 15 \left(\frac{2}{3}y^2 - \frac{2}{3}y^3 + \frac{2}{15}y^5 \right) \Big|_0^1 = 2.
 \end{aligned}$$

$$\begin{aligned}
 16.4.45 \quad \int_1^{\ln 8} \int_1^{\sqrt{z}} \int_{\ln y}^{\ln(2y)} e^{x+y^2-z} \, dx \, dy \, dz &= \int_1^{\ln 8} \int_1^{\sqrt{z}} e^{x+y^2-z} \Big|_{\ln y}^{\ln(2y)} dy \, dz = \\
 \int_1^{\ln 8} \int_1^{\sqrt{z}} (2ye^{y^2-z} - ye^{y^2-z}) \, dy \, dz &= \int_1^{\ln 8} \int_1^{\sqrt{z}} ye^{y^2-z} \, dy \, dz = \int_1^{\ln 8} \left(\frac{1}{2} e^{y^2-z} \right) \Big|_1^{\sqrt{z}} dz = \\
 \frac{1}{2} \int_1^{\ln 8} (1 - e^{1-z}) \, dz &= \frac{1}{2} (z + e^{1-z}) \Big|_1^{\ln 8} = \frac{1}{2} \left(\ln 8 + \frac{e}{8} - (1+1) \right) = \frac{1}{2} \ln 8 + \frac{e}{16} - 1.
 \end{aligned}$$

$$\begin{aligned}
 16.4.46 \quad \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{2-x} 4yz \, dz \, dy \, dx &= \int_0^1 \int_0^{\sqrt{1-x^2}} (2yz^2) \Big|_0^{2-x} dy \, dx = \int_0^1 \int_0^{\sqrt{1-x^2}} 2y(2-x)^2 \, dy \, dx = \\
 \int_0^1 y^2(2-x)^2 \Big|_0^{\sqrt{1-x^2}} dx &= \int_0^1 (1-x^2)(2-x)^2 \, dx = \int_0^1 (4-4x-3x^2+4x^3-x^4) \, dx = \\
 \left(4x - 2x^2 - x^3 + x^4 - \frac{1}{5}x^5 \right) \Big|_0^1 &= \frac{9}{5} - 0 = \frac{9}{5}.
 \end{aligned}$$

16.4.47 We can rewrite $\int_0^5 \int_{-1}^0 \int_0^{4x+4} dy \, dx \, dz$ as $\int_0^4 \int_{(y-4)/4}^0 \int_0^5 dz \, dx \, dy$. This is then evaluated as

$$\int_0^4 \int_{(y-4)/4}^0 5 \, dx \, dy = \int_0^4 -\frac{5}{4} (y-4) \, dy = -\frac{5}{4} \left(\frac{1}{2} y^2 - 4y \right) \Big|_0^4 = 10.$$

16.4.48 We can rewrite $\int_0^1 \int_{-2}^2 \int_0^{\sqrt{4-y^2}} dz \, dy \, dx$ as $\int_0^1 \int_0^2 \int_{-\sqrt{4-z^2}}^{\sqrt{4-z^2}} dy \, dz \, dx$. This can then be evaluated as

$$\int_0^1 \int_0^2 2\sqrt{4-z^2} \, dz \, dx = \int_0^1 \left(z\sqrt{4-z^2} + 4 \sin^{-1} \left(\frac{z}{2} \right) \right) \Big|_0^2 dx = \int_0^1 (2\pi) \, dx = (2\pi x) \Big|_0^1 = 2\pi.$$

16.4.49 We can rewrite $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2}} dy \, dz \, dx$ as $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2}} dz \, dy \, dx$. This is then evaluated as

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \sqrt{1-x^2} \, dy \, dx = \int_0^1 (\sqrt{1-x^2}) (\sqrt{1-x^2}) \, dx = \int_0^1 (1-x^2) \, dx = \left(x - \frac{1}{3} x^3 \right) \Big|_0^1 = \frac{2}{3}.$$

16.4.50 We can rewrite $\int_0^4 \int_0^{\sqrt{16-x^2}} \int_0^{\sqrt{16-x^2-y^2}} dy \, dz \, dx$ as $\int_0^4 \int_0^{\sqrt{16-z^2}} \int_0^{\sqrt{16-y^2-z^2}} dx \, dy \, dz$. This can then be evaluated as

$$\int_0^4 \int_0^{\sqrt{16-z^2}} \sqrt{16-y^2-z^2} \, dy \, dz. \text{ Converting to polar coordinates gives}$$

$$\int_0^{\pi/2} \int_0^4 \sqrt{16-r^2} r \, dr \, d\theta = \int_0^{\pi/2} \left(-\frac{1}{3} (16-r^2)^{3/2} \right) \Big|_0^4 d\theta = \int_0^{\pi/2} \frac{64}{3} d\theta = \frac{32\pi}{3}.$$

16.4.51 The volume of the box is $1 \cdot 2 \cdot \frac{\pi}{2} = \pi$. The average value is therefore

$$\frac{1}{\pi} \int_0^2 \int_0^1 \int_0^{\pi/2} 8xy \cos z \, dz \, dx \, dy = \frac{1}{\pi} \int_0^2 \int_0^1 8xy \, dx \, dy = \frac{1}{\pi} \int_0^2 4x^2 y \Big|_0^1 dy = \frac{1}{\pi} \cdot 2y^2 \Big|_0^2 = \frac{8}{\pi}.$$

16.4.52 The average value is $\frac{1}{\text{volume of } D} \iiint_D T(x, y, z) \, dV$. This can be written as

$$\begin{aligned} \frac{1}{\ln 2 \cdot \ln 4 \cdot \ln 8} \int_0^{\ln 2} \int_0^{\ln 4} \int_0^{\ln 8} 128 e^{-x-y-z} \, dz \, dy \, dx &= \frac{128}{6 (\ln 2)^3} \int_0^{\ln 2} \int_0^{\ln 4} \int_0^{\ln 8} e^{-x} e^{-y} e^{-z} \, dz \, dy \, dx = \\ \frac{64}{3 (\ln 2)^3} \int_0^{\ln 2} \int_0^{\ln 4} e^{-x} e^{-y} (-e^{-z}) \Big|_0^{\ln 8} dy \, dx &= \frac{64}{3 (\ln 2)^3} \int_0^{\ln 2} \int_0^{\ln 4} e^{-x} e^{-y} \left(1 - \frac{1}{8} \right) dy \, dx = \\ \frac{56}{3 (\ln 2)^3} \int_0^{\ln 2} \int_0^{\ln 4} e^{-x} e^{-y} \, dy \, dx &= \frac{56}{3 (\ln 2)^3} \int_0^{\ln 2} e^{-x} (-e^{-y}) \Big|_0^{\ln 4} dx = \frac{56}{3 (\ln 2)^3} \int_0^{\ln 2} e^{-x} \left(1 - \frac{1}{4} \right) dx = \\ \frac{14}{(\ln 2)^3} \int_0^{\ln 2} e^{-x} \, dx &= \frac{14}{(\ln 2)^3} (-e^{-x}) \Big|_0^{\ln 2} = \frac{14}{(\ln 2)^3} \left(1 - \frac{1}{2} \right) = \frac{7}{(\ln 2)^3}. \end{aligned}$$

16.4.53 The average value is given by $\frac{1}{\text{volume of } D} \iiint_D f(x, y, z) \, dV$. This can be written as

$$\begin{aligned}
& \frac{1}{(\pi \cdot 2^2 \cdot 2)} \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_0^2 (x^2 + y^2 + z^2) dz dy dx = \frac{1}{8\pi} \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \left((x^2 + y^2)z + \frac{1}{3}z^3 \right) \Big|_0^2 dy dx \\
&= \frac{1}{8\pi} \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \left(2(x^2 + y^2) + \frac{8}{3} \right) dy dx = \frac{1}{8\pi} \int_{-2}^2 \left(2x^2y + \frac{2}{3}y^3 + \frac{8}{3}y \right) \Big|_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dx \\
&= \frac{1}{2\pi} \int_{-2}^2 \left(x^2\sqrt{4-x^2} + \frac{1}{3}(4-x^2)^{3/2} + \frac{4}{3}\sqrt{4-x^2} \right) dx \\
&= \frac{1}{2\pi} \left(-\frac{x}{4}(x^2-2)\sqrt{4-x^2} + 2\sin^{-1}\left(\frac{x}{2}\right) + \frac{x}{12}(x^2-10) + 2\sin^{-1}\left(\frac{x}{2}\right) + \frac{x}{6}\sqrt{4-x^2} + \frac{8}{3}\sin^{-1}\left(\frac{x}{2}\right) \right) \Big|_{-2}^2 \\
&= \frac{1}{2\pi} \left(\frac{10\pi}{3} + \frac{10\pi}{3} \right) = \frac{10}{3}.
\end{aligned}$$

16.4.54 The average value is

$$\begin{aligned}
\frac{1}{\text{volume of } D} \iiint_D f(x, y, z) dV &= \frac{1}{\frac{1}{2}\left(\frac{4}{3}\pi \cdot 4^3\right)} \int_{-4}^4 \int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} \int_0^{\sqrt{16-x^2-y^2}} z dz dy dx \\
&= \frac{3}{128\pi} \int_{-4}^4 \int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} \left(\frac{1}{2}z^2 \right) \Big|_0^{\sqrt{16-x^2-y^2}} dy dx \\
&= \frac{3}{256\pi} \int_{-4}^4 \int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} (16-x^2-y^2) dy dx.
\end{aligned}$$

Converting to polar coordinates gives

$$\begin{aligned}
\frac{3}{256\pi} \int_0^{2\pi} \int_0^4 (16-r^2) r dr d\theta &= \frac{3}{256\pi} \int_0^{2\pi} \int_0^4 (16r-r^3) dr d\theta \\
&= \frac{3}{256\pi} \int_0^{2\pi} \left(8r^2 - \frac{1}{4}r^4 \right) \Big|_0^4 d\theta = \frac{3}{256\pi} \int_0^{2\pi} (128-64) d\theta \\
&= \frac{3}{4\pi} \int_0^{2\pi} d\theta = \frac{3}{2}.
\end{aligned}$$

16.4.55

- False. Only six iterations are possible.
- False. The outermost limits of integration must be constants.
- False. D is the intersection of two cylinders in first octant.

$$\begin{aligned}
\mathbf{16.4.56} \quad \int_1^4 \int_z^{4z} \int_0^{\pi^2} \frac{\sin \sqrt{yz}}{x^{3/2}} dy dx dz &= \int_0^{\pi^2} \int_1^4 \int_z^{4z} \frac{\sin \sqrt{yz}}{x^{3/2}} dx dz dy = \\
\int_0^{\pi^2} \int_1^4 \left(-\frac{2\sin \sqrt{yz}}{\sqrt{x}} \right) \Big|_z^{4z} dz dy &= \int_0^{\pi^2} \int_1^4 \left(-\frac{\sin \sqrt{yz}}{\sqrt{z}} + \frac{2\sin \sqrt{yz}}{\sqrt{z}} \right) dz dy = \int_0^{\pi^2} \int_1^4 \left(\frac{\sin \sqrt{yz}}{\sqrt{z}} \right) dz dy.
\end{aligned}$$

$$\begin{aligned}
&\text{Let } u = \sqrt{yz}, du = \frac{1}{2}(yz)^{-1/2} \cdot y dz. \text{ Substituting gives } \int_0^{\pi^2} \int_{\sqrt{y}}^{2\sqrt{y}} \left(\frac{2}{\sqrt{y}} \sin u \right) du dy = \\
&\int_0^{\pi^2} \left(-\frac{2}{\sqrt{y}} \cos u \right) \Big|_{u=\sqrt{y}}^{u=2\sqrt{y}} dy = \int_0^{\pi^2} \left(-\frac{2 \cos(2\sqrt{y})}{\sqrt{y}} + \frac{2 \cos(\sqrt{y})}{\sqrt{y}} \right) dy \text{ Let } v = \sqrt{y} \text{ so that } 2 dv = \frac{dy}{\sqrt{y}}. \\
&\text{Then we have } 4 \int_0^{\pi} (-\cos(2v) + \cos v) dv = 4 \left(-\frac{1}{2} \sin(2v) + \sin v \right) \Big|_{v=0}^{v=\pi} = 0.
\end{aligned}$$

$$\begin{aligned}
16.4.57 \quad V &= \int_{-1}^0 \int_{-x-1}^{x+1} \int_0^{1-x-y} 1 \, dz \, dy \, dx + \int_0^1 \int_{x-1}^{-x+1} \int_0^{1-x-y} 1 \, dz \, dy \, dx = \int_{-1}^0 \int_{-x-1}^{x+1} (1-x-y) \, dy \, dx + \\
&\int_0^1 \int_{x-1}^{-x+1} (1-x-y) \, dy \, dx = \int_{-1}^0 \left((1-x)y - \frac{1}{2}y^2 \right) \Big|_{-x-1}^{x+1} dx + \int_0^1 \left((1-x)y - \frac{1}{2}y^2 \right) \Big|_{x-1}^{-x+1} dx = \\
&\int_{-1}^0 (2-2x^2) \, dx + \int_0^1 (2-4x+2x^2) \, dx = \left(2x - \frac{2}{3}x^3 \right) \Big|_{-1}^0 + \left(2x - 2x^2 + \frac{2}{3}x^3 \right) \Big|_0^1 = \left(2 - \frac{2}{3} \right) + 2 - 2 + \frac{2}{3} = 2.
\end{aligned}$$

16.4.58 The surfaces intersect when $z_1 = z_2 \implies \sin x = \sin y \implies x = y$ or $x = \pi - y$. By symmetry, the volume equals 4 times the volume over the region, bounded by $y = x$ and $y = \pi - x$ from $x = 0$ to $x = \frac{\pi}{2}$ under $z = \sin y$. $V = 4 \int_0^{\pi/2} \int_x^{\pi-x} \int_0^{\sin y} 1 \, dz \, dy \, dx = 4 \int_0^{\pi/2} \int_x^{\pi-x} \sin y \, dy \, dx = 4 \int_0^{\pi/2} (-\cos y) \Big|_x^{\pi-x} dx =$

$$4 \int_0^{\pi/2} (\cos x - \cos(\pi - x)) \, dx = 4 \int_0^{\pi/2} (2 \cos x) \, dx = 8 (\sin x) \Big|_0^{\pi/2} = 8.$$

$$16.4.59 \quad V = \int_0^1 \int_z^{z+1} \int_0^1 dx \, dy \, dz = \int_0^1 \int_z^{z+1} 1 \, dy \, dz = \int_0^1 1 \, dz = 1.$$

$$16.4.60 \quad V = \int_0^1 \int_z^2 \int_0^2 dx \, dy \, dz = \int_0^1 \int_z^2 2 \, dy \, dz = \int_0^1 (4 - 2z) \, dz = (4z - z^2) \Big|_0^1 = 3.$$

$$\begin{aligned}
16.4.61 \quad V &= \int_0^2 \int_0^{4-2y} \int_0^{(4-z)/2} 1 \, dx \, dz \, dy = \int_0^2 \int_0^{4-2y} \left(\frac{4-z}{2} \right) dz \, dy = \int_0^2 \left(2z - \frac{1}{4}z^2 \right) \Big|_0^{4-2y} dy = \\
&\int_0^2 (4 - y^2) \, dy = \left(4y - \frac{1}{3}y^3 \right) \Big|_0^2 = \frac{16}{3}.
\end{aligned}$$

$$\begin{aligned}
16.4.62 \quad V &= \iiint_D 1 \, dV = \int_0^1 \int_0^{1-z} \int_{1-y-z}^{\sqrt{(1-z)^2 - y^2}} 1 \, dx \, dy \, dz = \int_0^1 \int_0^{1-z} (x) \Big|_{1-y-z}^{\sqrt{(1-z)^2 - y^2}} dy \, dz = \\
&\int_0^1 \int_0^{1-z} \left(\sqrt{(1-z)^2 - y^2} - 1 + y + z \right) dy \, dz = \\
&\int_0^1 \left(\frac{1}{2}y\sqrt{(1-z)^2 - y^2} + \frac{1}{2}(1-z)^2 \sin^{-1} \left(\frac{y}{1-z} \right) + \frac{1}{2}y^2 - (1-z)y \right) \Big|_0^{1-z} dz = \\
&\int_0^1 \left(\frac{1}{2}(1-z)^2 \sin^{-1} \left(\frac{1-z}{1-z} \right) - \frac{1}{2}(1-z)^2 \right) dz.
\end{aligned}$$

After a substitution, we have $\frac{1}{2} \int_0^1 \left(u^2 \sin^{-1} \left(\frac{u}{1-z} \right) - u^2 \right) dz$. Integration by parts yields

$$\frac{1}{2} \left(\frac{1}{3}u^3 \sin^{-1} \left(\frac{u}{1-z} \right) + \frac{1}{9}(8+u^2)\sqrt{4-u^2} - \frac{1}{3}u^3 \right) \Big|_{u=0}^{u=1} = \frac{\pi}{12} - \frac{1}{6}.$$

16.4.63 Due to symmetry, this region is made up of eight identical pieces, one in each octant. Consider the piece in the first octant. A particle moving through this region parallel to the positive y -axis would start on the xz -plane ($y = 0$) within the unit circle $x^2 + z^2 = 1$ and would end on the cylinder that runs parallel to the x -axis ($y^2 + z^2 = 1$). The total volume is thus $V = 8 \int_0^1 \int_0^{\sqrt{1-z^2}} \int_0^{\sqrt{1-z^2}} 1 \, dy \, dx \, dz =$

$$8 \int_0^1 \int_0^{\sqrt{1-z^2}} \sqrt{1-z^2} \, dx \, dz = 8 \int_0^1 (1-z^2) \, dz = \frac{16}{3}.$$

16.4.64 Due to symmetry, this region is made up of eight identical pieces, one in each octant. Consider the piece in the first octant. A particle moving through this region parallel to the positive y -axis would start on the xz -plane ($y = 0$) within the unit circle $x^2 + z^2 = 1$ and would either end on the

cylinder that runs parallel to the x -axis ($y^2 + z^2 = 1$) or the cylinder that runs parallel to the z -axis ($x^2 + y^2 = 1$). If the particle starts at a point on the xz -plane for which $x < z$, then $\sqrt{1 - z^2} < \sqrt{1 - x^2}$ and the particle ends on the cylinder $y^2 + z^2 = 1$. If the particle starts at a point on the xz -plane for which $z < x$, then $\sqrt{1 - x^2} < \sqrt{1 - z^2}$ and the particle ends on the cylinder $x^2 + y^2 = 1$.

The total volume is thus $V = 8 \left(\int_0^{\sqrt{2}/2} \int_x^{\sqrt{1-x^2}} \int_0^{\sqrt{1-z^2}} 1 \, dy \, dz \, dx + \int_0^{\sqrt{2}/2} \int_z^{\sqrt{1-z^2}} \int_0^{\sqrt{1-x^2}} 1 \, dy \, dx \, dz \right) =$
 $8 \left(\int_0^{\sqrt{2}/2} \int_x^{\sqrt{1-x^2}} \sqrt{1-z^2} \, dz \, dx + \int_0^{\sqrt{2}/2} \int_z^{\sqrt{1-z^2}} \sqrt{1-x^2} \, dx \, dz \right) =$
 $8 \left(\left(1 - \frac{\sqrt{2}}{2}\right) + \left(1 - \frac{\sqrt{2}}{2}\right) \right) = 8(2 - \sqrt{2}).$

16.4.65 The volume of the cheese is $\int_0^4 \int_0^4 \int_0^y 1 \, dz \, dy \, dx = \int_0^4 \int_0^4 y \, dy \, dx = \int_0^4 \left(\frac{1}{2} y^2 \right) \Big|_0^4 dx = \int_0^4 8 \, dx =$
 $(8x) \Big|_0^4 = 32 \cdot \frac{1}{2} (32) = \int_0^4 \int_0^a \int_0^y 1 \, dz \, dy \, dx = \int_0^4 \int_0^a y \, dy \, dx =$
 $\int_0^4 \left(\frac{1}{2} y^2 \right) \Big|_0^a dx = \int_0^4 \left(\frac{1}{2} a^2 \right) dx = \left(\left(\frac{1}{2} a^2 \right) x \right) \Big|_0^4 = 2a^2.$ Thus, $16 = 2a^2$, so $a = 2\sqrt{2}$.

16.4.66

a. $V = \iiint_{D_1} 1 \, dV = \int_0^1 \int_0^z \int_0^y 1 \, dx \, dy \, dz = \int_0^1 \int_0^z y \, dy \, dz = \int_0^1 \frac{1}{2} z^2 \, dz = \left(\frac{1}{6} z^3 \right) \Big|_0^1 = \frac{1}{6}.$

b. Let $D_2 = \{(x, y, z) : 0 \leq y \leq x \leq z \leq 1\}$. Then $V = \iiint_{D_2} 1 \, dV = \int_0^1 \int_0^z \int_0^x 1 \, dy \, dx \, dz = \frac{1}{6}$, by
 swapping x and y from part a. Likewise all other “cousins” have volume $\frac{1}{6}$.

c. The union includes all points with $0 \leq x \leq 1$, $0 \leq y \leq 1$ and $0 \leq z \leq 1$, which is the unit cube.

16.4.67 The equation of a cone with height h and whose base is centered at the origin with radius r in the xy -plane is $z = h - \frac{h}{r} \sqrt{x^2 + y^2}$. The volume is $V = \int_{-r}^r \int_{-\sqrt{r^2-x^2}}^{\sqrt{r^2-x^2}} \int_0^{h-\frac{h}{r}\sqrt{x^2+y^2}} 1 \, dz \, dy \, dx =$
 $\int_{-r}^r \int_{-\sqrt{r^2-x^2}}^{\sqrt{r^2-x^2}} \left(h - \frac{h}{r} \sqrt{x^2 + y^2} \right) dy \, dx.$ We switch to polar coordinates, using $x^2 + y^2 = a^2$ in order to avoid
 confusion with the constant r . Then we have $\int_0^{2\pi} \int_0^r \left(h - \frac{h}{r} a \right) a \, da \, d\theta = \int_0^{2\pi} \int_0^r \left(ha - \frac{h}{r} a^2 \right) da \, d\theta =$
 $\int_0^{2\pi} \left(\frac{1}{2} ha^2 - \frac{h}{3r} a^3 \right) \Big|_{a=0}^{a=r} d\theta = \int_0^{2\pi} \left(\frac{1}{2} r^2 h - \frac{hr^3}{3r} \right) d\theta = \int_0^{2\pi} \frac{1}{6} r^2 h \, d\theta = \frac{1}{3} \pi r^2 h.$

16.4.68 The equation of the plane through $(a, 0, 0)$, $(0, b, 0)$ and $(0, 0, c)$ is $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$. The volume is
 $\int_0^a \int_0^{b(1-x/a)} \int_0^{c(1-x/a-y/b)} 1 \, dz \, dy \, dx = \int_0^a \int_0^{b(1-x/a)} c \left(1 - \frac{x}{a} - \frac{y}{b} \right) dy \, dx =$
 $c \int_0^a \left(\left(1 - \frac{x}{a} \right) y - \frac{y^2}{2b} \right) \Big|_0^{b(1-x/a)} dx = \frac{bc}{2} \int_0^a \left(1 - \frac{x}{a} \right)^2 dx = \frac{bc}{2} \left(x - \frac{x^2}{a} + \frac{x^3}{3a^2} \right) \Big|_0^a = \frac{bc}{2} \left(a - a + \frac{a}{3} \right) =$
 $\frac{abc}{6}.$

16.4.69 The equation of a sphere with radius R is $x^2 + y^2 + z^2 = R^2$. The volume is given by

$$\begin{aligned} & \int_{-\sqrt{2Rh-h^2}}^{\sqrt{2Rh-h^2}} \int_{-\sqrt{2Rh-h^2-x^2}}^{\sqrt{2Rh-h^2-x^2}} \int_{R-h}^{\sqrt{R^2-x^2-y^2}} 1 \, dz \, dy \, dx \\ &= \int_{-\sqrt{2Rh-h^2}}^{\sqrt{2Rh-h^2}} \int_{-\sqrt{2Rh-h^2-x^2}}^{\sqrt{2Rh-h^2-x^2}} \left(\sqrt{R^2-x^2-y^2} - (R-h) \right) dy \, dx. \end{aligned}$$

Converting to polar coordinates gives

$$\begin{aligned} & \int_0^{2\pi} \int_0^{\sqrt{2Rh-h^2}} \left(\sqrt{R^2-r^2} - (R-h) \right) r \, dr \, d\theta = \int_0^{2\pi} \int_0^{\sqrt{2Rh-h^2}} \left(r \sqrt{R^2-r^2} - r(R-h) \right) dr \, d\theta \\ &= \int_0^{2\pi} \left(\left(-\frac{1}{3} (R^2-r^2)^{3/2} - \frac{1}{2} r^2 (R-h) \right) \right) \Big|_0^{\sqrt{2Rh-h^2}} d\theta \\ &= \int_0^{2\pi} \left[-\frac{1}{3} (R-h)^3 - \frac{1}{2} (2Rh-h^2) (R-h) + \frac{1}{3} R^3 \right] d\theta = \int_0^{2\pi} \left(\frac{h^2}{6} (3R-h) \right) d\theta \\ &= \frac{1}{3} \pi h^2 (3R-h). \end{aligned}$$

16.4.70 There are two volumes to consider: the volume V_1 of the cylinder of radius r inside the frustum, and the volume V_2 that remains when that cylinder is removed. The first volume can be computed without calculus: $V_1 = \pi r^2 h$. Note that the base of the volume V_2 is the annulus centered at the origin with inner radius r and outer radius R . Polar coordinates may be used (with a as the radius to avoid confusion with r) to show that the equation of the given frustum is $z = \frac{h}{R-r} (R-a)$. The volume

$$\begin{aligned} V_2 \text{ is } & \int_0^{2\pi} \int_r^R \int_0^{\frac{h}{R-r}(R-a)} a \, dz \, da \, d\theta = \int_0^{2\pi} \int_r^R \frac{ha}{R-r} (R-a) \, da \, d\theta = \frac{h}{R-r} \int_0^{2\pi} \int_r^R (Ra - a^2) \, da \, d\theta = \\ & \frac{h}{R-r} \int_0^{2\pi} \left(\frac{Ra^2}{2} - \frac{a^3}{3} \right) \Big|_{a=r}^R d\theta = \frac{h}{R-r} \int_0^{2\pi} \left(\frac{R^3}{6} - \frac{Rr^2}{2} + \frac{r^3}{3} \right) d\theta = \frac{h}{R-r} \cdot 2\pi \cdot \left(\frac{R^3}{6} - \frac{Rr^2}{2} + \frac{r^3}{3} \right) = \\ & \frac{h}{R-r} \cdot \frac{\pi R^3 - 3\pi Rr^2 + 2\pi r^3}{3}. \text{ The total volume } V \text{ is } V = V_1 + V_2 = \pi r^2 h + \frac{h}{R-r} \cdot \frac{\pi R^3 - 3\pi Rr^2 + 2\pi r^3}{3} = \\ & \frac{\pi h}{3} \cdot \frac{3r^2(R-r) + R^3 - 3Rr^2 + r^3}{R-r} = \frac{\pi h}{3} \cdot \frac{R^3 - r^3}{R-r} = \frac{\pi h}{3} (R^2 + Rr + r^2). \end{aligned}$$

16.4.71 The equation of the given ellipsoid is $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$. Using symmetry, the volume V of the ellipsoid is eight times the volume of the portion of the ellipsoid in the first octant. Thus $V =$

$$\begin{aligned} & 8 \int_0^a \int_0^{b\sqrt{1-x^2/a^2}} \int_0^{c\sqrt{1-x^2/a^2-y^2/b^2}} 1 \, dz \, dy \, dx = 8c \int_0^a \int_0^{b\sqrt{1-x^2/a^2}} \sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}} \, dy \, dx = \\ & 8c \int_0^a \left(\frac{y}{2} \sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}} + \frac{b(a^2-x^2)}{2a^2} \sin^{-1} \left(\frac{ay}{b\sqrt{a^2-x^2}} \right) \right) \Big|_0^{b\sqrt{1-x^2/a^2}} dx = 8c \int_0^a \frac{b\pi(a^2-x^2)}{4a^2} dx = \\ & \frac{2\pi bc}{a^2} \left(a^2x - \frac{x^3}{3} \right) \Big|_0^a = \frac{2\pi bc}{a^2} \cdot \frac{2a^3}{3} = \frac{4}{3} \pi abc \end{aligned}$$

16.4.72

$$\text{a. } p(0.75) = \int_0^{0.75} 0.8e^{-0.8t} dt = (-e^{-0.8t}) \Big|_{t=0}^{t=0.75} = 0.4512.$$

$$\begin{aligned} \text{b. } p(0.75) &= \int_0^{0.75} \int_0^{0.75} (0.8e^{-0.8t}) (0.1e^{-0.1s}) dt ds = \int_0^{0.75} (0.1e^{-0.1s}) (-e^{-0.8t}) \Big|_{t=0}^{t=0.75} ds = \\ & \int_0^{0.75} (0.1e^{-0.1s}) (0.4512) ds = (0.4512) (-e^{-0.1s}) \Big|_{s=0}^{s=0.75} = (0.4512) (0.0723) = 0.0326. \end{aligned}$$

c.

$$\begin{aligned}
p(0.75) &= \int_0^{0.75} \int_0^{0.75} \int_0^{0.75} (0.8e^{-0.8t}) (0.1e^{-0.1s}) (0.05e^{-0.05u}) dt ds du \\
&= \int_0^{0.75} \int_0^{0.75} (0.1e^{-0.1s}) (0.05e^{-0.05u}) (-e^{-0.8t}) \Big|_{t=0}^{t=0.75} ds du \\
&= (0.4512) \int_0^{0.75} \int_0^{0.75} (0.1e^{-0.1s}) (0.05e^{-0.05u}) ds du = (0.4512) \int_0^{0.75} (0.05e^{-0.05u}) (-e^{-0.1s}) \Big|_{s=0}^{s=0.75} du \\
&= (0.4512) (0.0723) \int_0^{0.75} (0.05e^{-0.05u}) du \\
&= (0.4512) (0.0723) (-e^{-0.05u}) \Big|_{u=0}^{u=0.75} = (0.4512) (0.0723) (0.0368) = 0.0012.
\end{aligned}$$

16.4.73 The Hypervolume is given by $\int_0^1 \int_0^{1-x} \int_0^{1-x-y} \int_0^{1-x-y-z} 1 dw dz dy dx$. This can be computed as

$$\begin{aligned}
&\int_0^1 \int_0^{1-x} \int_0^{1-x-y} (1-x-y-z) dz dy dx = \int_0^1 \int_0^{1-x} \left((1-x-y)z - \frac{1}{2}z^2 \right) \Big|_0^{1-x-y} dy dx = \\
&\int_0^1 \int_0^{1-x} \left((1-x-y)^2 - \frac{1}{2}(1-x-y)^2 \right) dy dx = \frac{1}{2} \int_0^1 \int_0^{1-x} (1-x-y)^2 dy dx = \\
&\frac{1}{2} \int_0^1 \left[-\frac{1}{3}(1-x-y)^3 \right]_0^{1-x} dx = \frac{1}{6} \int_0^1 (1-x)^3 dx = \frac{1}{6} \left(-\frac{1}{4}(1-x)^4 \right) \Big|_0^1 = \frac{1}{24}.
\end{aligned}$$

16.4.74 The region of integration for the integral $\int_0^1 \int_x^1 \int_x^y f(x) f(y) f(z) dz dy dx$ is the set of points (x, y, z) such that $0 < x < z < y < 1$. This region is one sixth of the unit cube. By rearranging x , y , and z , the other five sixths of the unit cube may be generated. Thus the integral of $f(x) f(y) f(z)$ over the unit cube is the sum of the integrals over these six regions. Now notice that the integrand $f(x) f(y) f(z)$ does not change as x , y , and z are rearranged, so the integral over each of the six regions is the same. Thus $\int_0^1 \int_0^1 \int_0^1 f(x) f(y) f(z) dz dy dx = 6 \int_0^1 \int_x^1 \int_x^y f(x) f(y) f(z) dz dy dx$. The integral of $f(x) f(y) f(z)$ over the unit cube can be calculated as follows:

$$\begin{aligned}
&\int_0^1 \int_0^1 \int_0^1 f(x) f(y) f(z) dz dy dx = \int_0^1 \int_0^1 f(x) f(y) \left(\int_0^1 f(z) dz \right) dy dx = \left(\int_0^1 f(z) dz \right) \int_0^1 \int_0^1 f(x) f(y) dy dx = \\
&\left(\int_0^1 f(z) dz \right) \int_0^1 f(x) \left(\int_0^1 f(y) dy \right) dx = \left(\int_0^1 f(z) dz \right) \left(\int_0^1 f(y) dy \right) \left(\int_0^1 f(x) dx \right).
\end{aligned}$$

Because the name of the variable is immaterial, we have $\int_0^1 \int_0^1 \int_0^1 f(x) f(y) f(z) dz dy dx =$

$$\left(\int_0^1 f(z) dz \right) \left(\int_0^1 f(y) dy \right) \left(\int_0^1 f(x) dx \right) = \left(\int_0^1 f(x) dx \right)^3.$$

Thus, $\int_0^1 \int_x^1 \int_x^y f(x) f(y) f(z) dz dy dx = \frac{1}{6} \int_0^1 \int_0^1 \int_0^1 f(x) f(y) f(z) dz dy dx = \frac{1}{6} \left(\int_0^1 f(x) dx \right)^3$.

16.5 Triple Integrals in Cylindrical and Spherical Coordinates

16.5.1 r measures the distance from the point to the z axis, θ is the angle that the segment from the point to the z -axis makes with the positive xz -plane, and z is the directed distance from the point to the xy -plane.

16.5.2 The triple (ρ, φ, θ) describes a point whose distance from the origin is ρ , such that the line from the origin to the point makes an angle of φ with the z -axis, and such that the projection of this line to the xy -plane makes an angle of θ with the positive x -axis.

16.5.3 A double cone (opening both upwards and downwards), where the radius is always 4 times the distance to the xy -plane.

16.5.4 A cone opening upwards making an angle of $\frac{\pi}{4}$ radian with the z -axis.

16.5.5 It approximates the volume of the cylindrical wedge formed by the changes Δr , $\Delta\theta$, and Δz .

16.5.6 It approximates the volume formed by the changes $\Delta\rho$, $\Delta\varphi$, and $\Delta\theta$.

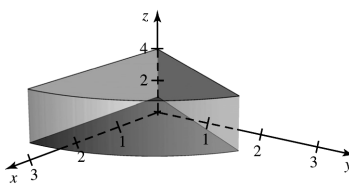
$$\mathbf{16.5.7} \quad \iiint_D w(r, \theta, z) dV = \int_{\alpha}^{\beta} \int_{g(\theta)}^{h(\theta)} \int_{G(r, \theta)}^{H(r, \theta)} w(r, \theta, z) dz r dr d\theta.$$

$$\mathbf{16.5.8} \quad \iiint_D w(\rho, \varphi, \theta) dV = \int_{\alpha}^{\beta} \int_a^b \int_{g(\varphi, \theta)}^{h(\varphi, \theta)} w(\rho, \varphi, \theta) \rho^2 \sin \varphi d\rho d\varphi d\theta.$$

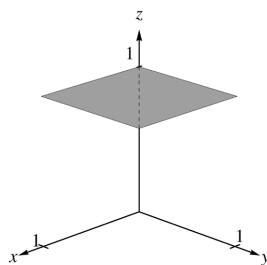
16.5.9 Cylindrical coordinates, because in cylindrical coordinates $x^2 + y^2$ simplifies to r^2 .

16.5.10 Spherical coordinates, because in spherical coordinates $x^2 + y^2 + z^2$ simplifies to ρ^2 .

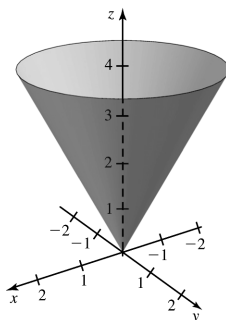
16.5.11 This is a wedge of a cylinder of radius 3 from $z = 1$ to $z = 4$, where the wedge angle is $\frac{\pi}{3}$.



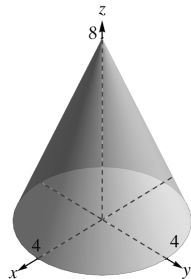
16.5.12 This is the first quadrant of the plane $z = 1$ for $x, y \geq 0$.



16.5.13 This is the solid upward pointing cone given by $z = 2r$ from $z = 0$ to $z = 4$.



16.5.14 This is a solid downward-pointing cone with vertex at $(0, 0, 8)$ between $z = 8$ and $z = 0$. Its intersection with the xy -plane is the circle $x^2 + y^2 = 16$.



$$16.5.15 \quad \int_0^{2\pi} \int_0^1 \int_{-1}^1 dz \, r \, dr \, d\theta = \int_0^{2\pi} \int_0^1 r z \Big|_{-1}^1 dr \, d\theta = \int_0^{2\pi} \int_0^1 2r \, dr \, d\theta = \int_0^{2\pi} 1 \, d\theta = 2\pi.$$

16.5.16 This is the integral over half of the cone whose equation is $z = 9 - 3r$, $0 \leq r \leq 3$. Thus

$$\begin{aligned} \int_0^3 \int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} \int_0^{9-3\sqrt{x^2+y^2}} dz \, dx \, dy &= \int_0^\pi \int_0^3 \int_0^{9-3r} r \, dz \, dr \, d\theta = \int_0^\pi \int_0^3 r z \Big|_0^{9-3r} dr \, d\theta = \\ \int_0^\pi \int_0^3 (9r - 3r^2) \, dr \, d\theta &= \int_0^\pi \left(\frac{9}{2}r^2 - r^3 \right) \Big|_0^3 d\theta = \int_0^\pi \left(\frac{81}{2} - 27 \right) d\theta = \frac{27}{2}\pi. \end{aligned}$$

16.5.17 This is the integral of $(x^2 + y^2)^{3/2}$ over a cylinder with radius 1 from $z = -1$ to $z = 1$. We convert to cylindrical coordinates, so the integrand becomes r^3 . Because $x^2 + y^2 = r^2$. The integral is thus

$$\int_0^{2\pi} \int_0^1 \int_{-1}^1 r^3 \, dz \, r \, dr \, d\theta = \int_0^{2\pi} \int_0^1 2r^4 \, dr \, d\theta = \frac{2}{5} \int_0^{2\pi} r^5 \Big|_0^1 d\theta = \frac{4\pi}{5}.$$

16.5.18 Converting to cylindrical coordinates, the integrand becomes $\frac{1}{1+r^2}$; the region of integration is the half-cylinder shown, so the integral is

$$\int_0^\pi \int_0^3 \int_0^2 \frac{1}{1+r^2} dz \, r \, dr \, d\theta = \int_0^\pi \int_0^3 \frac{2r}{1+r^2} dr \, d\theta = \int_0^\pi \ln(1+r^2) \Big|_0^3 d\theta = \int_0^\pi \ln(10) \, d\theta = \pi \ln(10).$$

16.5.19 The region of integration is a wedge from $\theta = \frac{\pi}{4}$ to $\theta = \frac{\pi}{2}$ with radius 1, between $z = 0$ and $z = 4$. This can be seen by noting that the limits of integration for y range from x to the boundary of the unit circle (y -coordinate $\sqrt{1-x^2}$) and that $x = \frac{\sqrt{2}}{2}$ corresponds to $\frac{\pi}{4}$. Thus the integral is

$$\begin{aligned} \int_{\pi/4}^{\pi/2} \int_0^1 \int_0^4 e^{-r^2} dz \, r \, dr \, d\theta &= \int_{\pi/4}^{\pi/2} \int_0^1 4r e^{-r^2} dr \, d\theta = -2 \int_{\pi/4}^{\pi/2} e^{-r^2} \Big|_0^1 d\theta = -2 \int_{\pi/4}^{\pi/2} (e^{-1} - 1) d\theta = \frac{\pi}{2} (1 - e^{-1}). \end{aligned}$$

16.5.20 The region of integration is the volume between the cone $x^2 + y^2 = z^2$ and the plane $z = 4$. To see this, note that x and y vary over the disk $x^2 + y^2 = 16$ and that z varies from the boundary of the cone to

4. Converting to cylindrical coordinates, we then have

$$\int_0^{2\pi} \int_0^4 \int_r^4 1 \, dz \, r \, dr \, d\theta = \int_0^{2\pi} \int_0^4 r(4-r) \, dr \, d\theta = \int_0^{2\pi} \left(2r^2 - \frac{1}{3}r^3 \right) \Big|_0^4 d\theta = \int_0^{2\pi} \frac{32}{3} d\theta = \frac{64\pi}{3}.$$

16.5.21 The region of integration is below the cone $x^2 + y^2 = z^2$ in the first octant, so the integral is

$$\int_0^{\pi/2} \int_0^3 \int_0^r \frac{1}{r} dz \, r \, dr \, d\theta = \int_0^{\pi/2} \int_0^3 r \, dr \, d\theta = \int_0^{\pi/2} \frac{9}{2} d\theta = \frac{9\pi}{4}.$$

16.5.22 The region of integration is a wedge from $\theta = 0$ to $\theta = \frac{\pi}{6}$ with radius 1, between $z = -1$ and $z = 1$.

The integral is then

$$\int_0^{\pi/6} \int_0^1 \int_{-1}^1 r \, dz \, r \, dr \, d\theta = \int_0^{\pi/6} \int_0^1 r^2 \, dz \, dr \, d\theta = \int_0^{\pi/6} \frac{2}{3} d\theta = \frac{\pi}{9}.$$

$$\mathbf{16.5.23} \quad \int_0^{2\pi} \int_0^4 \int_0^{10} \left(1 + \frac{z}{2}\right) dz r dr d\theta = \int_0^{2\pi} \int_0^4 \left(z + \frac{z^2}{4}\right) \Big|_0^{10} r dr d\theta = \int_0^{2\pi} \int_0^4 35r dr d\theta = 560\pi.$$

$$\mathbf{16.5.24} \quad \int_0^{2\pi} \int_0^3 \int_0^2 5e^{-r^2} dz r dr d\theta = -5 \int_0^{2\pi} \int_0^3 (-2re^{-r^2}) dr d\theta = 10\pi (1 - e^{-9}).$$

$$\mathbf{16.5.25} \quad \int_0^{2\pi} \int_0^6 \int_0^{6-r} (7-z) dz r dr d\theta = \int_0^{2\pi} \int_0^6 \left(7z - \frac{1}{2}z^2\right) \Big|_0^{6-r} r dr d\theta = \int_0^{2\pi} \int_0^4 \left(7r(6-r) - \frac{r}{2}(6-r)^2\right) dr d\theta = 396\pi.$$

$$\mathbf{16.5.26} \quad \int_0^{2\pi} \int_0^3 \int_0^{9-r^2} \left(1 + \frac{z}{9}\right) dz r dr d\theta = \int_0^{2\pi} \int_0^3 \left(z + \frac{z^2}{18}\right) \Big|_0^{9-r^2} r dr d\theta = \int_0^{2\pi} \int_0^3 \left(r(9-r^2) - \frac{r}{18}(9-r^2)^2\right) dr d\theta = 54\pi.$$

16.5.27 The base of both surfaces in the xy -plane is the area bounded by the unit circle $r = 1$. The mass of the solid bounded by the xy -plane and $z = 4 - 4r$ is given by $\int_0^{2\pi} \int_0^1 \int_0^{4-4r} (10 - 2z) dz r dr d\theta = \int_0^{2\pi} \int_0^1 (10z - z^2) \Big|_0^{4-4r} r dr d\theta = \int_0^{2\pi} \int_0^1 (10r(4-4r) - r(4-4r)^2) dr d\theta = \frac{32\pi}{3}$. The mass of the solid bounded by the xy -plane and $z = 4 - 4r^2$ is given by $\int_0^{2\pi} \int_0^1 \int_0^{4-4r^2} (10 - 2z) dz r dr d\theta = \int_0^{2\pi} \int_0^1 (10z - z^2) \Big|_0^{4-4r^2} r dr d\theta = \int_0^{2\pi} \int_0^1 (10r(4-4r^2) - r(4-4r^2)^2) dr d\theta = \frac{44\pi}{3}$. Thus the mass of the solid bounded by the paraboloid is larger. This can also be seen by noting that for $r \leq 1$, $r^2 \leq r$ so that $4 - 4r^2 \geq 4 - 4r$, so that the cone is contained in the paraboloid.

16.5.28 By the argument given above, again the solid bounded by the paraboloid will have greater mass. For the solid bounded by the cone, the mass is

$$\int_0^{2\pi} \int_0^1 \int_0^{4-4r} \left(\frac{8}{\pi}e^{-z}\right) dz r dr d\theta = \frac{8}{\pi} \int_0^{2\pi} \int_0^1 -r(e^{4r-4} - 1) dr d\theta = 5 - e^{-4}.$$

The mass of the solid bounded by the xy -plane and $z = 4 - 4r^2$ is $\int_0^{2\pi} \int_0^1 \int_0^{4-4r^2} \left(\frac{8}{\pi}e^{-z}\right) dz r dr d\theta = \frac{8}{\pi} \int_0^{2\pi} \int_0^1 -r(e^{4r^2-4} - 1) dr d\theta = 6 + 2e^{-4}$.

16.5.29 The base of this solid in the xy -plane is given by $3 = \sqrt{1+x^2+y^2}$ so is the area bounded by the circle $x^2 + y^2 = 8$ or $r = 2\sqrt{2}$. Thus $V = \int_0^{2\pi} \int_0^{2\sqrt{2}} \int_0^{3-\sqrt{1+r^2}} 1 dz r dr d\theta = \int_0^{2\pi} \int_0^{2\sqrt{2}} \left(3r - r\sqrt{1+r^2}\right) dr d\theta = 2\pi \left(\frac{3r^2}{2} - \frac{(1+r^2)^{3/2}}{3}\right) \Big|_0^{2\sqrt{2}} = 2\pi \left(12 - 9 - \left(0 - \frac{1}{3}\right)\right) = \frac{20\pi}{3}$.

16.5.30 This solid sits over the circle $r = 5$ in the xy -plane, so the volume is $\int_0^{2\pi} \int_0^5 \int_{r^2}^{25} 1 dz r dr d\theta = \int_0^{2\pi} \int_0^5 (25r - r^3) dr d\theta = \frac{625\pi}{2}$.

16.5.31 This solid sits over the circle $r = 5$ in the xy -plane, so the volume is $\int_0^{2\pi} \int_0^5 \int_{\sqrt{4+r^2}}^{\sqrt{29}} 1 dz r dr d\theta = \int_0^{2\pi} \int_0^5 (r\sqrt{29} - r\sqrt{4+r^2}) dr d\theta = \frac{\pi(17\sqrt{29} + 16)}{3}$.

16.5.32 The base is swept out completely as θ ranges from 0 to π , so the volume is

$$\int_0^\pi \int_0^{2\cos\theta} \int_0^4 1 \, dz \, r \, dr \, d\theta = \int_0^\pi \int_0^{2\cos\theta} 4r \, dr \, d\theta = 4\pi,$$

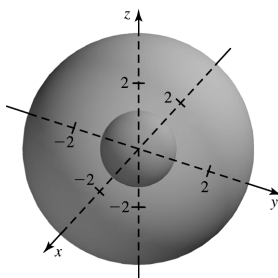
which is in fact the volume of a right circular cylinder of height 4 whose base is a unit circle.

16.5.33 The first octant is determined by $0 \leq \theta \leq \frac{\pi}{2}$, $0 \leq z$, and the condition $z = x$ in cylindrical coordinates becomes $z = r \cos \theta$, so the volume of the solid is

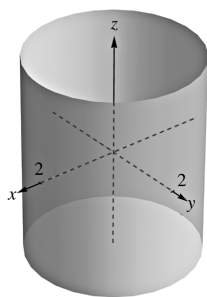
$$\int_0^{\pi/2} \int_0^1 \int_0^{r\cos\theta} 1 \, dz \, r \, dr \, d\theta = \int_0^{\pi/2} \int_0^1 r^2 \cos \theta \, dr \, d\theta = \frac{1}{3}.$$

16.5.34 The volume is $\int_0^{2\pi} \int_1^2 \int_0^{4-r(\cos\theta+\sin\theta)} 1 \, dz \, r \, dr \, d\theta = \int_0^{2\pi} \int_1^2 (4r - r^2(\cos\theta + \sin\theta)) \, dr \, d\theta = \int_0^{2\pi} \left(2r^2 - \frac{1}{3}r^3(\cos\theta + \sin\theta) \right) \Big|_1^2 d\theta = \int_0^{2\pi} \left(6 - \frac{7}{3}(\cos\theta + \sin\theta) \right) d\theta = 12\pi$

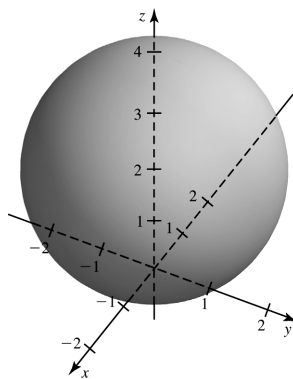
16.5.35 This is a spherical shell centered at the origin with outer radius 3 and inner radius 1.



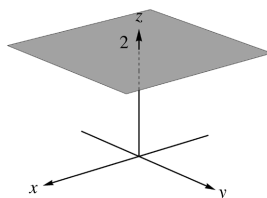
16.5.36 Because $\rho = 2 \csc \varphi$, $\rho \sin \varphi = 2$. Now $x^2 + y^2 = \rho^2 \sin^2 \varphi \cos^2 \theta + \rho^2 \sin^2 \varphi \sin^2 \theta = \rho^2 \sin^2 \varphi = 4$. Thus any point on the surface satisfies $x^2 + y^2 = 4$, so that the surface is a cylinder of radius 2 oriented along the z -axis.



16.5.37 This is a sphere of radius 2 centered at $(0, 0, 2)$. To see this, note that $\rho = 4 \cos \varphi$ implies that $\rho^2 = 4\rho \cos \varphi$. Converting to rectangular coordinates gives $x^2 + y^2 + z^2 = 4z$, and completing the square yields $x^2 + y^2 + (z - 2)^2 = 4$.



16.5.38 This is the plane $z = 2$, Because $\rho = 2 \sec \varphi$ implies that $z = \rho \cos \varphi = 2$.



16.5.39

- a. For Seattle, we have $\rho = 3960$, $\varphi = \frac{(90 - 47.6)\pi}{180} \approx 0.74$, and $\theta = \frac{-122.3\pi}{180} \approx -2.13$. So the coordinates are approximately $(3960, 0.74, -2.13)$. Then note that

$$u = (\rho \sin \varphi \cos \theta, \rho \sin \varphi \sin \theta, \rho \cos \varphi) = (-1426.85, -2257.05, 2924.28).$$

- b. For Rome, we have $\rho = 3960$, $\varphi = \frac{(90 - 41.9)\pi}{180} \approx 0.84$, and $\theta = \frac{12.5\pi}{180} \approx 0.22$. So the coordinates are approximately $(3960, 0.84, 0.22)$. Then note that

$$u = (\rho \sin \varphi \cos \theta, \rho \sin \varphi \sin \theta, \rho \cos \varphi) = (2877.61, 637.95, 2644.62).$$

- c. The distance is given by $\rho \cos^{-1} \left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}| |\mathbf{v}|} \right) \approx 5666$ miles.

16.5.40 For Los Angeles, we have $\rho = 3960$, $\varphi = \frac{(90 - 34.05)\pi}{180} \approx 0.98$, and $\theta = \frac{-118.24\pi}{180} \approx -2.06$. Then

$$u = (\rho \sin \varphi \cos \theta, \rho \sin \varphi \sin \theta, \rho \cos \varphi) = (-1552.48, -2890.52, 2217.27).$$

For New York City, we have $\rho = 3960$, $\varphi = \frac{(90 - 40.71)\pi}{180} \approx 0.86$, and $\theta = \frac{-74.01\pi}{180} \approx -1.29$. Then

$$v = (\rho \sin \varphi \cos \theta, \rho \sin \varphi \sin \theta, \rho \cos \varphi) = (826.89, -2885.62, 2582.83).$$

The distance between them is given by $\rho \cos^{-1} \left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}| |\mathbf{v}|} \right) = 2446$ miles.

$$\begin{aligned} \mathbf{16.5.41} \quad \iiint_D (x^2 + y^2 + z^2)^{5/2} dV &= \int_0^1 \int_0^{2\pi} \int_0^\pi \rho^5 \rho^2 \sin \varphi d\varphi d\theta d\rho = \int_0^1 \int_0^{2\pi} \int_0^\pi \rho^7 \sin \varphi d\varphi d\theta d\rho = \\ &= \int_0^1 \int_0^{2\pi} 2\rho^7 d\theta d\rho = 4\pi \int_0^1 \rho^7 d\rho = \frac{\pi}{2}. \end{aligned}$$

$$\begin{aligned}
 16.5.42 \quad \iiint_D e^{-(x^2+y^2+z^2)^{3/2}} dV &= \int_0^{2\pi} \int_0^\pi \int_0^1 e^{-\rho^3} \rho^2 \sin \varphi \, d\rho \, d\varphi \, d\theta = -\frac{1}{3} \int_0^{2\pi} \int_0^\pi e^{-\rho^3} \Big|_0^1 \sin \varphi \, d\varphi \, d\theta = \\
 &= -\frac{(e^{-1} - 1)}{3} \int_0^{2\pi} \int_0^\pi \sin \varphi \, d\varphi \, d\theta = -\frac{4\pi}{3} (e^{-1} - 1).
 \end{aligned}$$

16.5.43

$$\begin{aligned}
 \iiint_D (x^2 + y^2 + z^2)^{-3/2} dV &= \int_0^{2\pi} \int_0^\pi \int_1^2 \rho^{-3} \rho^2 \sin \varphi \, d\rho \, d\varphi \, d\theta \\
 &= \int_0^{2\pi} \int_0^\pi \int_1^2 \rho^{-1} \sin \varphi \, d\rho \, d\varphi \, d\theta = \ln 2 \int_0^{2\pi} \int_0^\pi \sin \varphi \, d\varphi \, d\theta = 4\pi \ln 2.
 \end{aligned}$$

$$16.5.44 \quad \int_0^{2\pi} \int_0^{\pi/3} \int_0^{4 \sec \varphi} \rho^2 \sin \varphi \, d\rho \, d\varphi \, d\theta = \frac{64}{3} \int_0^{2\pi} \int_0^{\pi/3} \sec^3 \varphi \sin \varphi \, d\varphi \, d\theta = 64\pi.$$

$$16.5.45 \quad \int_0^\pi \int_0^{\pi/6} \int_{2 \sec \varphi}^4 \rho^2 \sin \varphi \, d\rho \, d\varphi \, d\theta = \frac{1}{3} \int_0^\pi \int_0^{\pi/6} (64 - 8 \sec^3 \varphi) \sin \varphi \, d\varphi \, d\theta = \left(\frac{188}{9} - \frac{32}{3} \sqrt{3} \right) \pi.$$

16.5.46

$$\begin{aligned}
 \int_0^{2\pi} \int_0^{\pi/4} \int_1^{2 \sec \varphi} (\rho^{-3}) \rho^2 \sin \varphi \, d\rho \, d\varphi \, d\theta &= \int_0^{2\pi} \int_0^{\pi/4} \sin \varphi \ln(2 \sec \varphi) \, d\varphi \, d\theta \\
 &= 2\pi \left(\ln 2 \left(1 - \frac{3\sqrt{2}}{4} \right) + 1 - \frac{\sqrt{2}}{2} \right).
 \end{aligned}$$

$$16.5.47 \quad \int_0^{2\pi} \int_{\pi/6}^{\pi/3} \int_0^{2 \csc \varphi} \rho^2 \sin \varphi \, d\rho \, d\varphi \, d\theta = \frac{8}{3} \int_0^{2\pi} \int_{\pi/6}^{\pi/3} \csc^3 \varphi \sin \varphi \, d\varphi \, d\theta = \frac{32}{9} \pi \sqrt{3}.$$

16.5.48 A ball of radius a around the origin has volume given by

$$\int_0^{2\pi} \int_0^\pi \int_0^a \rho^2 \sin \varphi \, d\rho \, d\varphi \, d\theta = \frac{a^3}{3} \int_0^{2\pi} \int_0^\pi \sin \varphi \, d\varphi \, d\theta = \frac{4}{3} a^3.$$

16.5.49 The two spheres intersect where $2 \cos \varphi = 1$, i.e. when $\varphi = \frac{\pi}{3}$. That circle is then at $z = \frac{1}{2}$ (again looking at $\cos \varphi = \frac{1}{2}$ and using the fact that the lower sphere has radius 1). The volume in question is the sum of the volumes above and below that circle of intersection; because the circle is at height $\frac{1}{2}$ and both spheres have radius 1, the volumes above and below are identical. Thus we need only compute the upper volume and double it:

$$2 \int_0^{\pi/3} \int_{(\sec \varphi)/2}^1 \int_0^{2\pi} \rho^2 \sin \varphi \, d\theta \, d\rho \, d\varphi = \frac{4\pi}{3} \int_0^{\pi/3} \left(1 - \frac{\sec^3 \varphi}{8} \right) \sin \varphi \, d\varphi = \frac{5}{12} \pi.$$

16.5.50

$$\begin{aligned}
 \int_0^{2\pi} \int_0^\pi \int_0^{1+\cos \varphi} \rho^2 \sin \varphi \, d\rho \, d\varphi \, d\theta &= \frac{1}{3} \int_0^{2\pi} \int_0^\pi (1 + \cos \varphi)^3 \sin \varphi \, d\varphi \, d\theta \\
 &= -\frac{1}{12} \int_0^{2\pi} (1 + \cos \varphi)^4 \Big|_{\varphi=0}^{\varphi=\pi} d\theta = \frac{4}{3} \int_0^{2\pi} d\theta = \frac{8}{3} \pi.
 \end{aligned}$$

16.5.51 The portion of the sphere is determined by $\frac{\pi}{4} \leq \varphi \leq \frac{\pi}{2}$, so the integral is

$$\int_0^{2\pi} \int_{\pi/4}^{\pi/2} \int_0^{4 \cos \varphi} \rho^2 \sin \varphi \, d\rho \, d\varphi \, d\theta = \frac{64}{3} \int_0^{2\pi} \int_{\pi/4}^{\pi/2} \cos^3 \varphi \sin \varphi \, d\varphi \, d\theta = \frac{8}{3} \pi.$$

$$\begin{aligned}
 \mathbf{16.5.52} \quad & \int_0^{2\pi} \int_{\pi/6}^{\pi/3} \int_{\csc \varphi}^{2 \csc \varphi} \rho^2 \sin \varphi \, d\rho \, d\varphi \, d\theta = \frac{7}{3} \int_0^{2\pi} \int_{\pi/6}^{\pi/3} \csc^2 \varphi \, d\varphi \, d\theta = -\frac{7}{3} \int_0^{2\pi} \cot \varphi \Big|_{\varphi=\pi/6}^{\varphi=\pi/3} d\theta = \\
 & -\frac{7}{3} \int_0^{2\pi} \left(\frac{1}{\sqrt{3}} - \sqrt{3} \right) d\theta = \frac{28}{9} \pi \sqrt{3}.
 \end{aligned}$$

16.5.53 The value of φ corresponding to the circle of intersection of the sphere ρ and the plane $z = 2\sqrt{3}$ is found by noting that $\cos \varphi = \frac{2\sqrt{3}}{4} = \frac{\sqrt{3}}{2}$, so that $\varphi = \frac{\pi}{6}$. Similarly, the intersection with the plane $z = 2$ is where $\cos \varphi = \frac{2}{4} = \frac{1}{2}$ so that $\varphi = \frac{\pi}{3}$. We can find the volume of the required region by determining the volume of the regions above $z = 2$ and $z = 2\sqrt{3}$ and subtracting. The volume of the region above the plane $z = 2$ is

$$\int_0^{2\pi} \int_0^{\pi/3} \int_{2 \sec \varphi}^4 \rho^2 \sin \varphi \, d\rho \, d\varphi \, d\theta = \frac{1}{3} \int_0^{2\pi} \int_0^{\pi/3} (64 - 8 \sec^3 \varphi) \sin \varphi \, d\varphi \, d\theta = \frac{40}{3} \pi,$$

and the volume of the region above $z = 2\sqrt{3}$ is

$$\int_0^{2\pi} \int_0^{\pi/6} \int_{2\sqrt{3} \sec \varphi}^4 \rho^2 \sin \varphi \, d\rho \, d\varphi \, d\theta = \frac{1}{3} \int_0^{2\pi} \int_0^{\pi/6} (64 - 24\sqrt{3} \sec^3 \varphi) \sin \varphi \, d\varphi \, d\theta = \frac{128}{3} \pi - 24\sqrt{3} \pi,$$

so that the volume of the region between the two planes is $\left(24\sqrt{3} - \frac{88}{3} \right) \pi$.

16.5.54 This cone makes an angle of $\frac{\pi}{4}$ with the z -axis. We thus want

$$\int_0^{2\pi} \int_0^{\pi/4} \int_{\sec \varphi}^{2 \sec \varphi} \rho^2 \sin \varphi \, d\rho \, d\varphi \, d\theta = \frac{7}{3} \int_0^{2\pi} \int_0^{\pi/4} \sec^3 \varphi \sin \varphi \, d\varphi \, d\theta = \frac{7}{3} \pi.$$

A simpler method of arriving at this result is to recall that the volume of a cone is $\frac{1}{3}Ah$, where A is the area of the base and h is the height, so in this case the volume of the larger cone is $\frac{1}{3} \cdot 4\pi \cdot 2$ and of the smaller $\frac{1}{3} \cdot \pi \cdot 1$, so the difference is as above.

16.5.55

a. True. In either set of coordinates, any value of θ may be chosen.

b. True. Note that $r = z$ if and only if $\rho \sin \varphi = \rho \cos \varphi$, which happens if and only if $\varphi = \frac{\pi}{4}$.

16.5.56 $\rho^2 = \sec(2\varphi)$ implies that $1 = \rho^2 \cos(2\varphi) = \rho^2 (\cos^2 \varphi - \sin^2 \varphi) = z^2 - x^2 - y^2$ which is a hyperboloid of two sheets.

16.5.57 $\rho^2 = -\sec(2\varphi)$ implies that $-1 = \rho^2 \cos(2\varphi) = \rho^2 (\cos^2 \varphi - \sin^2 \varphi) = z^2 - x^2 - y^2$ so that $x^2 + y^2 - z^2 = 1$, which is a hyperboloid of one sheet. Because $\frac{\pi}{4} < \varphi < \frac{3\pi}{4}$, we only have the upper sheet.

$$\begin{aligned}
 \mathbf{16.5.58} \quad & \int_0^{2\pi} \int_0^{\pi} \int_0^4 (1 + \rho) \rho^2 \sin \varphi \, d\rho \, d\varphi \, d\theta = \int_0^{2\pi} \int_0^{\pi} \left(\frac{1}{3} \rho^3 + \frac{1}{4} \rho^4 \right) \Big|_0^4 \sin \varphi \, d\varphi \, d\theta = \\
 & \int_0^{2\pi} \int_0^{\pi} \frac{256}{3} \sin \varphi \, d\varphi \, d\theta \\
 & = \frac{1024}{3} \pi.
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{16.5.59} \quad & \int_0^{2\pi} \int_0^{\pi} \int_0^8 2e^{-\rho^3} \rho^2 \sin \varphi \, d\rho \, d\varphi \, d\theta = -\frac{2}{3} \int_0^{2\pi} \int_0^{\pi} e^{-\rho^3} \Big|_0^8 \sin \varphi \, d\varphi \, d\theta = \\
 & -\frac{2}{3} (e^{-512} - 1) \int_0^{2\pi} \int_0^{\pi/3} \sin \varphi \, d\varphi \, d\theta = \frac{8}{3} (1 - e^{-512}) \pi.
 \end{aligned}$$

16.5.60 The mass is

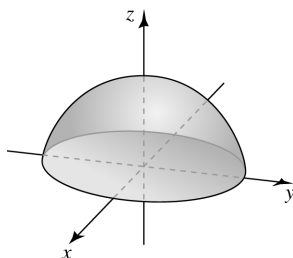
$$\begin{aligned}
 \int_0^{2\pi} \int_0^{4\sqrt{3}} \int_{r/\sqrt{3}}^4 (5-z)r \, dz \, dr \, d\theta &= \int_0^{2\pi} \int_0^{4\sqrt{3}} (5z - z^2/2)r \Big|_{r/\sqrt{3}}^4 \, dr \, d\theta \\
 &= \int_0^{2\pi} \int_0^{4\sqrt{3}} ((20-8) - (5r/\sqrt{3} - r^2/6))r \, dr \, d\theta \\
 &= \int_0^{2\pi} \int_0^{4\sqrt{3}} (12r - 5r^2/\sqrt{3} + r^3/6) \, dr \, d\theta \\
 &= \int_0^{2\pi} (6r^2 - 5r^3/3\sqrt{3} + r^4/24) \Big|_0^{4\sqrt{3}} \, d\theta \\
 &= 2\pi(288 - 320 + 96) = 128\pi.
 \end{aligned}$$

16.5.61 Note that because of the absolute value sign, the mass is symmetric around the xy -plane, so we can compute the mass for $0 \leq z \leq 1$ and double it. Using cylindrical coordinates, the mass is then

$$\begin{aligned}
 2 \int_0^1 \int_0^{2\pi} \int_0^2 (2-z)(4-r)r \, dr \, d\theta \, dz &= 2 \int_0^1 \int_0^{2\pi} \int_0^2 (2-z)(4r - r^2) \, dr \, d\theta \, dz = \\
 2 \int_0^1 \int_0^{2\pi} (2-z) \left(2r^2 - \frac{1}{3}r^3 \right) \Big|_0^2 \, d\theta \, dz &= \frac{64\pi}{3} \int_0^1 (2-z) \, dz = 32\pi.
 \end{aligned}$$

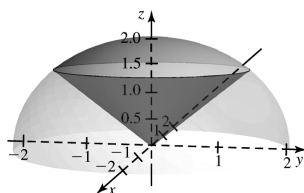
16.5.62 The cylinder $r = 1$ intersects the sphere $\rho = 5$ along the circle $x^2 + y^2 = 1$, $z = 2\sqrt{6}$. Note also that for any point (r, θ, z) on the sphere $r^2 + z^2 = x^2 + y^2 + z^2 = 25$. We have

$$\begin{aligned}
 \int_0^{2\pi} \int_1^5 \int_0^{\sqrt{25-r^2}} f(r, \theta, z) r \, dz \, dr \, d\theta &= \int_0^{2\pi} \int_0^{2\sqrt{6}} \int_1^{\sqrt{25-z^2}} f(r, \theta, z) r \, dr \, dz \, d\theta \\
 &= \int_1^5 \int_0^{\sqrt{25-r^2}} \int_0^{2\pi} f(r, \theta, z) r \, d\theta \, dz \, dr.
 \end{aligned}$$

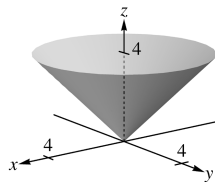


16.5.63

$$\begin{aligned}
 &\int_0^{2\pi} \int_0^{\sqrt{2}} \int_r^{\sqrt{4-r^2}} f(r, \theta, z) r \, dz \, dr \, d\theta \\
 &= \int_0^{2\pi} \int_0^{\sqrt{2}} \int_0^z f(r, \theta, z) r \, dr \, dz \, d\theta + \int_0^{2\pi} \int_{\sqrt{2}}^2 \int_0^{\sqrt{4-z^2}} f(r, \theta, z) r \, dr \, dz \, d\theta \\
 &= \int_0^{\sqrt{2}} \int_r^{\sqrt{4-r^2}} \int_0^{2\pi} f(r, \theta, z) r \, d\theta \, dz \, dr.
 \end{aligned}$$

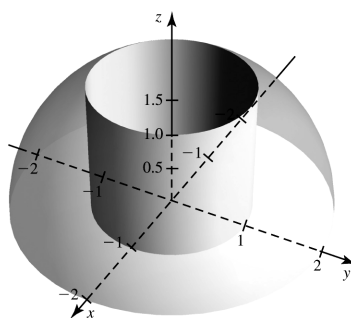


16.5.64 The region of integration is a cone with vertex at the origin making an angle of $\frac{\pi}{4}$ with the positive z -axis, from $z = 0$ to $z = 4$.



We have
$$\int_0^{\pi/4} \int_0^{2\pi} \int_0^{4 \sec \varphi} g(\rho, \varphi, \theta) \rho^2 \sin \varphi \, d\rho \, d\theta \, d\varphi = \int_0^{\pi/4} \int_0^{4 \sec \varphi} \int_0^{2\pi} g(\rho, \varphi, \theta) \rho^2 \sin \varphi \, d\theta \, d\rho \, d\varphi.$$

16.5.65 The region of integration is the solid between the upper half-sphere of radius 2 centered at the origin and a cylinder of radius 1 oriented along the z -axis.



We have
$$\int_{\pi/6}^{\pi/2} \int_0^{2\pi} \int_{\csc \varphi}^2 g(\rho, \varphi, \theta) \rho^2 \sin \varphi \, d\rho \, d\theta \, d\varphi = \int_{\pi/6}^{\pi/2} \int_{\csc \varphi}^2 \int_0^{2\pi} g(\rho, \varphi, \theta) \rho^2 \sin \varphi \, d\theta \, d\rho \, d\varphi.$$

16.5.66 Using spherical coordinates, this volume is

$$\int_{\pi/4}^{\pi/2} \int_0^{2\pi} \int_0^1 \rho^2 \sin \varphi \, d\rho \, d\theta \, d\varphi = \frac{2\pi}{3} \int_{\pi/4}^{\pi/2} \sin \varphi \, d\varphi = \frac{2\pi}{3} \cdot \frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{3} \pi.$$

16.5.67 This region is symmetric about the xy -plane, so we compute the volume of the region inside the solid cylinder for $z \geq 0$ that is below the cone $\varphi = \frac{\pi}{3}$ and double it. Use spherical coordinates. For a given value of φ , we have $0 \leq r \leq 2 \csc \varphi$, so the volume is

$$2 \int_{\pi/3}^{\pi/2} \int_0^{2 \csc \varphi} \int_0^{2\pi} \rho^2 \sin \varphi \, d\theta \, d\rho \, d\varphi = \frac{32\pi}{3} \int_{\pi/3}^{\pi/2} \csc^3 \varphi \sin \varphi \, d\varphi = \frac{32}{9} \pi \sqrt{3}.$$

16.5.68 This region is symmetric about the xy -plane, so we compute the volume of the region inside the upper half-sphere of radius 2 and below the cone $\varphi = \frac{\pi}{3}$ and double it. The volume is thus

$$2 \int_{\pi/3}^{\pi/2} \int_0^2 \int_0^{2\pi} \rho^2 \sin \varphi \, d\theta \, d\rho \, d\varphi = \frac{32\pi}{3} \int_{\pi/3}^{\pi/2} \sin \varphi \, d\varphi = \frac{16\pi}{3}.$$

16.5.69 Use cylindrical coordinates. Note that $x + y \geq 0$ for $-\frac{\pi}{4} \leq \theta \leq \frac{3\pi}{4}$, so this is the range of integration for θ .

$$\begin{aligned} \int_0^1 \int_{-\pi/4}^{3\pi/4} \int_0^r \cos \theta + r \sin \theta \, r \, dz \, d\theta \, dr &= \int_0^1 \int_{-\pi/4}^{3\pi/4} r^2 (\cos \theta + \sin \theta) \, d\theta \, dr \\ &= \int_0^1 r^2 (\sin \theta - \cos \theta) \Big|_{\theta=-\pi/4}^{\theta=3\pi/4} \, dr = 2\sqrt{2} \int_0^1 r^2 \, dr = \frac{2\sqrt{2}}{3}. \end{aligned}$$

16.5.70 Use cylindrical coordinates. The region is traced out for $0 \leq \theta \leq \pi$; however, it is symmetric around the xz -plane, so compute the volume for $0 \leq \theta \leq \frac{\pi}{2}$ to avoid negative values of r , and double it.

$$\begin{aligned} 2 \int_0^{\pi/2} \int_0^{2 \cos \theta} \int_0^{4-r \cos \theta} r \, dz \, dr \, d\theta &= 2 \int_0^{\pi/2} \int_0^{2 \cos \theta} r (4 - r \cos \theta) \, dr \, d\theta = \\ 2 \int_0^{\pi/2} \int_0^{2 \cos \theta} \left(2r^2 - \frac{1}{3} r^3 \cos \theta \right) \Big|_0^{2 \cos \theta} d\theta &= 2 \int_0^{\pi/2} \left(8 \cos^2 \theta - \frac{8}{3} \cos^4 \theta \right) d\theta = 3\pi. \end{aligned}$$

16.5.71 The planes $z = x - 2$ and $z = 2 - x$ intersect when $x = 2$, which is at the boundary of the cardioid, so we can simply integrate between those two planes. Thus the volume is $\int_0^{2\pi} \int_0^{1+\cos \theta} \int_{r \cos \theta - 2}^{2-r \cos \theta} r \, dz \, dr \, d\theta = \int_0^{2\pi} \int_0^{1+\cos \theta} r (4 - 2r \cos \theta) \, dr \, d\theta = \frac{7\pi}{2}$.

16.5.72 Use cylindrical coordinates. $\int_1^2 \int_0^{\sqrt{4-r^2}} \int_0^{2\pi} r \, d\theta \, dz \, dr = 2\pi \int_1^2 r \sqrt{4-r^2} \, dr = 2\pi\sqrt{3}$.

16.5.73 $\int_0^{2\pi} \int_0^2 \int_0^8 r (1 - 0.05e^{-0.01r^2}) \, dz \, dr \, d\theta \approx 95.60362$.

16.5.74

$$\begin{aligned} \text{a. } \int_1^\infty \int_0^\pi \int_0^{2\pi} \frac{2 \cdot 10^{-4}}{\rho^4} \rho^2 \sin \varphi \, d\theta \, d\varphi \, d\rho &= 4\pi \cdot 10^{-4} \int_1^\infty \int_0^\pi \rho^{-2} \sin \varphi \, d\varphi \, d\rho = 8\pi \cdot 10^{-4} \int_1^\infty \rho^{-2} \, d\rho = \\ \lim_{b \rightarrow \infty} \left(8\pi \cdot 10^{-4} \int_1^b \rho^{-2} \, d\rho \right) &= 8\pi \cdot 10^{-4} \lim_{b \rightarrow \infty} \left(-\rho^{-1} \right) \Big|_1^b = 8\pi \cdot 10^{-4} \lim_{b \rightarrow \infty} \left(\frac{-1}{b} + 1 \right) = 8\pi \cdot 10^{-4}. \end{aligned}$$

$$\begin{aligned} \text{b. } 2 \cdot 10^{-4} \int_0^\infty \int_0^\pi \int_0^{2\pi} e^{-0.01\rho^3} \rho^2 \sin \varphi \, d\theta \, d\varphi \, d\rho &= 4\pi \cdot 10^{-4} \int_0^\infty \int_0^\pi \rho^2 e^{-0.01\rho^3} \sin \varphi \, d\varphi \, d\rho = \\ \lim_{b \rightarrow \infty} \left(-8\pi \cdot 10^{-4} \int_0^b \rho^2 e^{-0.01\rho^3} \, d\rho \right) &= \lim_{b \rightarrow \infty} \left(-8\pi \cdot 10^{-4} \left(-\frac{100}{3} e^{-0.01\rho^3} \right) \Big|_0^b \right) = \\ \lim_{b \rightarrow \infty} \left(\frac{8\pi}{3} \cdot 10^{-6} (1 - e^{-0.01b^3}) \right) &= \frac{8\pi}{3} \cdot 10^{-6} \end{aligned}$$

16.5.75

a. With $x = \cos \varphi$ we have $\sin \varphi = \sqrt{1-x^2}$ and $dx = -\sin \varphi \, d\varphi$ so that $d\varphi = -\frac{1}{\sqrt{1-x^2}} dx$. The limits of integration for φ become 1 to -1 , so the integral then becomes

$$\begin{aligned} F(d) &= -\frac{GMm}{4\pi} \int_0^{2\pi} \int_{-1}^1 \frac{(d-Rx) \sqrt{1-x^2}}{(R^2+d^2-2Rdx)^{3/2} (-\sqrt{1-x^2})} dx \, d\theta \\ &= \frac{GMm}{4\pi} \int_{-1}^1 \int_0^{2\pi} \frac{d-Rx}{(R^2+d^2-2Rdx)^{3/2}} d\theta \, dx \\ &= \frac{GMm}{2} \int_{-1}^1 \left[\frac{d}{(R^2+d^2-2Rdx)^{3/2}} - \frac{Rx}{(R^2+d^2-2Rdx)^{3/2}} \right] dx \\ &= \frac{GMm}{2} \left(\frac{1}{R\sqrt{R^2+d^2-2Rdx}} - \frac{R^2+d^2-Rdx}{Rd^2\sqrt{R^2+d^2-2Rdx}} \right) \Big|_{-1}^1 \\ &= \frac{GMm}{2} \frac{Rdx - R^2}{Rd^2\sqrt{R^2+d^2-2Rdx}} \Big|_{-1}^1. \end{aligned}$$

Assuming $d > R$, this simplifies to $F(d) = \frac{GMm}{2} \left(\frac{Rd-R^2}{Rd^2(d-R)} + \frac{Rd+R^2}{Rd^2(d+R)} \right) = \frac{GMm}{2} \left(\frac{1}{d^2} + \frac{1}{d^2} \right) = \frac{GMm}{d^2}$.

b. Suppose $d < R$. Then we have

$$F(d) = \frac{GMm}{2} \frac{Rdx - R^2}{Rd^2\sqrt{R^2 + d^2 - 2Rdx}} \Big|_{-1}^1 = \frac{GMm}{2} \left(\frac{Rd - R^2}{Rd^2(R-d)} + \frac{Rd + R^2}{Rd^2(R+d)} \right) = \frac{GMm}{2} \left(-\frac{1}{d^2} + \frac{1}{d^2} \right) = 0.$$

16.5.76 Use cylindrical coordinates with the origin in the center of one of the tank ends, the z -axis parallel to the long axis of the tank and the y -axis parallel to the surface of the water such that the positive x -axis points through the water. The lines from the origin to the point where the water meets the tank make angles of $\pm \frac{\pi}{3}$ with the x -axis. Because each of these angles has cosine $\frac{1}{2}$ (the height of the water is one half the radius). Thus the volume of water (in cubic feet) is

$$\int_{-\pi/3}^{\pi/3} \int_{(\sec \theta)/2}^1 \int_0^2 r \, dz \, dr \, d\theta = \int_{-\pi/3}^{\pi/3} \left(1 - \frac{1}{4} \sec^2 \theta \right) d\theta = \frac{2\pi}{3} - \frac{\sqrt{3}}{2}.$$

This integral is perhaps just as easy to work in Cartesian coordinates: the area of the region of one end of the tank that contains water is simply

$$\int_{1/2}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (1) \, dy \, dx = \frac{\pi}{3} - \frac{\sqrt{3}}{4},$$

which gets multiplied by 2 for the length of the tank.

16.5.77 Assume the base of the cone lies in the xy -plane and that the center of the base is at the origin, with the vertex of the cone on the positive z -axis. Then the equation of the cone in cylindrical coordinates (a, θ, z) is $z = h - \frac{h}{r}a$ where h and r are the given constants. Thus the volume is $\int_0^r \int_0^{h-\frac{h}{r}a} \int_0^{2\pi} a \, d\theta \, dz \, da = 2\pi \int_0^r a \left(h - \frac{h}{r}a \right) da = 2\pi \left(\frac{hr^2}{2} - \frac{hr^2}{3} \right) = \frac{1}{3}\pi hr^2$.

16.5.78 In spherical coordinates, $0 \leq \varphi \leq \cos^{-1} \left(\frac{R-h}{R} \right)$. $\int_0^R \int_{(R-h)\sec \varphi}^R \int_0^{2\pi} \rho^2 \sin \varphi \, d\theta \, d\rho \, d\varphi = \frac{2}{3} \int_0^{\cos^{-1}((R-h)/R)} \left(R^3 - (R-h)^3 \sec^3 \varphi \right) \sin \varphi \, d\varphi = \frac{1}{3}\pi h^2 (3R-h)$.

16.5.79 Using similar triangles, if the frustum is extended to a complete cone, the height of the cone is $\frac{Rh}{R-r}$. Thus the equation of the cone (in cylindrical coordinates (a, θ, z)) is $z = \frac{Rh}{R-r} - \frac{h}{R-r}a$ so that $a = R - z\frac{R-r}{h}$ and the volume of the frustum is $\int_0^h \int_0^{R-z\frac{R-r}{h}} \int_0^{2\pi} a \, d\theta \, da \, dz = \pi \int_0^r \left(R - z\frac{R-r}{h} \right) dz = \frac{\pi}{3} (R^2 + rR + r^2) h$.

16.5.80 The easiest way to do this problem is using Cartesian coordinates with a change of variable. The volume of the ellipsoid is 8 times the volume of the first-octant portion, so we have

$$8 \int_0^1 \int_0^{\sqrt{1-x^2/a^2}} \int_0^{\sqrt{1-x^2/a^2-y^2/b^2}} 1 \, dz \, dy \, dx. \text{ Now make a change of variables } x = au, y = bv, z = cw \text{ so}$$

$$\text{that } dx = a \, du, dy = b \, dv, dz = c \, dw \text{ to get } 8 \int_0^1 \int_0^{\sqrt{1-u^2}} \int_0^{\sqrt{1-u^2-v^2}} abc \, dw \, dv \, du =$$

$$8abc \int_0^1 \int_0^{\sqrt{1-u^2}} \int_0^{\sqrt{1-u^2-v^2}} dw \, dv \, du, \text{ which is just } abc \text{ times the first-octant volume of a sphere of radius } 1, \text{ so is equal to } \frac{4}{3}\pi abc.$$

16.5.81 The two spheres are $x^2 + y^2 + z^2 = R^2$, $x^2 + y^2 + (z-r)^2 = r^2$. The equation of the second sphere simplifies to $x^2 + y^2 + z^2 - 2zr = 0$, so the two spheres meet when $R^2 - 2zr = 0$ or $z = \frac{R^2}{2r}$. This is the plane of intersection of the spheres. The volume in question now consists of two spherical caps, one on either side

of this plane. The upper one is a spherical cap of the sphere of radius R with height $R - \frac{R^2}{2r} = \frac{2Rr - R^2}{2r}$, so by problem 78 has volume $\frac{\pi}{3} \left(\frac{2Rr - R^2}{2r} \right)^2 \left(3R - \frac{2Rr - R^2}{2r} \right)$. The lower one is a spherical cap of the sphere of radius r with height $\frac{R^2}{2r}$, so again by problem 78 it has volume $\frac{\pi}{3} \frac{R^4}{4r^2} \left(3r - \frac{R^2}{2r} \right)$. Adding these two and simplifying gives for the volume $\frac{\pi R^3 (8r - 3R)}{12r}$.

16.6 Integrals for Mass Calculations

16.6.1 By definition, the system will balance when the pivot is located at the center of mass of the two people.

16.6.2 Its mass is $1 \cdot 50 + 2 \cdot 50 = 150$ g. Its center of mass is given by $\frac{1}{150} \left(\int_0^{50} x \, dx + \int_{50}^{100} 2x \, dx \right) = \frac{1}{150} (1250 + 7500) = \frac{175}{3}$. So the center of mass is $\frac{175}{3}$ cm from the less dense end of the rod.

16.6.3 Integrate the density function over the region to find the mass; then integrate x times the density function and divide by the mass to get the y -coordinate and integrate y times the density function and divide by the mass to get the x -coordinate.

16.6.4 Because to compute M_x , we need to compute a weighted average of distances from the x -axis, which is y .

16.6.5 Integrate the density function over the region to find the mass. To find M_{yz} , integrate x times the density function over the region; the x -coordinate is then $\frac{M_{yz}}{M}$. Similarly for the other coordinates.

16.6.6 Because to compute M_{xz} we are finding a weighted sum of distances from the xz -plane, which is y .

16.6.7 The center of mass is $\frac{1}{13} (10 \cdot 3 + 3(-1)) = \frac{27}{13}$.

16.6.8 The center of mass is $\frac{1}{8 + 4 + 1} (8 \cdot 2 + 4 \cdot (-4) + 1 \cdot 0) = 0$.

16.6.9 The mass is $\int_0^\pi (1 + \sin x) \, dx = \pi + 2$, and the center of mass is then $\frac{1}{\pi + 2} \int_0^\pi x (1 + \sin x) \, dx = \frac{1}{\pi + 2} \left(\frac{1}{2} \pi^2 + \pi \right) = \frac{\pi}{2}$.

16.6.10 The mass is $\int_0^1 (1 + x^3) \, dx = \frac{5}{4}$, so the center of mass is $\frac{1}{5/4} \int_0^1 x (1 + x^3) \, dx = \frac{4}{5} \left(\frac{1}{2} + \frac{1}{5} \right) = \frac{14}{25}$.

16.6.11 The mass is $\int_0^4 \left(2 - \frac{x^2}{16} \right) \, dx = 8 - \frac{4}{3} = \frac{20}{3}$, so the center of mass is $\frac{1}{20/3} \int_0^4 x \left(2 - \frac{x^2}{16} \right) \, dx = \frac{3}{20} (16 - 4) = \frac{9}{5}$.

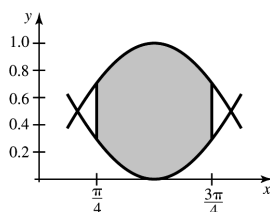
16.6.12 The mass is $\int_0^\pi (2 + \cos x) \, dx = 2\pi$, so the center of mass is $\frac{1}{2\pi} \int_0^\pi x (2 + \cos x) \, dx = \frac{\pi^2 - 2}{2\pi}$.

16.6.13 The mass is $1 \cdot 2 + \int_2^4 (1 + x) \, dx = 10$ so that the center of mass is

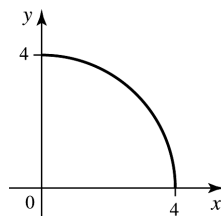
$$\frac{1}{10} \left(\int_0^2 x \, dx + \int_2^4 x (1 + x) \, dx \right) = \frac{8}{3}.$$

16.6.14 The mass is $\int_0^1 x^2 dx + \int_1^2 x(2-x) dx = 1$, so the center of mass is $\int_0^1 x^3 dx + \int_1^2 x^2(2-x) dx = \frac{7}{6}$.

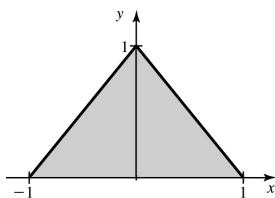
16.6.15 The region is symmetric with respect to $x = \frac{\pi}{2}$ and $y = \frac{1}{2}$, so its center of mass is at $\left(\frac{\pi}{2}, \frac{1}{2}\right)$.



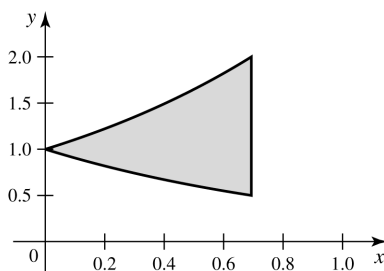
16.6.16 Assume density 1. Clearly $M_x = M_y$, so we need only compute one of them. The mass is one quarter the area of the circle, or 4π . Then using polar coordinates we have $\bar{x} = \frac{M_y}{M} = \frac{1}{4\pi} \int_0^4 \int_0^{\pi/2} (r \sin \theta) r d\theta dr = \frac{1}{4\pi} \int_0^4 r^2 dr = \frac{16}{3\pi}$, so that $\bar{y} = \frac{16}{3\pi}$ as well.



16.6.17 Assume density 1. By symmetry, $\bar{x} = 0$. The mass of the region is 1, because the two triangles together make a unit square. Thus $\bar{y} = \frac{M_x}{M} = \int_{-1}^0 \int_0^{1+x} y dy dx + \int_0^1 \int_0^{1-x} y dy dx = \frac{1}{3}$



16.6.18 Assume density 1. The mass is $\int_0^{\ln 2} \int_{e^{-x}}^{e^x} 1 dy dx = \int_0^{\ln 2} (e^x - e^{-x}) dx = (e^x + e^{-x}) \Big|_0^{\ln 2} = \frac{1}{2}$. Then $\bar{x} = \frac{M_y}{M} = \frac{1}{1/2} \int_0^{\ln 2} \int_{e^{-x}}^{e^x} x dy dx = 2 \int_0^{\ln 2} x(e^x - e^{-x}) dx = 5 \ln(2) - 3$. Also, $\bar{y} = \frac{M_x}{M} = \frac{1}{1/2} \int_0^{\ln 2} \int_{e^{-x}}^{e^x} y dy dx = \int_0^{\ln 2} (e^{2x} - e^{-2x}) dx = \frac{9}{8}$, so the center of mass is $\left(5 \ln(2) - 3, \frac{9}{8}\right)$.



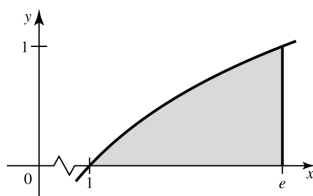
16.6.19 Assume density 1. The mass is $\int_1^e \int_0^{\ln x} 1 \, dy \, dx = \int_1^e \ln x \, dx = 1$. Then

$$\bar{x} = \frac{M_y}{M} = \frac{1}{1} \int_1^e \int_0^{\ln x} x \, dy \, dx = \int_1^e x \ln x \, dx = \frac{1}{4} (e^2 + 1)$$

and

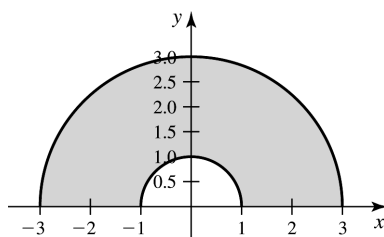
$$\bar{y} = \frac{M_x}{M} = \frac{1}{1} \int_1^e \int_0^{\ln x} y \, dy \, dx = \frac{1}{2} \int_1^e (\ln x)^2 \, dx = \frac{1}{2} e - 1,$$

so that the center of mass is $\left(\frac{1}{4} (e^2 + 1), \frac{1}{2} e - 1 \right)$.



16.6.20 Assume density 1. The mass is $\int_0^\pi \int_1^3 r \, dr \, d\theta = 4\pi$. Then clearly $\bar{x} = 0$ by symmetry, and

$$\bar{y} = \frac{M_x}{M} = \frac{1}{4\pi} \int_0^\pi \int_1^3 r^2 \sin \theta \, dr \, d\theta = \frac{13}{6\pi} \int_0^\pi \sin \theta \, d\theta = \frac{13}{3\pi}, \text{ so the center of mass is } \left(0, \frac{13}{3\pi} \right).$$



16.6.21 The mass is $\int_0^4 \int_0^2 \left(1 + \frac{x}{2} \right) \, dy \, dx = 2 \int_0^4 \left(1 + \frac{x}{2} \right) \, dx = 16$. Then

$$\bar{x} = \frac{M_y}{M} = \frac{1}{16} \int_0^4 \int_0^2 \left(x + \frac{x^2}{2} \right) \, dy \, dx = \frac{1}{8} \int_0^4 \left(x + \frac{x^2}{2} \right) \, dx = \frac{7}{3},$$

and

$$\bar{y} = \frac{M_x}{M} = \frac{1}{16} \int_0^4 \int_0^2 y \left(1 + \frac{x}{2} \right) \, dy \, dx = \frac{1}{8} \int_0^4 \left(1 + \frac{x}{2} \right) \, dx = 1.$$

Thus, the center of mass is at $\left(\frac{7}{3}, 1 \right)$. The density of the plate increases as you move toward the right.

16.6.22 The mass is $\int_0^5 \int_0^1 2e^{-y/2} dx dy = 2 \int_0^5 2e^{-y/2} dy = 4 - 4e^{-5/2}$. So we have

$$\bar{x} = \frac{M_y}{M} = \frac{1}{4 - 4e^{-5/2}} \int_0^5 \int_0^1 2xe^{-y/2} dx dy = \frac{1}{4 - 4e^{-5/2}} \int_0^5 e^{-y/2} dy = \frac{1}{2}.$$

$$\bar{y} = \frac{M_x}{M} = \frac{1}{4 - 4e^{-5/2}} \int_0^5 \int_0^1 2ye^{-y/2} dx dy = \frac{1}{4 - 4e^{-5/2}} \int_0^5 2ye^{-5y/2} dy = \frac{2e^{5/2} - 7}{e^{5/2} - 1},$$

so that the center of mass is at $\left(\frac{1}{2}, \frac{2e^{5/2} - 7}{e^{5/2} - 1}\right)$. The density of the plate decreases as you move up the plate.

16.6.23 The mass is $\int_0^4 \int_0^{4-x} (1+x+y) dy dx = \int_0^4 \left(4-x+x(4-x) + \frac{(4-x)^2}{2}\right) dx = \frac{88}{3}$. By symmetry, $\bar{x} = \bar{y}$ (both the region and the density function are symmetric around $x = y$), and $\bar{x} = \frac{M_y}{M} = \frac{1}{88/3} \int_0^4 \int_0^{4-x} (x+x^2+xy) dy dx = \frac{3}{88} \int_0^4 \left(x(4-x) + x^2(4-x) + x\frac{(4-x)^2}{2}\right) dx = \frac{16}{11}$, so the center of mass is $\left(\frac{16}{11}, \frac{16}{11}\right)$. The density of the plate increases as you move right and/or up.

16.6.24 The mass is $\int_0^\pi \int_0^2 \left(r + \frac{1}{2}r^2 \sin \theta\right) dr d\theta = \frac{6\pi + 8}{3}$. The density function does not depend on x , and the region is symmetric about the y -axis, so $\bar{x} = 0$. Then

$$\bar{y} = \frac{M_x}{M} = \frac{1}{(6\pi + 8)/3} \int_0^\pi \int_0^2 r \sin \theta \left(1 + \frac{1}{2}r \sin \theta\right) r dr d\theta = \frac{3\pi + 16}{6\pi + 8},$$

so the center of mass is $\left(0, \frac{3\pi + 16}{6\pi + 8}\right)$. The density increases as you move up the plate.

16.6.25 The mass is $\int_{-3}^3 \int_0^{\sqrt{9-x^2}/3} (1+y) dy dx = \int_{-3}^3 \left(\frac{\sqrt{9-x^2}}{3} + \frac{9-x^2}{18}\right) dx = \frac{3}{2}\pi + 2 = \frac{3\pi + 4}{2}$. The density function does not depend on x , and the region is symmetric about the y -axis, so $\bar{x} = 0$. Then $\bar{y} = \frac{M_x}{M} = \frac{1}{(3\pi + 4)/2} \int_{-3}^3 \int_0^{\sqrt{9-x^2}/3} (y+y^2) dy dx = \frac{3\pi + 16}{12\pi + 16}$, so the center of mass is $\left(0, \frac{3\pi + 16}{12\pi + 16}\right)$. The density increases as you move up the plate.

16.6.26 The mass is $\int_0^{\pi/2} \int_0^2 r(1+r^2) dr d\theta = 3\pi$. Because both the density function and the region are symmetric around $x = y$, $\bar{x} = \bar{y}$. We have $\bar{x} = \frac{M_y}{M} = \frac{1}{3\pi} \int_0^{\pi/2} \int_0^2 r^2(1+r^2) \cos \theta dr d\theta = \frac{1}{3\pi} \int_0^{\pi/2} \frac{136}{15} \cos \theta d\theta = \frac{136}{45\pi}$, so that the center of mass is $\left(\frac{136}{45\pi}, \frac{136}{45\pi}\right)$. The density of the plate increases as you move away from the origin.

16.6.27 Assuming density 1, the mass is the volume of a half-sphere of radius 4, which is $\frac{1}{2} \cdot \frac{4}{3}\pi \cdot 4^3 = \frac{128\pi}{3}$.

By symmetry, $\bar{x} = \bar{y} = 0$. Also, $\bar{z} = \frac{1}{(128\pi)/3} \int_0^4 \int_0^{2\pi} \int_0^{\pi/2} \rho \cos \varphi \cdot \rho^2 \sin \varphi d\varphi d\theta d\rho = \frac{3}{128\pi} \int_0^4 \int_0^{2\pi} \int_0^{\pi/2} \rho^3 \cos \varphi \sin \varphi d\varphi d\theta d\rho = \frac{3}{128} \int_0^4 \rho^3 d\rho = \frac{3}{2}$, so the center of mass is at $\left(0, 0, \frac{3}{2}\right)$.

16.6.28 Assuming density 1, the mass is $\int_0^{2\pi} \int_0^5 \int_{r^2}^{25} r \, dz \, dr \, d\theta = 2500\pi$. By symmetry, $\bar{x} = \bar{y} = 0$, and $\bar{z} = \frac{1}{2500\pi} \int_0^{2\pi} \int_0^5 \int_{r^2}^{25} r z \, dz \, dr \, d\theta = \frac{1}{2500\pi} \cdot \frac{312500\pi}{7} = \frac{125}{7}$, so that the center of mass is at $\left(0, 0, \frac{125}{7}\right)$.

16.6.29 Assuming density 1, the mass is the volume of a pyramid with height 1 and base area $\frac{1}{2}$, so the volume is $\frac{1}{6}$. The region is symmetric with respect to the line $x = y = z$, $\bar{x} = \bar{y} = \bar{z}$, and $\bar{z} = \frac{1}{1/6} \int_0^1 \int_0^{1-x} \int_0^{1-x-y} z \, dz \, dy \, dx = 3 \int_0^1 \int_0^{1-x} (1-x-y)^2 \, dy \, dx = \frac{1}{4}$, so the center of mass is $\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)$.

16.6.30 Assuming density 1, the mass is the volume of a cone with base area 256π and height 16, so is $\frac{4096\pi}{3}$. By symmetry, $\bar{x} = \bar{y} = 0$, and $\bar{z} = \frac{1}{(4096\pi)/3} \int_0^{16} \int_0^{16-r} \int_0^{2\pi} r z \, d\theta \, dz \, dr = \frac{3}{4096} \int_0^{16} r (16-r)^2 \, dr = 4$, so the center of mass is $(0, 0, 4)$.

16.6.31 Assuming density 1, the mass is

$$\int_0^1 \int_0^{2\pi} \int_0^{1-r \sin \theta} r \, dz \, d\theta \, dr = \int_0^1 \int_0^{2\pi} r (1-r \sin \theta) \, d\theta \, dr = \pi.$$

The region is symmetric around the yz -plane, so $\bar{x} = 0$. We have

$$\begin{aligned} \bar{y} &= \frac{1}{\pi} \int_0^1 \int_0^{2\pi} \int_0^{1-r \sin \theta} r^2 \sin \theta \, dz \, d\theta \, dr = \frac{1}{\pi} \int_0^1 \int_0^{2\pi} r^2 (1-r \sin \theta) \sin \theta \, d\theta \, dr = -\frac{1}{4}, \\ \bar{z} &= \frac{1}{\pi} \int_0^1 \int_0^{2\pi} \int_0^{1-r \sin \theta} r z \, dz \, d\theta \, dr = \frac{1}{2\pi} \int_0^1 \int_0^{2\pi} r (1-r \sin \theta)^2 \, d\theta \, dr = \frac{5}{8}, \end{aligned}$$

so that the center of mass is $\left(0, -\frac{1}{4}, \frac{5}{8}\right)$.

16.6.32 Assuming density 1, the mass is $\int_0^2 \int_0^{2\pi} \int_0^{2\sqrt{4-r^2}} r \, dz \, d\theta \, dr = 4\pi \int_0^2 r \sqrt{4-r^2} \, dr = \frac{32\pi}{3}$. By symmetry, $\bar{x} = \bar{y} = 0$, and $\bar{z} = \frac{1}{(32\pi/3)} \int_0^2 \int_0^{2\pi} \int_0^{2\sqrt{4-r^2}} r z \, dz \, d\theta \, dr = \frac{3}{8} \int_0^2 r (4-r^2) \, dr = \frac{3}{2}$, so that the center of mass is $\left(0, 0, \frac{3}{2}\right)$.

16.6.33 The mass is $\int_0^4 \int_0^1 \int_0^1 \left(1 + \frac{x}{2}\right) \, dz \, dy \, dx = \int_0^4 \left(1 + \frac{x}{2}\right) \, dx = 8$. By symmetry, Because the density depends only on x , $\bar{y} = \bar{z} = \frac{1}{2}$, while $\bar{x} = \frac{1}{8} \int_0^4 \int_0^1 \int_0^1 \left(x + \frac{x^2}{2}\right) \, dz \, dy \, dx = \frac{1}{8} \int_0^4 \left(x + \frac{x^2}{2}\right) \, dx = \frac{7}{3}$, so the center of mass is $\left(\frac{7}{3}, \frac{1}{2}, \frac{1}{2}\right)$.

16.6.34 The mass is

$$\int_0^4 \int_0^{\sqrt{4-z}} \int_0^{2\pi} r (5-z) \, d\theta \, dr \, dz = 2\pi \int_0^4 \int_0^{\sqrt{4-z}} r (5-z) \, dr \, dz = \pi \int_0^4 (5-z) (4-z) \, dz = \frac{88\pi}{3}.$$

By symmetry, because the density depends only on z , $\bar{x} = \bar{y} = 0$, while

$$\begin{aligned} \bar{z} &= \frac{1}{(88\pi)/3} \int_0^4 \int_0^{\sqrt{4-z}} \int_0^{2\pi} r z (5-z) \, d\theta \, dr \, dz = \frac{3}{44} \int_0^4 \int_0^{\sqrt{4-z}} r z (5-z) \, dr \, dz \\ &= \frac{3}{88} \int_0^4 (5-z) (4-z) \, dz = \frac{12}{11}, \end{aligned}$$

so the center of mass is $\left(0, 0, \frac{12}{11}\right)$.

16.6.35 The mass of the sphere is given by $\int_0^6 \int_0^{\pi/2} \int_0^{2\pi} \left(1 + \frac{\rho}{4}\right) \rho^2 \sin \varphi \, d\theta \, d\varphi \, d\rho = 2\pi \int_0^6 \left(1 + \frac{\rho}{4}\right) \rho^2 \, d\rho = 306\pi$. By symmetry, because the density function depends only on ρ , $\bar{x} = \bar{y} = 0$. Also,
 $\bar{z} = \frac{1}{306\pi} \int_0^6 \int_0^{\pi/2} \int_0^{2\pi} \left(1 + \frac{\rho}{4}\right) \rho^2 \sin \varphi \cdot \rho \cos \varphi \, d\theta \, d\varphi \, d\rho = \frac{1}{153} \int_0^6 \int_0^{\pi/2} \rho^3 \left(1 + \frac{\rho}{4}\right) \sin \varphi \cos \varphi \, d\varphi \, d\rho = \frac{1}{306} \int_0^6 \rho^3 \left(1 + \frac{\rho}{4}\right) \, d\rho = \frac{198}{85}$, so the center of mass is $\left(0, 0, \frac{198}{85}\right)$.

16.6.36 The mass of the cube is given by

$$\int_0^1 \int_0^1 \int_0^1 (2 + x + y + z) \, dz \, dy \, dx = \int_0^1 \int_0^1 \left(\frac{5}{2} + x + y\right) \, dy \, dx = \int_0^1 (3 + x) \, dx = \frac{7}{2}.$$

Both the region and the density function are symmetric with respect to the line $x = y = z$, so $\bar{x} = \bar{y} = \bar{z}$, and

$$\begin{aligned} \bar{x} &= \frac{1}{7/2} \int_0^1 \int_0^1 \int_0^1 (2x + x^2 + xy + xz) \, dz \, dy \, dx = \frac{2}{7} \int_0^1 \int_0^1 \left(\frac{5}{2}x + x^2 + xy\right) \, dy \, dx \\ &= \frac{2}{7} \int_0^1 (3x + x^2) \, dx = \frac{11}{21}, \end{aligned}$$

so that the center of mass is $\left(\frac{11}{21}, \frac{11}{21}, \frac{11}{21}\right)$.

16.6.37 The mass is $\int_0^1 \int_0^4 \int_0^x (2 + y) \, dz \, dy \, dx = \int_0^1 \int_0^4 (2x + xy) \, dy \, dx = \int_0^1 16x \, dx = 8$. Then

$$\begin{aligned} \bar{x} &= \frac{1}{8} \int_0^1 \int_0^4 \int_0^x (2x + xy) \, dz \, dy \, dx = \frac{1}{8} \int_0^1 \int_0^4 x^2 (y + 2) \, dy \, dx = 2 \int_0^1 x^2 \, dx = \frac{2}{3}. \\ \bar{y} &= \frac{1}{8} \int_0^1 \int_0^4 \int_0^x (2y + y^2) \, dz \, dy \, dx = \frac{1}{8} \int_0^1 \int_0^4 x (y + 2y^2) \, dy \, dx = \frac{14}{3} \int_0^1 x \, dx = \frac{7}{3}. \\ \bar{z} &= \frac{1}{8} \int_0^1 \int_0^4 \int_0^x (2z + yz) \, dz \, dy \, dx = \frac{1}{8} \int_0^1 \int_0^4 \left(x^2 + \frac{x^2 y}{2}\right) \, dy \, dx = \frac{1}{8} \int_0^1 8x^2 \, dx = \frac{1}{3}, \end{aligned}$$

so that the center of mass is $\left(\frac{2}{3}, \frac{7}{3}, \frac{1}{3}\right)$.

16.6.38 The mass is $\int_0^9 \int_0^{9-r} \int_0^{2\pi} r(1 + z) \, d\theta \, dz \, dr = 2\pi \int_0^9 r \left(9 - r + \frac{(9-r)^2}{2}\right) \, dr = \frac{3159\pi}{4}$. By symmetry and the fact that the density depends only on z , we have $\bar{x} = \bar{y} = 0$. We also have
 $\bar{z} = \frac{1}{(3159\pi)/4} \int_0^9 \int_0^{9-r} \int_0^{2\pi} r(z + z^2) \, d\theta \, dz \, dr = \frac{8}{3159} \int_0^9 r \left(\frac{(9-r)^2}{2} + \frac{(9-r)^3}{3}\right) \, dr = \frac{207}{65}$, so that the center of mass is $\left(0, 0, \frac{207}{65}\right)$.

16.6.39

- False. It has a center of mass with a y -coordinate of zero.
- True. Because every point is balanced by the corresponding point on the other side of the origin.

c. False. For example, the annulus $1 \leq r \leq 3$ has center of mass at the origin.

d. False. For example, the solid resulting from revolving the annulus in part (c) about the x -axis is connected, but its center of mass is at the origin.

16.6.40 The mass of the rod is $\int_0^L 2e^{-x/3} dx = 6(1 - e^{-L/3})$, so its center of mass is given by

$$\frac{1}{6(1 - e^{-L/3})} \int_0^L 2x e^{-x/3} dx = \frac{18 - 18e^{-L/3} - 6Le^{-L/3}}{6(1 - e^{-L/3})},$$

so that as $L \rightarrow \infty$, the center of mass approaches

$$\lim_{L \rightarrow \infty} \frac{18 - 18e^{-L/3} - 6Le^{-L/3}}{6(1 - e^{-L/3})} = 3, \text{ because all terms involving } L \text{ approach } 0 \text{ in the limit.}$$

16.6.41 The mass of the rod is $\int_0^L \frac{10}{1+x^2} dx = 10 \tan^{-1}(L)$, so its center of mass is

$$\frac{1}{\tan^{-1}(L)} \int_0^L \frac{10x}{1+x^2} dx = \frac{\ln(1+L^2)}{\tan^{-1}(L)}.$$

As $L \rightarrow \infty$, the numerator grows without bound while the denominator approaches $\frac{\pi}{2}$, so $\bar{x} \rightarrow \infty$ as $L \rightarrow \infty$.

16.6.42 The mass of the plate is $\int_0^L \int_{-e^{-x}}^{e^{-x}} dy dx = \int_0^L 2e^{-x} dx = 2 - 2e^{-L}$, so that

$$\lim_{L \rightarrow \infty} \bar{x} = \lim_{L \rightarrow \infty} \frac{1}{2 - 2e^{-L}} \int_0^L \int_{-e^{-x}}^{e^{-x}} x dy dx = \lim_{L \rightarrow \infty} \frac{1}{2 - 2e^{-L}} \int_0^L 2x e^{-x} dx = \lim_{L \rightarrow \infty} \frac{2 - 2e^{-L}(L+1)}{2 - 2e^{-L}} = 1.$$

Also, $\lim_{L \rightarrow \infty} \bar{y} = \lim_{L \rightarrow \infty} \frac{1}{2 - 2e^{-L}} \int_0^L \int_{-e^{-x}}^{e^{-x}} y dy dx = \lim_{L \rightarrow \infty} 0 = 0$. Thus as $L \rightarrow \infty$, the center of mass approaches $(1, 0)$.

16.6.43 The mass of the plate, assuming density 1, is $8 + 4 = 12$. Because the area of the rectangle is 8 and the two triangles together to form a 2×2 square. By symmetry, $\bar{x} = 0$. Compute $\bar{y}$ by computing M_x for each of the three pieces. The moments around the x -axis are equal for the two triangles. So we need only compute M_x for one of the triangles. This is $M_x = \int_2^4 \int_0^{4-x} y dy dx = \int_2^4 \frac{1}{2} (4-x)^2 dx = \frac{4}{3}$

The moment around the x -axis for the rectangle is

$$M_x = \int_{-2}^2 \int_0^2 y dy dx = 8,$$

so that $\bar{y} = \frac{1}{12} \left(\frac{4}{3} + 8 + \frac{4}{3} \right) = \frac{8}{9}$. Thus the center of mass is at $\left(0, \frac{8}{9} \right)$.

16.6.44 Assuming density 1, the mass of the plate is the area of the outer rectangle (48) less the area of the inner, missing, rectangle (4), so the mass is 44. By symmetry, $\bar{x} = 0$. To compute $\bar{y}$, compute the moment around the x -axis by computing the moment for the outer rectangle and that for the missing rectangle, and subtract one from the other. Thus $\bar{y} = \frac{1}{44} \left(\int_{-4}^4 \int_{-4}^2 y dy dx - \int_{-2}^2 \int_{-1}^0 y dy dx \right) = \frac{1}{44} (-48 + 2) = -\frac{23}{22}$.

Thus the center of mass is at $\left(0, -\frac{23}{22} \right)$.

16.6.45 The mass of the region (assuming density 1) is $\frac{1}{2} \pi r^2 = 2\pi$, by symmetry, $\bar{x} = 0$, and $\bar{y} = \frac{1}{2\pi} \int_0^\pi \int_0^2 r^2 \sin \theta dr d\theta = \frac{1}{2\pi} \int_0^\pi \frac{8}{3} \sin \theta d\theta = \frac{8}{3\pi}$, so the center of mass is at $\left(0, \frac{8}{3\pi} \right)$.

16.6.46 The mass of the region is π ; by symmetry, $\bar{x} = \bar{y}$, and

$$\bar{x} = \frac{1}{\pi} \int_0^{\pi/2} \int_0^2 r^2 \cos \theta \, dr \, d\theta = \frac{8}{3\pi} \int_0^{\pi/2} \cos \theta \, d\theta = \frac{8}{3\pi},$$

thus the center of mass is at $\left(\frac{8}{3\pi}, \frac{8}{3\pi}\right)$.

16.6.47 The mass of the cardioid is $\int_0^{2\pi} \int_0^{1+\cos\theta} r \, dr \, d\theta = \frac{1}{2} \int_0^{2\pi} (1+\cos\theta)^2 \, d\theta = \frac{3\pi}{2}$. By symmetry, $\bar{y} = 0$, and $\bar{x} = \frac{1}{(3\pi)/2} \int_0^{2\pi} \int_0^{1+\cos\theta} r^2 \cos \theta \, dr \, d\theta = \frac{2}{9\pi} \int_0^{2\pi} (1+\cos\theta)^3 \cos \theta \, d\theta = \frac{5}{6}$, so that the center of mass is at $\left(\frac{5}{6}, 0\right)$.

16.6.48 The mass of the cardioid is $\int_0^{2\pi} \int_0^{3-3\cos\theta} r \, dr \, d\theta = \frac{9}{2} \int_0^{2\pi} (1-\cos\theta)^2 \, d\theta = \frac{27\pi}{2}$. By symmetry, $\bar{y} = 0$, and $\bar{x} = \frac{1}{(27\pi)/2} \int_0^{2\pi} \int_0^{3-3\cos\theta} r^2 \cos \theta \, dr \, d\theta = \frac{2}{3\pi} \int_0^{2\pi} (1-\cos\theta)^3 \cos \theta \, d\theta = -\frac{5}{2}$, so that the center of mass is at $\left(-\frac{5}{2}, 0\right)$.

16.6.49 The mass of the leaf is $\int_0^{\pi/2} \int_0^{\sin 2\theta} r \, dr \, d\theta = \frac{1}{2} \int_0^{\pi/2} \sin^2 2\theta \, d\theta = \frac{\pi}{8}$. By symmetry, $\bar{x} = \bar{y}$, and $\bar{x} = \frac{1}{\pi/8} \int_0^{\pi/2} \int_0^{\sin 2\theta} r^2 \cos \theta \, dr \, d\theta = \frac{8}{3\pi} \int_0^{\pi/2} \sin^3 2\theta \cos \theta \, d\theta = \frac{128}{105\pi}$, so that the center of mass is at $\left(\frac{128}{105\pi}, \frac{128}{105\pi}\right)$.

16.6.50 The mass of the limaçon is $\int_0^{2\pi} \int_0^{2+\cos\theta} r \, dr \, d\theta = \frac{1}{2} \int_0^{2\pi} (2+\cos\theta)^2 \, d\theta = \frac{9\pi}{2}$. By symmetry, $\bar{y} = 0$, and $\bar{x} = \frac{1}{(9\pi)/2} \int_0^{2\pi} \int_0^{2+\cos\theta} r^2 \cos \theta \, dr \, d\theta = \frac{1}{2} \int_0^{2\pi} (2+\cos\theta)^3 \cos \theta \, d\theta = \frac{17}{18}$, so that the center of mass is at $\left(\frac{17}{18}, 0\right)$.

16.6.51 Assume the origin is at the midpoint of the diameter with the y -axis pointing up. Assume the density is one. The mass is πr (the length of the wire). The moment is $\int_0^\pi (r \sin \theta) \, d\theta = 2r$. Therefore $\bar{y} = \frac{2r}{\pi}$, while $\bar{x} = 0$ by symmetry.

16.6.52 The line $y = b$ intersects the parabola at $x = \pm\sqrt{b/a}$, so the mass of the plate (assuming density 1) is

$$\int_{-\sqrt{b/a}}^{\sqrt{b/a}} \int_{ax^2}^b 1 \, dy \, dx = \int_{-\sqrt{b/a}}^{\sqrt{b/a}} (b - ax^2) \, dx = \frac{4b}{3} \sqrt{b/a}.$$

By symmetry, $\bar{x} = 0$, and

$$\bar{y} = \frac{1}{\left(\frac{4b}{3}\sqrt{b/a}\right)} \int_{-\sqrt{b/a}}^{\sqrt{b/a}} \int_{ax^2}^b y \, dy \, dx = \frac{3}{8b} \sqrt{\frac{a}{b}} \int_{-\sqrt{b/a}}^{\sqrt{b/a}} (b^2 - a^2 x^4) \, dx = \frac{3b}{5},$$

so that the center of mass is at $\left(0, \frac{3b}{5}\right)$ and is independent of a .

16.6.53 The mass of the region, assuming density 1, is $a^2 - \frac{\pi}{4}a^2 = \frac{4-\pi}{4}a^2$. By symmetry, $\bar{x} = \bar{y}$, and $\bar{x} = \frac{1}{a^2(4-\pi)/4} \int_0^a \int_{\sqrt{a^2-x^2}}^a x \, dy \, dx = \frac{4}{a^2(4-\pi)} \int_0^a x \left(a - \sqrt{a^2-x^2} \right) dx = \frac{2a}{3(4-\pi)}$, so that the center of mass is at $\left(\frac{2a}{3(4-\pi)}, \frac{2a}{3(4-\pi)} \right)$.

16.6.54 Clearly the center of mass is at the center of the box - halfway between each opposite pair of faces. The easiest way to see this is to place the origin at the center of the box; then (for example)

$$\bar{x} = \frac{1}{m} \int_{-c/2}^{c/2} \int_{-b/2}^{b/2} \int_{-a/2}^{a/2} x \, dx \, dy \, dz = \frac{1}{m} \int_{-c/2}^{c/2} \int_{-b/2}^{b/2} \frac{1}{2} x^2 \Big|_{-a/2}^{a/2} dy \, dz = 0.$$

16.6.55 Place the origin at the vertex of the cone and let the z -axis be the axis of the cone. Then the cone has equation (in cylindrical coordinates (a, θ, z)) $z = \frac{h}{r}a$. The mass of the cone is $\frac{1}{3}\pi r^2 h$, and by symmetry $\bar{x} = \bar{y} = 0$. Now $\bar{z} = \frac{1}{\pi r^2 h/3} \int_0^r \int_0^{(h/r)a} \int_0^{2\pi} az \, d\theta \, dz \, da = \frac{3h}{r^4} \int_0^r a^3 \, da = \frac{3h}{4}$, so that the center of mass is one quarter of the way from the base to the vertex.

16.6.56 Place the origin at the center of the hemisphere. The sphere has mass $\frac{2}{3}\pi a^3$, so (in spherical coordinates)

$$\bar{z} = \frac{1}{(2\pi a^3)/3} \int_0^a \int_0^{\pi/2} \int_0^{2\pi} \rho^3 \sin \varphi \cos \varphi \, d\theta \, d\varphi \, d\rho = \frac{3}{a^3} \int_0^a \int_0^{\pi/2} \rho^3 \sin \varphi \cos \varphi \, d\varphi \, d\rho = \frac{3}{8}a,$$

so that the center of mass is $\frac{3}{8}$ of the way from the origin to the top of the hemisphere.

16.6.57 Place the origin at the middle of the base of the triangle. Then the y -coordinate of the center of mass can be determined. If h is the height of the triangle, its area is $\frac{bh}{2}$, $\bar{x} = 0$ by symmetry, and $\bar{y} = \frac{1}{(bh)/2} \int_0^h \int_{b(y-h)/(2h)}^{-b(y-h)/(2h)} y \, dx \, dy = \frac{2}{bh} \int_0^h \frac{-b(y-h)}{h} dy = -\frac{2}{h^2} \int_0^h (y^2 - hy) \, dy = \frac{h}{3}$, so that the center of mass is $\frac{1}{3}$ of the way from the base to the vertex.

16.6.58 The tetrahedron has height a , and the base has area $\frac{a^2}{2}$, so the volume of the tetrahedron is $\frac{a^3}{6}$. By symmetry, $\bar{x} = \bar{y} = \bar{z}$, and $\bar{x} = \frac{1}{a^3/6} \int_0^a \int_0^{a(1-x/a)} \int_0^{a(1-x/a-y/a)} x \, dz \, dy \, dx$. Make the change of variable $u = \frac{x}{a}$, $v = \frac{y}{a}$, $w = \frac{z}{a}$; then $dx = a \, du$, $du = a \, dv$, $dz = a \, dw$ and then

$$\bar{x} = \frac{1}{a^3/6} \int_0^a \int_0^{1-u} \int_0^{1-u-v} a^4 u \, dw \, dv \, du = 6a \int_0^1 \int_0^{1-u} u(1-u-v) \, dv \, du = \frac{a}{4},$$

so the center of mass is $\left(\frac{a}{4}, \frac{a}{4}, \frac{a}{4} \right)$.

16.6.59 Place the origin at the center of the ellipse with the circular base of radius r in the xy -plane, so that the top of the ellipsoid is on the positive z -axis, at $(0, 0, a)$. Use cylindrical coordinates (ρ, θ, z) ; then the equation for the top half of the ellipsoid is $z = a\sqrt{1 - \frac{\rho^2}{r^2}}$. We know (Problem 80, Section 13.5) that

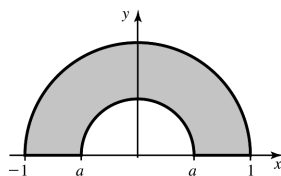
the volume of the top half of this ellipsoid is $\frac{2\pi r^2 a}{3}$, so $\bar{z} = \frac{1}{(2\pi r^2 a)/3} \int_0^r \int_0^{a\sqrt{1-\rho^2/r^2}} \int_0^{2\pi} \rho z \, d\theta \, dz \, d\rho = \frac{3a}{2r^2} \int_0^r \rho \left(1 - \frac{\rho^2}{r^2} \right) d\rho = \frac{3}{8}a$, so the center of mass is $\frac{3}{8}$ of the way from the base to the top of the ellipsoid.

16.6.60 The area of the country is (adding up the large rectangle and the two small squares) $48 + 2 \cdot 4 = 56$. $\bar{x} = 0$ by symmetry. To compute $\bar{y}$, we subtract the moment of the missing small rectangle around the x -axis from that of the large rectangle. But the large rectangle is symmetric around the x -axis, so its moment is zero and thus $\bar{y} = \frac{1}{56} \left(0 - \int_{-4}^{-2} \int_{-2}^2 y \, dx \, dy \right) = -\frac{1}{14} \int_{-4}^{-2} y \, dy = \frac{3}{7}$, so that the geographical center is at $\left(0, \frac{3}{7} \right)$. The population center is a discrete computation. Count population in thousands: $\bar{x} = \frac{10 \cdot (-2) + 15 \cdot 2 + 20 \cdot 2 + 5 \cdot 4 + 15 \cdot (-2)}{10 + 15 + 20 + 5 + 15} = \frac{40}{65} = \frac{8}{13}$, $\bar{y} = \frac{10 \cdot 2 + 15 \cdot 3 + 20 \cdot 0 + 5 \cdot (-4) + 15 \cdot (-2)}{10 + 15 + 20 + 5 + 15} = \frac{15}{65} = \frac{3}{13}$, so that the population center is at $\left(\frac{8}{13}, \frac{3}{13} \right)$.

16.6.61

- a. The mass of the plate is the difference of the area of the two semicircles, so is $\frac{1}{2}\pi(1 - a^2)$. The y -coordinate of the center of mass is then

$$\bar{y} = \frac{1}{\pi(1 - a^2)/2} \int_a^1 \int_0^\pi r^2 \sin \theta \, d\theta \, dr = \frac{4}{3\pi(1 - a^2)} \int_a^1 r^2 \, dr = \frac{4(1 - a^3)}{3\pi(1 - a^2)} = \frac{4(a^2 + a + 1)}{3(a + 1)\pi}.$$



- b. The center of mass always has x coordinate 0, so it lies on the edge of the plate exactly when $\frac{4(a^2 + a + 1)}{3(a + 1)\pi} = a$ or 1. Solving for equality with 1 gives no solutions in the range $0 \leq a \leq 1$. Solving for equality with a gives $a = -\frac{1}{2} \left(1 - \sqrt{\frac{3(\pi + 4)}{3\pi - 4}} \right) \approx 0.49366$ while the other solution is outside of the range $0 \leq a \leq 1$.

16.6.62

- a. The mass of the solid is the difference of the volumes of the two hemispheres, so is $\frac{2}{3}\pi(1 - a^3)$. The

$$z\text{-coordinate of the center of mass is then } \bar{z} = \frac{1}{2\pi(1 - a^3)/3} \int_0^1 \int_0^{\pi/2} \int_0^{2\pi} r^3 \cos \varphi \sin \varphi \, d\theta \, d\varphi \, dr = \frac{3}{2(1 - a^3)} \int_0^1 r^3 \, dr = \frac{3(1 - a^4)}{8(1 - a^3)}.$$

- b. The center of mass always has x and y -coordinates 0, so it lies on the edge of the plate exactly when $\frac{3(1 - a^4)}{8(1 - a^3)} = a$ or 1. Solving for equality with 1 gives no solutions in the range $0 \leq a \leq 1$; solving for equality with a gives $a = \frac{(1450 + 450\sqrt{11})^{2/3} - 5(1450 + 450\sqrt{11})^{1/3} - 50}{15(1450 + 450\sqrt{11})^{1/3}} \approx 0.38936$.

16.6.63 Place the origin at the center of the bottom of the soda can. If the height of soda in the can is h , for $0 \leq h \leq 12$, the mass of the can is $16\pi h + \frac{16}{1000}\pi(12 - h) = \frac{6}{125}\pi(333h + 4)$. To compute $\bar{z}$, compute the moments around the x -axis separately for the soda and the air:

$$\begin{aligned}
\bar{z} &= \frac{1}{6\pi(333h+4)/125} \int_0^4 \int_0^{2\pi} \left(\frac{1}{1000} \int_h^{12} zr \, dz + \int_0^h zr \, dz \right) d\theta \, dr \\
&= \frac{125}{6\pi(333h+4)} \int_0^4 \int_0^{2\pi} \left(\frac{1}{2000} (144 - h^2) r + \frac{1}{2} h^2 r \right) d\theta \, dr \\
&= \frac{125}{6\pi(333h+4)} \cdot \frac{\pi(144 + 999h^2)}{125} = \frac{3}{2} \cdot \frac{111h^2 + 16}{333h + 4}.
\end{aligned}$$

The center of mass is at its lowest point when the derivative of this function is zero, i.e. when $\frac{333h}{333h+4} - \frac{999(16+111h^2)}{2(333h+4)^2} = 0$. Placing over a common denominator, setting the numerator to zero and solving gives $h = \frac{40\sqrt{10}-4}{333} \approx 0.3678$ cm.

16.6.64

a. There are two cases: either $0 < a \leq b$ or $0 < b \leq a$. In either case, the area of the triangle is $\frac{1}{2}bh$.

Case 1: $0 < a \leq b$.

$$\begin{aligned}
\bar{x} &= \frac{1}{bh/2} \left(\int_0^a \int_0^{(h/a)x} x \, dy \, dx + \int_a^b \int_0^{(h/(a-b))(x-b)} x \, dy \, dx \right) = \\
&\frac{2}{bh} \left(\frac{h}{a} \int_0^a x^2 \, dx + \frac{h}{a-b} \int_a^b (x^2 - bx) \, dx \right) = \frac{2}{bh} \left(\frac{h}{3a} a^3 + \frac{h(b^2 + ab - 2a^2)}{6} \right) = \frac{a+b}{3}. \\
\bar{y} &= \frac{1}{bh/2} \left(\int_0^a \int_0^{(h/a)x} y \, dy \, dx + \int_a^b \int_0^{(h/(a-b))(x-b)} y \, dy \, dx \right) = \\
&\frac{2}{bh} \left(\frac{h^2}{2a^2} \int_0^a x^2 \, dx + \frac{h^2}{2(a-b)^2} \int_a^b (x-b)^2 \, dx \right) = \frac{2}{bh} \left(\frac{ah^2}{6} - \frac{h^2(a-b)}{6} \right) = \frac{2}{bh} \cdot \frac{bh^2}{6} = \frac{h}{3}.
\end{aligned}$$

Case 2: $0 < b \leq a$.

$$\begin{aligned}
\bar{x} &= \frac{1}{bh/2} \left(\int_0^b \int_0^{(h/a)x} x \, dy \, dx + \int_b^a \int_{(h/(a-b))(x-b)}^{(h/a)x} x \, dy \, dx \right) = \\
&\frac{2}{bh} \left(\frac{h}{a} \int_0^b x^2 \, dx + \int_b^a x \left(\frac{h}{a} x - \frac{h}{a-b} (x-b) \right) dx \right) = \frac{2}{bh} \left(\frac{hb^3}{3a} + \frac{(a^2 + ab - 2b^2)bh}{6a} \right) = \frac{a+b}{3}. \\
\bar{y} &= \frac{1}{bh/2} \left(\int_0^b \int_0^{(h/a)x} y \, dy \, dx + \int_b^a \int_{(h/(a-b))(x-b)}^{(h/a)x} y \, dy \, dx \right) = \\
&\frac{1}{bh} \left(\frac{h^2}{a^2} \int_0^b x^2 \, dx + \int_b^a \frac{h^2}{a^2} x^2 - \frac{h^2}{(a-b)^2} (x-b)^2 \, dx \right) = \frac{1}{bh} \left(\frac{h^2b^3}{3a^2} + \frac{h^2b(a^2 - b^2)}{3a^2} \right) = \frac{h}{3}.
\end{aligned}$$

In either case, the centroid is at $\left(\frac{a+b}{3}, \frac{h}{3} \right)$.

b. Because each median bisects the triangle, the centroid must lie on each median. Because the triangle will be balanced with respect to the axis determined by the median. Because this is true of each median, the centroid must be at the intersection of the medians.

16.6.65

- a. Place the origin at Q , with the circles to the right of the y -axis. Then using polar coordinates (a, θ) , the equation of the circles are $a = 2R \cos \theta$ and $a = 2r \cos \theta$ for $0 \leq \theta \leq \pi$. The mass of the earring is $\pi(R^2 - r^2)$, and $\bar{y} = 0$.

$$\bar{x} = \frac{1}{\pi(R^2 - r^2)} \int_0^\pi \int_{2r \cos \theta}^{2R \cos \theta} a^2 \cos \theta \, da \, d\theta = \frac{8(R^3 - r^3)}{3\pi(R^2 - r^2)} \int_0^\pi \cos^4 \theta \, d\theta = \frac{R^2 + Rr + r^2}{R + r}.$$

With the origin instead at the center of the large circle, the equations of the circles are $x^2 + y^2 = R^2$ and $(x - (R - r))^2 + y^2 = r^2$. Then the moment around the y -axis of the large circle is zero, so

$$\text{to compute } \bar{x} \text{ for the earring, we compute } \bar{x} = \frac{1}{\pi(R^2 - r^2)} \left(0 - \int_{R-2r}^R \int_{-\sqrt{r^2 - (x - (R-r))^2}}^{\sqrt{r^2 - (x - (R-r))^2}} x \, dy \, dx \right) =$$

$$- \frac{1}{\pi(R^2 - r^2)} \int_{R-2r}^R x \sqrt{r^2 - (x - (R-r))^2} \, dx = - \frac{r^2}{R + r}.$$

- b. With the origin at the center of the large circle, point P is $(R - 2r, 0)$, so we want $R - 2r = -\frac{r^2}{R + r}$.

Multiplying both sides by $R + r$, dividing by r^2 and letting $x = \frac{R}{r}$, we find that $x = \frac{1}{x - 1}$ or

$$x^2 - x - 1 = 0, \text{ which has roots } \frac{1 \pm \sqrt{5}}{2}. \text{ Because } R, r > 0, \text{ it must be the positive value, so}$$

$$x = \frac{1 + \sqrt{5}}{2} \text{ satisfies the condition.}$$

16.7 Change of Variables in Multiple Integrals

16.7.1 It is the square with vertices $(0, 0)$, $(2, 0)$, $(2, 2)$, and $(0, 2)$.

16.7.2 The Jacobian is $J(u, v) = \begin{vmatrix} \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} \\ \frac{\partial h}{\partial u} & \frac{\partial h}{\partial v} \end{vmatrix}.$

16.7.3 The Jacobian is $\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2$, so $\iint_R f(x, y) \, dA = \iint_S f(u + v, u - v) |J(u, v)| \, dA =$

$$\iint_S 2f(u + v, u - v) \, dA = \int_0^1 \int_0^1 2f(u + v, u - v) \, dv \, du.$$

16.7.4 It is the cube of side length $\frac{1}{2}$ in the first octant with one vertex at the origin.

16.7.5 $u = \frac{x}{2}$, so $0 \leq u \leq 1$ means $0 \leq \frac{x}{2} \leq 1$ or $0 \leq x \leq 2$. $v = 2y$, so $0 \leq v \leq 1$ means $0 \leq 2y \leq 1$, or $0 \leq y \leq \frac{1}{2}$. Thus the image is the region $\left\{ (x, y) : 0 \leq x \leq 2, 0 \leq y \leq \frac{1}{2} \right\}$, which is a rectangle in the first quadrant with vertices $(0, 0)$, $(2, 0)$, $\left(2, \frac{1}{2}\right)$, $\left(0, \frac{1}{2}\right)$.

16.7.6 $0 \leq u \leq 1 \Rightarrow 0 \leq -x \leq 1 \Rightarrow 0 \geq x \geq -1$, and $0 \leq v \leq 1 \Rightarrow 0 \leq -y \leq 1 \Rightarrow 0 \geq y \geq -1$, so we get a unit square in the third quadrant with one vertex at the origin.

16.7.7 Solving for u and v gives $u = x + y$ and $v = x - y$. The region is thus the square bounded by the lines $x + y = 0$, $x + y = 1$, $x - y = 0$, and $x - y = 1$.

16.7.8 Solving for u and v gives $u = \frac{y}{2}$ and $v = x - y$. The region is thus the parallelogram bounded by the lines $y = 0$, $y = 2$, $x - y = 0$, and $x - y = 1$.

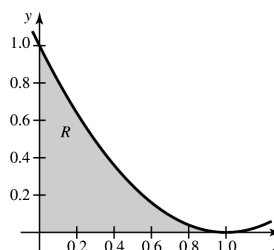
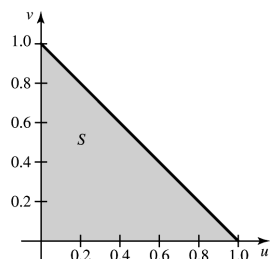
16.7.9 From $(0, 0)$ to $(1, 0)$, $x = u^2$ and $y = 0$, so this traces out the segment from $(0, 0)$ to $(1, 0)$. Similarly, from $(0, 1)$ to $(0, 0)$, $x = -v^2$ and $y = 0$, so this traces out the segment from $(-1, 0)$ to $(0, 0)$. From $(1, 0)$ to $(1, 1)$, $(x, y) = (1 - v^2, 2v)$, so that (x, y) satisfies the equation $y^2 = 4 - 4x$. From $(1, 1)$ to $(0, 1)$, $(x, y) = (u^2 - 1, 2u)$, so that (x, y) satisfies the equation $y^2 = 4 + 4x$. So the result is the region enclosed by the x -axis and the parabolas $y^2 = 4 - 4x$ and $y^2 = 4 + 4x$.

16.7.10 This is the result of interchanging x and y in the previous exercise, so the result is the region enclosed by the y -axis and the parabolas $x^2 = 4 - 4y$ and $x^2 = 4 + 4y$.

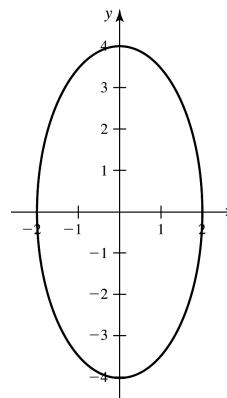
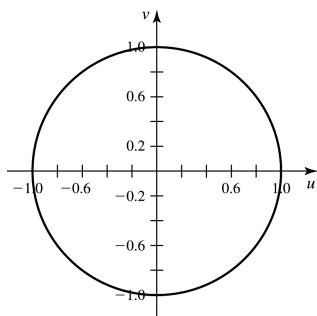
16.7.11 As (u, v) goes from $(0, 0)$ to $(1, 0)$, (x, y) goes from $(0, 0)$ to $(1, 0)$ along the x -axis. As (u, v) goes from $(1, 0)$ to $(1, 1)$, (x, y) traces out the upper half of the unit circle. As (u, v) goes from $(1, 1)$ to $(0, 1)$, (x, y) goes from $(-1, 0)$ to $(0, 0)$ along the x -axis. Finally, as (u, v) goes from $(0, 1)$ to $(0, 0)$, (x, y) is stationary at the origin. Thus the region swept out is the upper half of the unit circle.

16.7.12 As (u, v) goes from $(0, 0)$ to $(1, 0)$, (x, y) is stationary at the origin. As (u, v) goes from $(1, 0)$ to $(1, 1)$, (x, y) traces out the segment from $(0, 0)$ to $(0, -1)$. As (u, v) goes from $(1, 1)$ to $(0, 1)$, (x, y) traces out the right half of the unit circle. Finally, as (u, v) goes from $(0, 1)$ to $(0, 0)$, (x, y) traces out the segment from $(1, 0)$ to $(0, 0)$. The result is the right half of the unit circle.

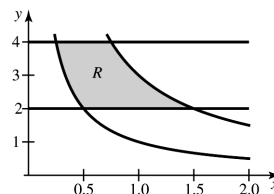
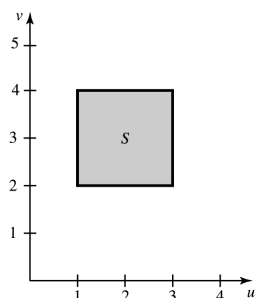
16.7.13 $R = \{(x, y) : \sqrt{y} \leq 1 - x, x \geq 0, y \geq 0\} = \{(x, y) : y \leq (1 - x)^2, x \geq 0, y \geq 0\}.$



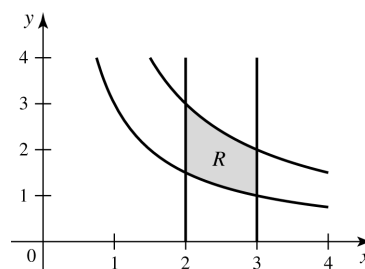
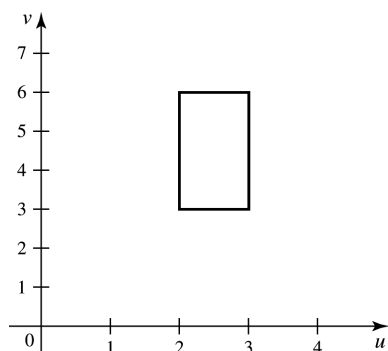
16.7.14 $R = \left\{ (x, y) : \left(\frac{x}{2}\right)^2 + \left(\frac{y}{4}\right)^2 \leq 1 \right\} = \{(x, y) : 4x^2 + y^2 \leq 16\}.$



16.7.15 $R = \{(x, y) : 1 \leq xy \leq 3, 2 \leq y \leq 4\} = \left\{ (x, y) : 2 \leq y \leq 4, \frac{1}{y} \leq x \leq \frac{3}{y} \right\}.$



$$16.7.16 \quad R = \{(x, y) : 2 \leq x \leq 3, 3 \leq xy \leq 6\} = \left\{ (x, y) : 2 \leq x \leq 3, \frac{3}{x} \leq y \leq \frac{6}{x} \right\}.$$



$$16.7.17 \quad J = \begin{vmatrix} 3 & 0 \\ 0 & -3 \end{vmatrix} = -9.$$

$$16.7.18 \quad J = \begin{vmatrix} 0 & 4 \\ -2 & 0 \end{vmatrix} = 8.$$

$$16.7.19 \quad J = \begin{vmatrix} 2v & 2u \\ 2u & -2v \end{vmatrix} = -4(u^2 + v^2).$$

$$16.7.20 \quad J = \begin{vmatrix} \cos(\pi v) & -u\pi \sin(\pi v) \\ \sin(\pi v) & u\pi \cos(\pi v) \end{vmatrix} = u\pi.$$

$$16.7.21 \quad J = \begin{vmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{vmatrix} = -1.$$

$$16.7.22 \quad J = \begin{vmatrix} v^{-1} & -uv^{-2} \\ 0 & 1 \end{vmatrix} = \frac{1}{v}.$$

16.7.23 Add the two equations to get $3x = u + v$, so $x = \frac{u+v}{3}$, and $y = u - \frac{u+v}{3} = \frac{2u-v}{3}$. The Jacobian

$$\text{is then } J = \begin{vmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & -\frac{1}{3} \end{vmatrix} = -\frac{1}{3}$$

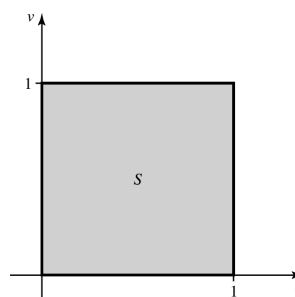
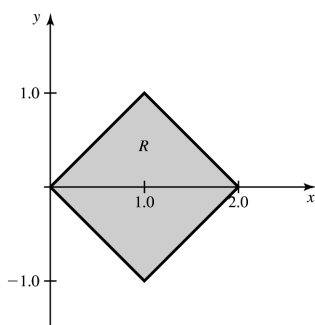
$$16.7.24 \quad x = v, \text{ so } y = \frac{u}{v}. \text{ The Jacobian is then } J = \begin{vmatrix} 0 & 1 \\ v^{-1} & -uv^{-2} \end{vmatrix} = -\frac{1}{v}.$$

16.7.25 Add twice the second equation to the first to get $-y = u + 2v$ so that $y = -u - 2v$, and then $x = -u - 3v$. The Jacobian is $J = \begin{vmatrix} -1 & -3 \\ -1 & -2 \end{vmatrix} = -1$.

16.7.26 Subtract twice the second equation from the first to get $u - 2v = -5x$ so that $x = \frac{2v - u}{5}$, and then $2y = v - 3x = v - 3\frac{2v - u}{5} = \frac{3u - v}{5}$, so that $y = \frac{3u - v}{10}$. The Jacobian is $J = \begin{vmatrix} -\frac{1}{5} & \frac{2}{5} \\ \frac{3}{10} & -\frac{1}{10} \end{vmatrix} = -\frac{1}{10}$.

16.7.27

a.



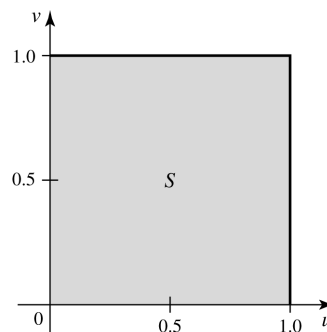
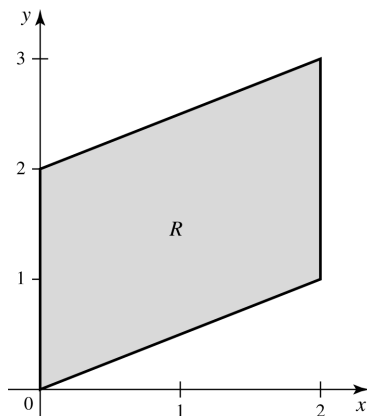
b. The region S is the first-quadrant unit square with one vertex at the origin, so the limits of integration are $0 \leq u, v \leq 1$.

c. The Jacobian of the transformation is $\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2$.

$$\text{d. } \iint_R xy \, dA = \int_0^1 \int_0^1 (u+v) \cdot (u-v) |-2| \, dv \, du = 2 \int_0^1 \int_0^1 (u^2 - v^2) \, dv \, du = 2 \int_0^1 \left(u^2 - \frac{1}{3} \right) du = 0.$$

16.7.28

a. $S = \{(u, v) : 0 \leq 2u \leq 2, 2u \leq 4v + 2u \leq 2u + 4\} = \{(u, v) : 0 \leq u \leq 1, u \leq 2v + u \leq u + 2\} = \{(u, v) : 0 \leq u \leq 1, 0 \leq v \leq 1\}$.



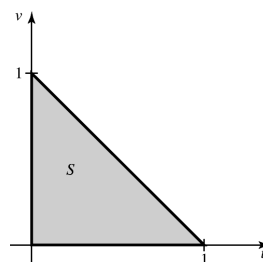
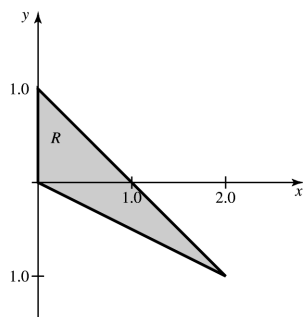
b. From the above computation, the new integration limits are $0 \leq u, v \leq 1$.

c. The Jacobian is $\begin{vmatrix} 2 & 0 \\ 2 & 4 \end{vmatrix} = 8$.

$$\text{d. } \iint_R x^2 y \, dA = 64 \int_0^1 \int_0^1 u^2 (2v + u) \, dv \, du = 64 \int_0^1 (u^2 + u^3) \, du = \frac{112}{3}.$$

16.7.29

a. $S = \{(u, v) : 0 \leq 2u \leq 2, -u \leq v - u \leq 1 - 2u\} = \{(u, v) : 0 \leq u \leq 1, 0 \leq v \leq 1 - u\}$



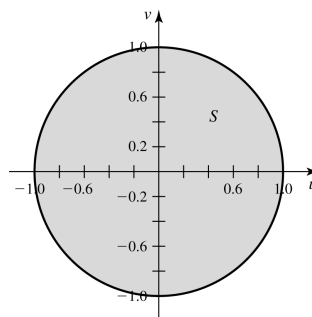
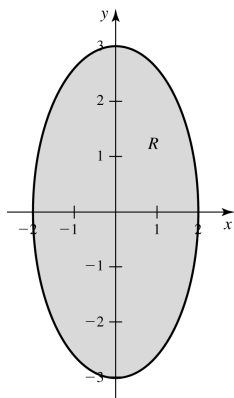
b. From the above, the new limits of integration are $0 \leq u \leq 1, 0 \leq v \leq 1 - u$.

c. The Jacobian is $\begin{vmatrix} 2 & 0 \\ -1 & 1 \end{vmatrix} = 2$.

$$\text{d. } \iint_R x^2 \sqrt{x + 2y} \, dA = 8 \int_0^1 \int_0^{1-u} u^2 \sqrt{2v} \, dv \, du = \frac{16\sqrt{2}}{3} \int_0^1 u^2 (1 - u)^{3/2} \, du = \frac{256\sqrt{2}}{945}.$$

16.7.30

a. Under the given transformation, $9x^2 + 4y^2 = 36$ becomes $9(2u)^2 + 4(3v)^2 = 36u^2 + 36v^2 = 36$, which is the unit circle $u^2 + v^2 = 1$.



b. The limits of integration over the unit circle are $-1 \leq u \leq 1, -\sqrt{1 - u^2} \leq v \leq \sqrt{1 - u^2}$.

c. The Jacobian is $\begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix} = 6$.

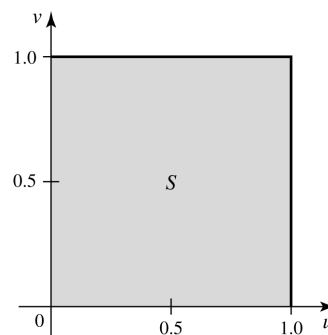
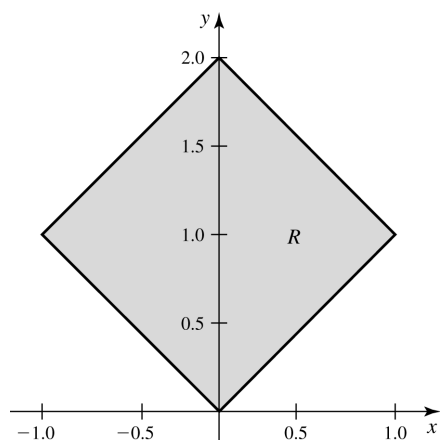
$$\text{d. } \iint_R xy \, dA = 36 \int_{-1}^1 \int_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}} uv \, dv \, du = 0.$$

16.7.31 Use the substitution $y = v$, $x = u + v$. Then $S = \{(u, v) : 0 \leq v \leq 1, v \leq u + v \leq v + 2\} = \{(u, v) : 0 \leq v \leq 1, 0 \leq u \leq 2\}$.

The Jacobian of this transformation is $J(u, v) = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1$ so that

$$\int_0^1 \int_y^{y+2} \sqrt{x-y} \, dx \, dy = \int_0^1 \int_0^2 \sqrt{u} \, du \, dv = \frac{2}{3} 2^{3/2} = \frac{4\sqrt{2}}{3}.$$

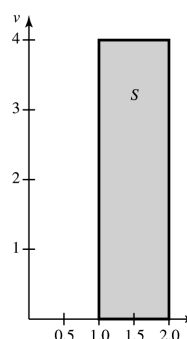
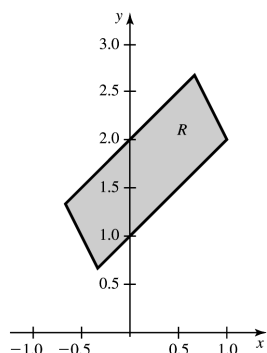
16.7.32 Use the substitution $x = u - v$, $y = u + v$. Then S becomes the unit square in the first quadrant with one vertex at the origin, as can be seen by tracing each edge.



The Jacobian of this transformation is $J(u, v) = \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = 2$ so that

$$\iint_R \sqrt{y^2 - x^2} \, dA = 2 \int_0^1 \int_0^1 \sqrt{(u+v)^2 - (u-v)^2} \, dv \, du = 2 \int_0^1 \int_0^1 \sqrt{4uv} \, dv \, du = \frac{16}{9}.$$

16.7.33 The points of intersection of the given lines are $\left(-\frac{1}{3}, \frac{2}{3}\right)$, $\left(-\frac{2}{3}, \frac{4}{3}\right)$, $\left(\frac{2}{3}, \frac{8}{3}\right)$, and $(1, 2)$. Setting $u = y - x$, $v = y + 2x$ sends these points into the rectangle with vertices $(1, 0)$, $(2, 0)$, $(2, 4)$, and $(1, 4)$.



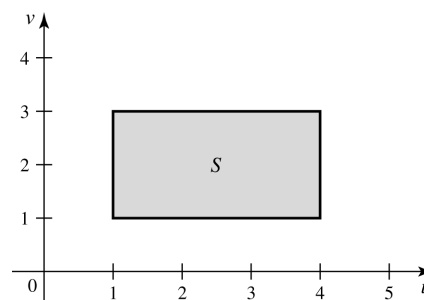
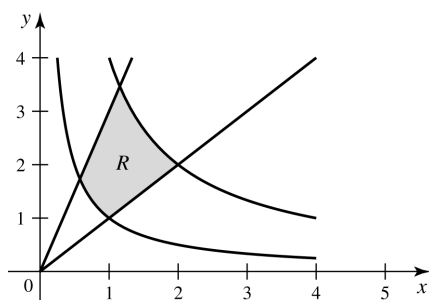
So use the transformation (solving for x, y) $x = \frac{v-u}{3}$, $y = \frac{v+2u}{3}$, $J(u, v) = \begin{vmatrix} -\frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} \end{vmatrix} = -\frac{1}{3}$, so that

$$\iint_R \left(\frac{y-x}{y+2x+1} \right)^4 dA = \frac{1}{3} \int_1^2 \int_0^4 \left(\frac{u}{v+1} \right)^4 dv du = \frac{1}{3} \int_1^2 \frac{124}{375} u^4 du = \frac{3844}{5625}.$$

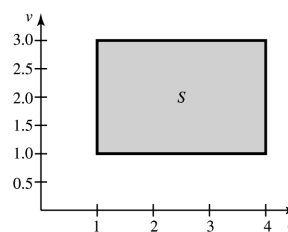
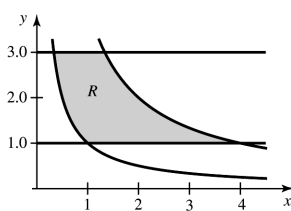
16.7.34 Use the transformation $u = xy$, $v = \frac{y}{x}$, so that $x = \sqrt{\frac{u}{v}}$, $y = \sqrt{uv}$. The Jacobian is

$$\left| \frac{u^{-1/2}v^{-1/2}}{\frac{u^{-1/2}v^{1/2}}{2}} - \frac{u^{-1/2}v^{-3/2}}{\frac{u^{1/2}v^{-1/2}}{2}} \right| = \frac{1}{2v}, \text{ and the new region is the square with vertices } (1, 1), (1, 3), (4, 1),$$

and $(4, 3)$, so that $\iint_R e^{xy} dA = \int_1^4 \int_1^3 e^u \frac{1}{2v} dv du = \frac{\ln 3}{2} \int_1^4 e^u du = \frac{\ln 3}{2} (e^4 - e).$

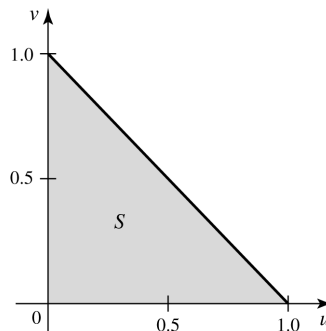
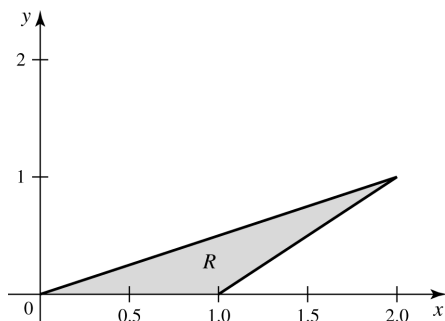


16.7.35 Use the transformation $u = xy$, $v = y$, so that $x = \frac{u}{v}$, $y = v$. The new region is the square with vertices $(1, 1)$, $(4, 1)$, $(4, 3)$, and $(1, 3)$. The Jacobian of the transformation is $\begin{vmatrix} \frac{1}{v} & -\frac{u}{v^2} \\ 0 & 1 \end{vmatrix} = \frac{1}{v}.$



$$\text{Thus } \iint_R xy dA = \int_1^4 \int_1^3 \frac{u}{v} dv du = \int_1^4 u \ln 3 du = \frac{15 \ln 3}{2}.$$

16.7.36 R is the interior of the triangle with vertices $(0, 0)$, $(1, 0)$, and $(2, 1)$. Looking at the form of the integrand, try $u = x - 2y$, $v = y$ so that $x = u + 2v$, $y = v$. Then S is the triangle bounded by $(0, 0)$, $(1, 0)$, and $(0, 1)$.



The Jacobian of this transformation is $\begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} = 1$, and the integral is equal to $\iint_R (x-y) \sqrt{x-2y} dA =$

$$\int_0^1 \int_0^{1-u} (u+v) \sqrt{u} dv du = \frac{4}{21}.$$

$$\mathbf{16.7.37} \quad J(u, v, w) = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} = 2.$$

$$\mathbf{16.7.38} \quad J(u, v, w) = \begin{vmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{vmatrix} = -4.$$

$$\mathbf{16.7.39} \quad J(u, v, w) = \begin{vmatrix} 0 & w & v \\ w & 0 & u \\ 2u & -2v & 0 \end{vmatrix} = 2w(u^2 - v^2).$$

16.7.40 Solving for x , y , and z gives $x = \frac{u+v+w}{2}$, $y = \frac{-u+v+w}{2}$, $z = \frac{u-v+w}{2}$, so that

$$J(u, v, w) = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{vmatrix} = \frac{1}{2}.$$

16.7.41 Let $u = y - x$, $v = z - y$, $w = z$; then $x = w - v - u$, $y = w - v$, $z = w$ and the Jacobian is

$$J(u, v, w) = \begin{vmatrix} -1 & -1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 1. \quad \text{The new region is clearly } 0 \leq u \leq 2, 0 \leq v \leq 1, 0 \leq w \leq 3, \text{ so we obtain}$$

$$\iiint_D xy dV = \int_0^2 \int_0^1 \int_0^3 (w-v-u)(w-v) dw dv du = 5.$$

16.7.42 Let $u = y - 2x$, $v = z - 3y$, $w = z - 4x$. Then $x = -\frac{3}{2}u - \frac{1}{2}v + \frac{1}{2}w$, $y = -2u - v + w$,

$z = -6u - 2v + 3w$, so the Jacobian is $J(u, v, w) = \begin{vmatrix} -\frac{3}{2} & -\frac{1}{2} & \frac{1}{2} \\ -2 & -1 & 1 \\ -6 & -2 & 3 \end{vmatrix} = \frac{1}{2}$, and the new region of integration

is $0 \leq u \leq 1, 0 \leq v \leq 1, 0 \leq w \leq 3$. Thus $\iiint_D dV = \int_0^3 \int_0^1 \int_0^1 \frac{1}{2} du dv dw = \frac{3}{2}$.

16.7.43 Using the given change of variables, the Jacobian is $\begin{vmatrix} 4 \cos v & -4u \sin v & 0 \\ 2 \sin v & 2u \cos v & 0 \\ 0 & 0 & 1 \end{vmatrix} = 8u$, and the new range of integration is $0 \leq u \leq 1$, $0 \leq v \leq 2\pi$, $0 \leq w \leq 16 - 16u^2$. Thus

$$\iiint_D z \, dV = \int_0^1 \int_0^{2\pi} \int_0^{16-16u^2} 8uw \, dw \, dv \, du = \int_0^1 2048\pi u (u^2 - 1)^2 \, du = \frac{1024\pi}{3}.$$

(Note that this change of variables is essentially cylindrical coordinates, with an adjustment for the fact that we are integrating over a paraboloid with differently sized axes.)

16.7.44 Using the given change of variables, the Jacobian is $\begin{vmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 6$. In uvw -coordinates, the equation

becomes $u^2 + v^2 + w^2 = 1$, so the integral is half the volume of the unit sphere, or $\frac{2\pi}{3}$. Multiplying by the Jacobian gives 4π .

16.7.45

- True. This is because $g(u, v)$ and $h(u, v)$ are of the form $au + bv$ so their partial derivatives are constants.
- True. This is because the transformation maps lines to lines.
- True. It simply halves lengths and reflects in the x -axis.

16.7.46 Expand the determinant about the third row. $J(r, \theta, Z) = \begin{vmatrix} \cos \theta & -r \sin \theta & 0 \\ \sin \theta & r \cos \theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r \cos^2 \theta + r \sin^2 \theta = r$.

16.7.47 Expand the determinant about the third row. $J(\rho, \varphi, \theta) = \begin{vmatrix} \sin \varphi \cos \theta & \rho \cos \varphi \cos \theta & -\rho \sin \varphi \sin \theta \\ \sin \varphi \sin \theta & \rho \cos \varphi \sin \theta & \rho \sin \varphi \cos \theta \\ \cos \varphi & -\rho \sin \varphi & 0 \end{vmatrix}$
 $= \cos \varphi (\rho^2 \sin \varphi \cos \varphi \cos^2 \theta + \rho^2 \sin \varphi \cos \varphi \sin^2 \theta) + \rho \sin \varphi (\rho \sin^2 \varphi \cos^2 \theta + \rho \sin^2 \varphi \sin^2 \theta) = \rho^2 \sin \varphi \cos^2 \varphi + \rho^2 \sin^3 \varphi = \rho^2 \sin \varphi$.

16.7.48 Under T , the ellipse becomes $u^2 + v^2 = 1$, the unit circle, with area π . The Jacobian of the transformation is ab , so the ellipse area is πab .

16.7.49 This integral is four times the integral over the first quadrant. This is $4ab \int_0^1 \int_0^{\sqrt{1-u^2}} abuv \, dv \, du = 2a^2b^2 \int_0^1 u(1-u^2) \, du = \frac{a^2b^2}{2}$.

16.7.50 The area of R is given by Exercise 48 ($\frac{\pi ab}{2}$), and $\bar{x} = 0$ by symmetry. To find $\bar{y}$, we compute using the same change of variables (where R^+ is the upper half) $\frac{1}{(\pi ab)/2} \int_{-1}^1 \int_0^{\sqrt{1-u^2}} ab^2v \, dv \, du = \frac{b}{\pi} \int_{-1}^1 (1-u^2) \, du = \frac{4b}{3\pi}$.

16.7.51 The distance of a point from the origin is $\sqrt{x^2 + y^2} = \sqrt{a^2u^2 + b^2v^2}$, so the average squared distance is $\frac{1}{\pi ab} \int_{-1}^1 \int_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}} ab(a^2u^2 + b^2v^2) dv du = \frac{a^2 + b^2}{4}$.

16.7.52 The distance of a point in the upper half of R and the x -axis is simply the point's y -coordinate, so the average distance is $\frac{1}{(\pi ab)/2} \int_{-1}^1 \int_0^{\sqrt{1-u^2}} ab^2v dv du = \frac{4b}{3\pi}$.

16.7.53 Under the given transformation, the equation becomes $u^2 + v^2 + w^2 = 1$, the unit sphere. The Jacobian of the transformation is abc , so the volume of D is abc times the volume of the unit sphere, or $\frac{4}{3}\pi abc$.

16.7.54 This integral is eight times the integral over the first octant, so it is

$$8abc \int_0^1 \int_0^{\sqrt{1-u^2}} \int_0^{\sqrt{1-u^2-v^2}} abcuvw dw dv du = \frac{a^2b^2c^2}{6}.$$

16.7.55 The mass of the upper half is $\frac{2\pi abc}{3}$ by Problem 53, and $\bar{x} = \bar{y} = 0$ by symmetry.

$$\bar{z} = \frac{1}{(2\pi abc)/3} \int_{-1}^1 \int_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}} \int_0^{\sqrt{1-u^2-v^2}} abc^2w dw dv du = \frac{3c}{2\pi} \int_{-1}^1 \int_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}} (1-u^2-v^2) dv du = \frac{3c}{8}.$$

16.7.56 The distance of a point from the origin is $\sqrt{x^2 + y^2 + z^2} = \sqrt{a^2u^2 + b^2v^2 + c^2w^2}$, so the average squared distance is $\frac{3}{4\pi abc} \int_{-1}^1 \int_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}} \int_{-\sqrt{1-u^2-v^2}}^{\sqrt{1-u^2-v^2}} (a^2u^2 + b^2v^2 + c^2w^2) abc dw dv du$

To evaluate this integral, switch to cylindrical coordinates; we obtain

$$\begin{aligned} & \frac{3}{2\pi} \int_0^1 \int_0^{2\pi} \int_0^{\sqrt{1-r^2}} r(a^2r^2 \cos^2 \theta + b^2r^2 \sin^2 \theta + c^2z^2) dz d\theta dr = \\ & \frac{1}{2\pi} \int_0^1 \int_0^{2\pi} r\sqrt{1-r^2} (c^2 - c^2r^2 + 3b^2r^2 + 3a^2r^2 \cos^2 \theta - 3b^2r^2 \cos^2 \theta) d\theta dr = \frac{a^2 + b^2 + c^2}{5}. \end{aligned}$$

16.7.57

a. The line $u = a$ is the set $\{(a, v)\}$, which maps under T to $\{(a^2 - v^2, 2av)\}$. But points of this form satisfy the equation $a^2 - x = \frac{1}{4a^2}y^2$, or $x = -\frac{1}{4a^2}y^2 + a^2$, which is a parabola opening in the negative x direction. The vertex of the parabola $x = Ay^2 + By + C$ is at $\left(C - \frac{B^2}{4A}, -\frac{B}{2A}\right)$, which for this parabola is $(a^2, 0)$, which lies on the positive x -axis.

b. The line $v = b$ is the set $\{(u, b)\}$, which maps under T to $\{(u^2 - b^2, 2ub)\}$. But points of this form satisfy the equation $b^2 + x = \frac{1}{4b^2}y^2$, or $x = \frac{1}{4b^2}y^2 - b^2$, which is a parabola opening in the positive x direction. The vertex of the parabola $x = Ay^2 + By + C$ is at $\left(C - \frac{B^2}{4A}, -\frac{B}{2A}\right)$, which for this parabola is $(-b^2, 0)$, which lies on the negative x -axis.

c. $J(u, v) = \begin{vmatrix} 2u & -2v \\ 2v & 2u \end{vmatrix} = 4(u^2 + v^2).$

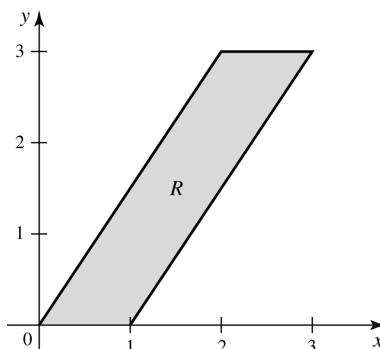
d. Use the transformation $x = v^2 - u^2$, $y = 2uv$. The Jacobian of this transformation is $\begin{vmatrix} -2u & 2v \\ 2v & 2u \end{vmatrix} = -4(u^2 + v^2)$. $x = 4 - \frac{1}{16}y^2$ corresponds to the lines $u = \pm 2$ (by an analysis similar to part (a)), while

$x = \frac{1}{4}y^2 - 1$ corresponds to the line $v = \pm 1$, so that the rectangle with vertices $(-2, -1)$, $(-2, 1)$, $(2, 1)$, $(2, -1)$ is mapped to the region bounded by the parabolas. However, note that the area of that rectangle to the left of the v -axis and the area to the right of the v -axis are each mapped onto that region (that is, the map is 2 : 1; it is a simple computation to see that (a, b) and $(-a, -b)$ map to the same xy point). Thus to determine the area of the original region, we want to integrate over only (say) the right half of the rectangle. So the area is $\int_0^2 \int_{-1}^1 4(u^2 + v^2) dv du = \frac{80}{3}$.

- e. As in part (d), use the transformation $x = v^2 - u^2$, $y = 2uv$, with Jacobian $-4(u^2 + v^2)$. The correspondences are: $x = \frac{y^2}{4} - 1 \Leftrightarrow u = 1x = \frac{y^2}{64} - 16 \Leftrightarrow u = 4x = 9 - \frac{y^2}{36} \Leftrightarrow v = \pm 3x = 4 - \frac{y^2}{16} \Leftrightarrow v = \pm 2$. Because we are looking at the positive portion of the bounded piece, the new range of integration is $1 \leq u \leq 4$, $2 \leq v \leq 3$. Thus the area is $\int_1^4 \int_2^3 4(u^2 + v^2) dv du = 160$.
- f. This simply reverses the roles of x and y in parts (a) and (b). Thus lines $u = a$ in the uv plane map to parabolas in the xy plane that open in the negative y direction with vertices on the positive y -axis, while lines $v = b$ in the uv plane map to parabolas in the xy plane that open in the positive y direction with vertices on the positive y -axis.

16.7.58

- a. Here is the effect of the shear transformation with $a = 1$, $b = 2$, $c = 3$ ($x = u + 2v$; $y = 3v$):



The y coordinates are all multiplied by c , while the x coordinates are gotten by expanding by a and then adding b times the y coordinate. This has the effect of pushing the square further and further to the right as the y coordinate increases.

- b. $J(u, v) = \begin{vmatrix} a & b \\ 0 & c \end{vmatrix} = ac$
- c. R is a parallelogram with vertices $(0, 0)$, $(a, 0)$, $(a + b, c)$, and (b, c) , so it has base a and height c , so has area ac (which is exactly what we would expect because the Jacobian is ac).
- d. By symmetry, $\bar{y} = \frac{c}{2}$ and $\bar{x} = \frac{a + b}{2}$.
- e. The analogous shear transformation is $x = au$, $y = bu + cv$.

16.7.59

- a. The z coordinate remains the same, while the other coordinates are stretched by an amount depending on all three coordinates. The result is thus a parallelepiped with its base in the xy -plane.

$$\text{b. } J(u, v, w) = \begin{vmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & 1 \end{vmatrix} = ad$$

c. D has a height of 1 Because S and the z coordinate remains unchanged. Its base is in the xy plane, and is the parallelogram that is the result of the shear transformation $x = au + by$, $y = dv$ (set $w = 0$ in the original equations to see this). The area of this parallelogram, from Problem 58, is ad , so the total volume is ad (again what we would expect from the Jacobian).

d. By symmetry, the center of mass is $\left(\frac{a+b+c}{2}, \frac{d+e}{2}, \frac{1}{2}\right)$.

16.7.60

$$\text{a. } J(u, v) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

b. R is the parallelogram with vertices $(0, 0)$, (a, c) , (b, d) , and $(a+b, c+d)$. Compute its area by thinking of (a, c) and (b, d) as vectors in 3-space with zero z coordinate; then the area of the parallelogram is the cross-product $|\langle a, c, 0 \rangle \times \langle b, d, 0 \rangle| = |\langle 0, 0, ad - bc \rangle| = |ad - bc| = |J(u, v)|$

c. Let $P = (q, r)$ and let ℓ be the line given by $(q + et, r + ft)$; $0 \leq t \leq 1$ so that $Q = (q + e, r + f)$. Then $T(P) = (aq + br, cq + dr)$, while $T(Q) = (aq + ae + br + bf, cq + ce + dr + df)$. Then $T((q + et, r + ft)) = (aq + br, (ae + bf)t, cq + dr + (ce + df)t) = (aq + br, cq + dr) + (ae + bf, ce + df)t = T(P) + (ae + bf, ce + df)t$. Now, when $t = 1$, this is $T(Q)$, so that ℓ maps to a line from $T(P)$ to $T(Q)$.

d. By moving and reflecting the x and y axes if necessary, we may assume that the vertices of the parallelogram are $(0, 0)$, $(r, 0)$, (s, t) , and $(r + s, t)$ for $r, s, t > 0$. The area of this parallelogram is then rt (base times height). Then the vertices of S are $(0, 0)$, (ar, cr) , $(as + bt, cs + dt)$, and $(ar + as + bt, cr + cs + dt)$. Again regarding (ar, cr) and $(as + bt, cs + dt)$ as vectors in 3-space, the area of the parallelogram is $|\langle ar, cr, 0 \rangle \times \langle as + bt, cs + dt, 0 \rangle| = |\langle 0, 0, ar(cs + dt) - cr(as + bt) \rangle| = |\langle 0, 0, (ad - bc)rt \rangle| = |J(u, v)|rt = |J(u, v)|\text{ area}(S)$.

16.7.61

a. This is just the definition of the transformation T , which is given by $x = g(u, v)$, $y = h(u, v)$, so that the image of (x, y) is $T((x, y)) = (g(u, v), h(u, v))$. Apply this to the coordinates of the points O , P , Q .

b. The Taylor expansions of g and h at $(0, 0)$ are

$$g(u, v) = g(0, 0) + u g_u(0, 0) + v g_v(0, 0) + \text{terms involving higher derivatives}$$

$$h(u, v) = h(0, 0) + u h_u(0, 0) + v h_v(0, 0) + \text{terms involving higher derivatives}.$$

Substituting the two points $(\Delta u, 0)$ and $(0, \Delta v)$ into these equations and considering only the terms up through the first derivative gives the desired result.

c. From part (b),

$$P' \approx (g(0, 0) + g_u(0, 0)\Delta u, h(0, 0) + h_u(0, 0)\Delta u)$$

$$Q' \approx (g(0, 0) + g_v(0, 0)\Delta v, h(0, 0) + h_v(0, 0)\Delta v),$$

so that $\overrightarrow{O'P'} \approx \Delta u \langle g_u(0, 0), h_u(0, 0) \rangle$ and $\overrightarrow{O'Q'} \approx \Delta v \langle g_v(0, 0), h_v(0, 0) \rangle$. The area of the parallelogram determined by $\overrightarrow{O'P'}$ and $\overrightarrow{O'Q'}$ is the magnitude of the cross product of these vectors (considered as vectors in 3-space with zero z coordinate), so the area of the resulting region is approximately

$$\begin{aligned}
& \left| \Delta u \langle g_u(0, 0), h_u(0, 0), 0 \rangle \times \Delta v \langle g_v(0, 0), h_v(0, 0), 0 \rangle \right| \\
&= \Delta u \Delta v \left| \langle g_u(0, 0), h_u(0, 0), 0 \rangle \times \langle g_v(0, 0), h_v(0, 0), 0 \rangle \right| \\
&= \Delta u \Delta v |g_u(0, 0) h_v(0, 0) - g_v(0, 0) h_u(0, 0)| \\
&= |J(u, v)| \Delta u \Delta v.
\end{aligned}$$

d. Because the area of R is $\Delta u \Delta v$, the ratio is approximately $|J(u, v)|$.

16.7.62

- Let $n_1 = \langle at, bt, 0 \rangle$, $n_2 = \langle ct, 0, dt \rangle$, $n_3 = \langle 0, et, ft \rangle$ for $t \in \mathbb{R}$. Then n_1 is normal to the first pair, n_2 to the second pair, and n_3 to the third pair.
- The triple scalar product of any three vectors is the volume of the parallelepiped that they determine. This volume is zero if and only if they are coplanar.
- The triple scalar product is $\langle at, bt, 0 \rangle \cdot (\langle ct, 0, dt \rangle \times \langle 0, et, ft \rangle) = \langle at, bt, 0 \rangle \cdot t^2 \langle -de, cf, ce \rangle = t^3(-ade - bcf)$, so the normal vectors are coplanar if and only if $ade + bcf = 0$.
- If the three vectors are coplanar, then the cross product of any two of them is perpendicular to the plane they determine. Thus, for example, from part (c), $N = t(de, cf, -ce)$ is normal to the plane. So any line in the direction of N is parallel to all six planes. By choosing the line appropriately, we can ensure that it does not lie in any of the six planes, and thus it does not intersect any of them. Thus the six planes do not form a bounded region if $ade + bcf = 0$.

e. The Jacobian is $\begin{vmatrix} a & b & 0 \\ c & 0 & d \\ 0 & e & f \end{vmatrix} = -ade - bcf$. The Jacobian is zero if R is unbounded.

Chapter Sixteen Review

1

a. False. For example, if $g(x, y) = 2$, then $\int_c^d \int_a^b 2 \, dx \, dy = 2(b-a)(d-c)$ while

$$\left(\int_c^d 2 \, dy \right) \left(\int_a^b 2 \, dx \right) = 4(b-a)(d-c).$$

- True. The first set is the set whose φ coordinate is $\frac{\pi}{2}$; φ is the angle the line to the point makes with the z -axis, so this is the set of points in the xy -plane.
- False. The integrand doesn't change.
- False. For example, it maps the standard unit square into the square with vertices $(0, 0)$, $(0, -1)$, $(1, -1)$, $(1, 0)$.

2

$$\begin{aligned}
\int_1^2 \int_1^4 \frac{xy}{(x^2 + y^2)^2} \, dx \, dy &= \frac{1}{2} \int_1^2 \int_1^4 \frac{(2x)y}{(x^2 + y^2)^2} \, dx \, dy = -\frac{1}{2} \int_1^2 \frac{y}{x^2 + y^2} \Big|_1^4 \, dy \\
&= -\frac{1}{2} \int_1^2 \left(\frac{y}{y^2 + 16} - \frac{y}{y^2 + 1} \right) \, dy = \frac{1}{4} \ln 17 - \frac{3}{4} \ln 2.
\end{aligned}$$

$$3 \int_1^3 \int_1^{e^x} \frac{x}{y} dy dx = \int_1^3 x \ln y \Big|_1^{e^x} dx = \int_1^3 x^2 dx = \frac{26}{3}.$$

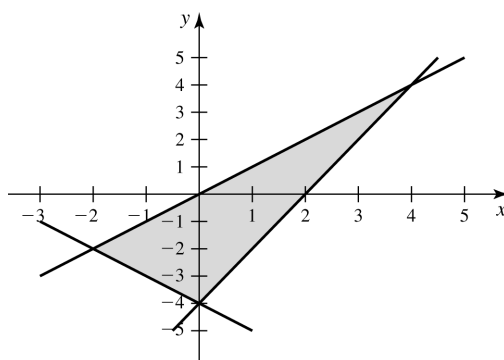
$$4 \int_1^2 \int_0^{\ln x} x^3 e^y dy dx = \int_1^2 x^3 e^y \Big|_0^{\ln x} dx = \int_1^2 (x^4 - x^3) dx = \frac{49}{20}.$$

$$5 \int_0^1 \int_{-\sqrt{y}}^{\sqrt{y}} f(x, y) dx dy.$$

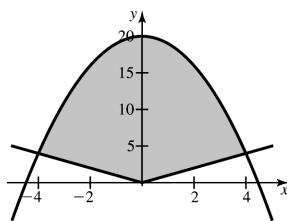
$$6 \int_{-1}^1 \int_0^{x+1} f(x, y) dy dx.$$

$$7 \int_0^1 \int_0^{\sqrt{1-x^2}} f(x, y) dy dx.$$

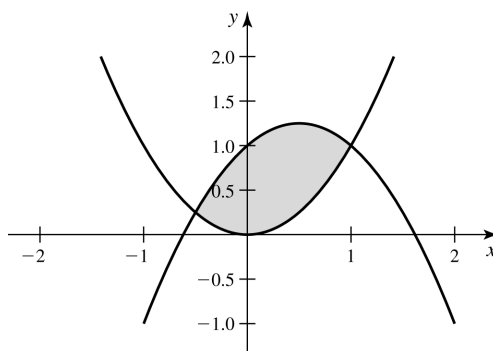
$$8 \int_{-2}^0 \int_{-x-4}^x 1 dy dx + \int_0^4 \int_{2x-4}^x 1 dy dx = \int_{-2}^0 (2x+4) dx + \int_0^4 (4-x) dx = 12.$$



$$9 \ 2 \int_0^4 \int_x^{20-x^2} 1 dy dx = 2 \int_0^4 (20 - x^2 - x) dx = \frac{304}{3}.$$



$$10 \int_{-1/2}^1 \int_{x^2}^{1+x-x^2} 1 dy dx = \int_{-1/2}^1 (1 + x - 2x^2) dx = \frac{9}{8}.$$



$$11 \int_1^2 \int_0^{x^{3/2}} \frac{2y}{\sqrt{x^4+1}} dy dx = \int_1^2 \frac{x^3}{\sqrt{x^4+1}} dx = \frac{\sqrt{17}-\sqrt{2}}{2}.$$

$$12 \int_1^4 \int_0^{\sqrt{x}} x^{-1/2} e^y dy dx = \int_1^4 x^{-1/2} (e^{x^{1/2}} - 1) dx = 2e^2 - 2e - 2.$$

$$13 \int_0^\pi \int_0^{4 \sin \theta} r^2 (\cos \theta + \sin \theta) dr d\theta = \frac{64}{3} \int_0^\pi \sin^3 \theta (\cos \theta + \sin \theta) d\theta = 8\pi.$$

$$14 \int_0^2 \int_0^x (x^2 + y^2) dy dx = \int_0^2 \frac{4}{3} x^3 dx = \frac{16}{3}.$$

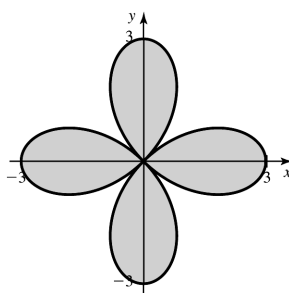
$$15 \int_0^1 \int_{y^{1/3}}^1 x^{10} \cos(\pi x^4 y) dx dy = \int_0^1 \int_0^x x^{10} \cos(\pi x^4 y) dy dx = \int_0^1 \frac{x^{10}}{\pi x^4} \sin(\pi x^7) dx = \frac{1}{\pi} \int_0^1 x^6 \sin(\pi x^7) dx = \frac{2}{7\pi^2}.$$

$$16 \int_0^2 \int_{y^2}^4 x^8 y \sqrt{1+x^4 y^2} dx dy = \int_0^4 \int_0^{\sqrt{x}} x^8 y \sqrt{1+x^4 y^2} dy dx = \frac{1}{3} \int_0^4 x^4 (-1 + (1+x^5)^{3/2}) dx = \frac{420250}{3} \sqrt{41} - \frac{5122}{25}.$$

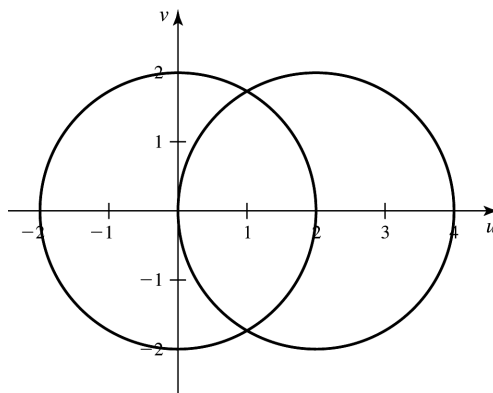
$$17 \int_0^1 \int_0^{\pi/2} 3(r \cos \theta)^2 r \sin \theta r d\theta dr = \int_0^1 \int_0^{\pi/2} 3r^4 \cos^2 \theta \sin \theta d\theta dr = \int_0^1 r^4 dr = \frac{1}{5}.$$

$$18 \int_1^4 \int_0^\pi \frac{1}{(1+r^2)^2} r d\theta dr = \pi \int_1^4 \frac{r}{(1+r^2)^2} dr = \frac{15}{68} \pi.$$

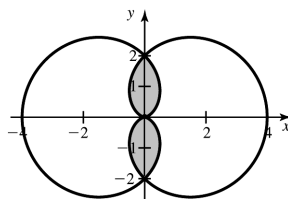
$$19 \text{ The area is four times the area of one leaf, so is } 4 \int_{-\pi/4}^{\pi/4} \int_0^{\cos 2\theta} r dr d\theta = 18 \int_{-\pi/4}^{\pi/4} \cos^2 2\theta d\theta = \frac{9\pi}{2}.$$



20 The area is twice the area of the region above the x -axis. The two circles intersect above the x axis at $\cos \theta = \frac{1}{2}$, or $\theta = \frac{\pi}{3}$. Thus the area of the region is $2 \left(\int_0^{\pi/3} \int_0^2 r dr d\theta + \int_{\pi/3}^{\pi/2} \int_0^{\cos \theta} r dr d\theta \right) = 2 \left(\int_0^{\pi/3} 2 d\theta + \int_{\pi/3}^{\pi/2} 8 \cos^2 \theta d\theta \right) = \frac{4\pi}{3} + \frac{4\pi}{3} - 2\sqrt{3} = \frac{8\pi}{3} - 2\sqrt{3}.$



- 21** The area is four times the area bounded by the cardioid $2 - 2\cos\theta$ and the y -axis for positive x (i.e. for $0 \leq \theta \leq \frac{\pi}{2}$). Because all four portions of the area are congruent. Thus it is $4 \int_0^{\pi/2} \int_0^{2-2\cos\theta} r \, dr \, d\theta = 8 \int_0^{\pi/2} (1 - \cos\theta)^2 \, d\theta = 6\pi - 16$.



- 22** The area of the disk is $\pi \cdot 4^2 = 16\pi$. Using polar coordinates, the average value is then $\frac{1}{16\pi} \int_0^4 \int_0^{2\pi} r \sqrt{16 - r^2} \, d\theta \, dr = \frac{1}{8} \int_0^4 r \sqrt{16 - r^2} \, dr = \frac{8}{3}$.

- 23** The volume of the cone is $\frac{1}{3}$ times the area of the base times the height. The base (at $z = 8$) is a circle of radius 4, so the volume is $\frac{1}{3}\pi \cdot 4^2 \cdot 8 = \frac{128\pi}{3}$. Use cylindrical coordinates to integrate; then the distance to the z -axis is r and the average is $\frac{1}{(128\pi)/3} \int_0^8 \int_0^{z/2} \int_0^{2\pi} r^2 \, d\theta \, dr \, dz = \frac{1}{64} \int_0^8 \frac{z^3}{8} \, dz = 2$.

24 $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2}} f(x, y, z) \, dz \, dy \, dx.$

25 $\int_0^1 \int_{2y}^2 \int_0^{\sqrt{z^2-4y^2}/2} f(x, y, z) \, dx \, dz \, dy.$

26 $\int_0^2 \int_y^2 \int_0^{9-x^2} f(x, y, z) \, dz \, dx \, dy.$

27 $\int_0^1 \int_{-z}^z \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dy \, dx \, dz = 2 \int_0^1 \int_{-z}^z \sqrt{1-x^2} \, dx \, dz = 2 \int_0^1 \left(z\sqrt{1-z^2} + \arcsin(z) \right) dz = \pi - \frac{4}{3}.$

28 $\int_0^\pi \int_0^y \int_0^{\sin x} dz \, dx \, dy = \int_0^\pi \int_0^y \sin x \, dx \, dy = \int_0^\pi (1 - \cos y) \, dy = \pi.$

29 The region in the xy -plane can be restated as $0 \leq x \leq 2$, $0 \leq y \leq \frac{x}{2}$. Reordering the integral gives

$$\int_1^9 \int_0^2 \int_0^{x/2} \frac{4 \sin(x^2)}{\sqrt{z}} dy dx dz = 2 \int_1^9 \int_0^2 \frac{x \sin(x^2)}{\sqrt{z}} dx dz = (1 - \cos 4) \int_1^9 z^{-1/2} dz = 4 - 4 \cos 4.$$

30 This is the integral of a function over the right half of the ellipse $\frac{x^2}{2} + y^2 = 2$. Change the order of integration of x and y so that $-\sqrt{2} \leq y \leq \sqrt{2}$, $0 \leq x \leq \sqrt{4 - 2y^2}$. We have

$$\int_{-\sqrt{2}}^{\sqrt{2}} \int_0^{\sqrt{4-2y^2}} \int_{x^2+3y^2}^{8-x^2-y^2} dz dx dy = \int_{-\sqrt{2}}^{\sqrt{2}} \int_0^{\sqrt{4-2y^2}} (8 - 2x^2 - 4y^2) dx dy = \int_{-\sqrt{2}}^{\sqrt{2}} \left((8 - 4y^2) \sqrt{4 - 2y^2} - \frac{2}{3} (4 - 2y^2)^{3/2} \right) dy = 4\pi\sqrt{2}.$$

31 Reorder to integrate with respect to x last. Because $0 \leq y \leq 2$, we have $0 \leq x \leq 4$. The integral is

$$\begin{aligned} \int_0^4 \int_{\sqrt{x}}^2 \int_0^{y^{1/3}} yz^5 (1 + x + y^2 + z^6)^2 dz dy dx &= \frac{1}{18} \int_0^4 \int_{\sqrt{x}}^2 (7y^7 + 9y^5 + 9xy^5 + 3y^3 + 6xy^3 + 3x^2y^3) dy dx \\ &= \frac{1}{18} \int_0^4 \left(332 - \frac{25}{8}x^4 - 3x^3 + \frac{45}{4}x^2 + 120x \right) dx = \frac{848}{9}. \end{aligned}$$

$$\begin{aligned} \textbf{32} \quad \int_1^2 \int_0^1 (12 - x^2 - 2y^2) dy dx &= \int_1^2 \left(12y - x^2y - \frac{2}{3}y^3 \right) \Big|_0^1 dx = \int_1^2 \left(\frac{34}{3} - x^2 \right) dx = \\ \left(\frac{34}{3}x - \frac{x^3}{3} \right) \Big|_1^2 &= 9. \end{aligned}$$

$$\begin{aligned} \textbf{33} \quad \text{This is given by } \int_0^1 \int_0^{1-x^2} 2(1 - y - x^2) dy dx &= \int_0^1 (2y - y^2 - 2x^2y) \Big|_0^{1-x^2} dx = \int_0^1 (2 - 2x^2 - (1 - x^2)^2 - \\ 2x^2(1 - x^2)) dx &= \int_0^1 (1 - 2x^2 + x^4) dx = \left(x - \frac{2x^3}{3} + \frac{x^5}{5} \right) \Big|_0^1 = \frac{8}{15}. \end{aligned}$$

$$\textbf{34} \quad \int_0^1 \int_0^{3-3x} \int_0^2 1 dz dy dx = \int_0^1 (6 - 6x) dx = 3.$$

$$\textbf{35} \quad \int_0^2 \int_0^\pi \int_0^{\sin \theta} r dz d\theta dr = \int_0^2 \int_0^\pi r^2 \sin \theta d\theta dr = \int_0^2 2r^2 dr = \frac{16}{3}.$$

36 Note that $2 - x^2 - (1 + y^2) = 1 - (x^2 + y^2)$. Note also that the projection of the intersection of the surfaces to the plane $z = 0$ is the unit circle. We use cylindrical coordinates to obtain

$$\begin{aligned} \int_0^{2\pi} \int_0^1 (1 - r^2) r dr d\theta &= \int_0^{2\pi} \int_0^1 (r - r^3) dr d\theta = 2\pi \left(\frac{r^2}{2} - \frac{r^4}{4} \right) \Big|_0^1 = \frac{\pi}{2}. \end{aligned}$$

37 Look at the intersection of the cylinders from the positive z -axis. The vertical sides of the region lie on the cylinder $x^2 + y^2 = 4$, and the top and bottom lie on $x^2 + z^2 = 4$. Thus the region is $\{(x, y, z) : x^2 + y^2 \leq 4, -\sqrt{4 - x^2} \leq z \leq \sqrt{4 - x^2}\}$. Thus the volume is

$$\begin{aligned} \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} 1 dz dy dx &= \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} 2\sqrt{4 - x^2} dy dx = \int_{-2}^2 4(4 - x^2) dx = \frac{128}{3}. \end{aligned}$$

$$\textbf{38} \quad \int_0^1 \int_0^x \int_0^y 1 dz dy dx = \int_0^1 \int_0^x y dy dx = \int_0^1 \frac{x^2}{2} dx = \frac{1}{6}.$$

39 Rewrite the integral as $\int_0^{1/2} \int_{\sin^{-1} x}^{\sin^{-1} 2x} 1 \, dy \, dx$ and then change the order of integration. This results in the integral breaking up into two integrals, and we obtain

$$\int_0^{\pi/6} \int_{(\sin y)/2}^{\sin y} 1 \, dx \, dy + \int_{\pi/6}^{\pi/2} \int_{(\sin y)/2}^{1/2} 1 \, dx \, dy = \int_0^{\pi/6} \frac{\sin y}{2} \, dy + \int_{\pi/6}^{\pi/2} \left(\frac{1}{2} - \frac{\sin y}{2} \right) \, dy = \frac{1 - \sqrt{3}}{2} + \frac{\pi}{6}.$$

40 The plane of the slanted surface of the tetrahedron is $6x + 3y + 2z = 6$, so the possibilities are

$$\begin{aligned} \int_0^1 \int_0^{2-2x} \int_0^{(6-6x-3y)/2} dz \, dy \, dx &= \int_0^1 \int_0^{3-3x} \int_0^{(6-6x-2z)/3} dy \, dz \, dx \\ &= \int_0^2 \int_0^{(2-y)/2} \int_0^{(6-6x-3y)/2} dz \, dx \, dy \\ &= \int_0^2 \int_0^{2-2x} \int_0^{(6-3y-2z)/6} dx \, dz \, dy \\ &= \int_0^3 \int_0^{(3-z)/3} \int_0^{(6-6x-2z)/3} dy \, dx \, dz \\ &= \int_0^3 \int_0^{2(3-z)/3} \int_0^{(6-3y-2z)/6} dx \, dy \, dz. \end{aligned}$$

41

$$\begin{aligned} \int_0^{\pi/4} \int_0^{\sec \theta} r^3 \, dr \, d\theta &= \int_0^1 \int_0^x (x^2 + y^2) \, dy \, dx = \int_0^1 \left(x^2 y + \frac{1}{3} y^3 \right) \Big|_0^x \, dx \\ &= \int_0^1 \left(\frac{4}{3} x^3 \right) \, dx = \left(\frac{1}{3} x^4 \right) \Big|_0^1 = \frac{1}{3}. \end{aligned}$$

42 Use cylindrical coordinates. The volume of the paraboloid is $\int_0^2 \int_0^{2\pi} \int_0^{4-r^2} r \, dz \, d\theta \, dr =$

$2\pi \int_0^2 (4r - r^3) \, dr = 8\pi$. The distance of a point from the origin is $\sqrt{r^2 + z^2}$, so the average squared distance is $\frac{1}{8\pi} \int_0^2 \int_0^{2\pi} \int_0^{4-r^2} r(r^2 + z^2) \, dz \, d\theta \, dr = \frac{1}{4} \int_0^2 \left(\frac{1}{3} r(4-r^2)^3 + r^3(4-r^2) \right) \, dr = 4$.

43 The volume of the prism is $\int_0^1 \int_0^{3-3x} \int_0^2 1 \, dz \, dy \, dx = \int_0^1 (6-6x) \, dx = 3$. Thus the average x coordinate is $\frac{1}{3} \int_0^1 \int_0^{3-3x} \int_0^2 x \, dz \, dy \, dx = \frac{1}{3} \int_0^1 2x(3-3x) \, dx = \frac{1}{3}$.

44 This is a quarter of a cylinder of radius 3 and length 3 oriented along the z -axis, so $\int_0^3 \int_0^{\pi/2} \int_0^3 r^3 \cdot r \, dz \, d\theta \, dr = \frac{3\pi}{2} \int_0^3 r^4 \, dr = \frac{729\pi}{10}$.

45 Use cylindrical coordinates. We have $\int_{-2}^2 \int_{-\pi/2}^{\pi/2} \int_0^1 \frac{1}{(1+r^2)^2} r \, dr \, d\theta \, dz$. This can be written as $(2 - (-2)) \cdot \left(\frac{\pi}{2} - \left(-\frac{\pi}{2} \right) \right) \int_0^1 \frac{r}{(1+r^2)^2} \, dr = 4\pi \int_1^2 \frac{1}{2} \left(\frac{1}{u^2} \right) \, du = 2\pi \left(-\frac{1}{u} \right) \Big|_1^2 = \pi$.

46 $\sqrt{9 - x^2 - y^2} = \sqrt{1 + x^2 + y^2}$ for $x^2 + y^2 = 4$. We have

$$\begin{aligned} \int_0^{2\pi} \int_0^2 \int_{\sqrt{1+r^2}}^{\sqrt{9-r^2}} dz r dr d\theta &= 2\pi \int_0^2 \left(r\sqrt{9-r^2} - r\sqrt{1+r^2} \right) dr \\ &= -\frac{2\pi}{3} \left((9-r^2)^{3/2} + (1+r^2)^{3/2} \right) \Big|_0^2 \\ &= \frac{56\pi}{3} - \frac{20\sqrt{5}\pi}{3}. \end{aligned}$$

47 $\int_0^4 \int_0^\pi \int_0^{2\cos\theta} r dr d\theta dz = 2 \int_0^4 \int_0^\pi \cos^2\theta d\theta dz = 4\pi.$

48 $\int_0^{2\pi} \int_0^{\pi/2} \int_0^{2\cos\varphi} \rho^2 \sin\varphi d\rho d\varphi d\theta = \frac{8}{3} \int_0^{2\pi} \int_0^{\pi/2} \cos^3\varphi \sin\varphi d\varphi d\theta = \frac{4\pi}{3}.$

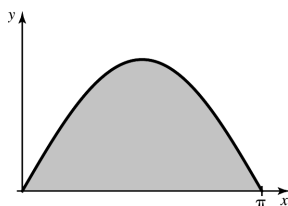
49 $\int_0^\pi \int_0^{\pi/4} \int_{2\sec\varphi}^{4\sec\varphi} \rho^2 \sin\varphi d\rho d\varphi d\theta = \frac{56}{3} \int_0^\pi \int_0^{\pi/4} \sec^3\varphi \sin\varphi d\varphi d\theta = \frac{28\pi}{3}.$

50 $\int_0^\pi \int_0^{(1-\cos\varphi)/2} \int_0^{2\pi} \rho^2 \sin\varphi d\theta d\rho d\varphi = \frac{\pi}{12} \int_0^\pi (1-\cos\varphi)^3 \sin\varphi d\varphi = \frac{\pi}{3}.$

51 $\int_0^{\pi/2} \int_0^{4\sin 2\varphi} \int_0^{2\pi} \rho^2 \sin\varphi d\theta d\rho d\varphi = \frac{128\pi}{3} \int_0^{\pi/2} \sin^3 2\varphi \sin\varphi d\varphi = \frac{2048\pi}{105}.$

52 $\int_0^{\pi/4} \int_0^{4\cos\varphi} \int_0^{2\pi} \rho^2 \sin\varphi d\theta d\rho d\varphi = \frac{128\pi}{3} \int_0^{\pi/4} \cos^3\varphi \sin\varphi d\varphi = 8\pi.$

53

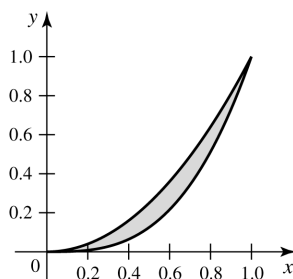


The mass of the plate is $\int_0^\pi \sin x dx = -\cos \pi + \cos 0 = 2$. By symmetry, $\bar{x} = \frac{\pi}{2}$,

$$\bar{y} = \frac{1}{2} \int_0^\pi \int_0^{\sin x} y dy dx = \frac{1}{4} \int_0^\pi \sin^2 x dx = \frac{\pi}{8}.$$

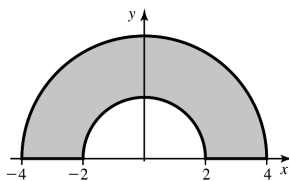
The center of mass is $\left(\frac{\pi}{2}, \frac{\pi}{8}\right)$.

54



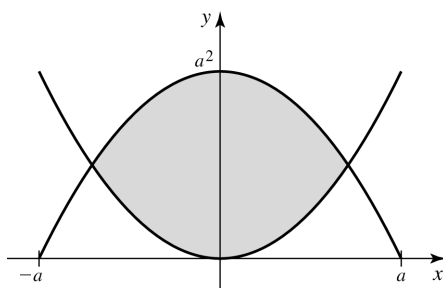
The mass of the plate is $\int_0^1 \int_{x^3}^{x^2} 1 \, dy \, dx = \int_0^1 (x^2 - x^3) \, dx = \frac{1}{12}$, so that $\bar{x} = \frac{1}{1/12} \int_0^1 \int_{x^3}^{x^2} x \, dy \, dx = 12 \int_0^1 (x^3 - x^4) \, dx = \frac{3}{5}$. $\bar{y} = \frac{1}{1/12} \int_0^1 \int_{x^3}^{x^2} y \, dy \, dx = 6 \int_0^1 (x^4 - x^6) \, dx = \frac{12}{35}$. The center of mass is at $\left(\frac{3}{5}, \frac{12}{35}\right)$.

55



By symmetry, $\bar{x} = 0$. The mass of the region is $8\pi - 2\pi = 6\pi$, so $\bar{y} = \frac{1}{6\pi} \int_0^\pi \int_2^4 r^2 \sin \theta \, dr \, d\theta = \frac{28}{9\pi} \int_0^\pi \sin \theta \, d\theta = \frac{56}{9\pi}$. The center of mass is at $\left(0, \frac{56}{9\pi}\right)$.

56



By symmetry, $\bar{x} = 0$. The two parabolas intersect where $x^2 = a^2 - x^2$, so where $x = \pm \frac{a}{\sqrt{2}}$. The mass is then

$$\int_{-a/\sqrt{2}}^{a/\sqrt{2}} \int_{x^2}^{a^2-x^2} 1 \, dy \, dx = \int_{-a/\sqrt{2}}^{a/\sqrt{2}} (a^2 - 2x^2) \, dx = \frac{2}{3} a^3 \sqrt{2}$$

so that $\bar{y} = \frac{1}{2a^3\sqrt{2}/3} \int_{-a/\sqrt{2}}^{a/\sqrt{2}} \int_{x^2}^{a^2-x^2} y \, dy \, dx = \frac{3}{4a^3\sqrt{2}} \int_{-a/\sqrt{2}}^{a/\sqrt{2}} ((a^2 - x^2)^2 - x^4) \, dx = \frac{1}{2} a^2$. The center of mass is at $\left(0, \frac{1}{2} a^2\right)$.

57 By symmetry, $\bar{x} = \bar{y} = 0$. The mass (using cylindrical coordinates) is

$$\int_0^6 \int_{r^2}^{36} \int_0^{2\pi} r \, d\theta \, dz \, dr = 2\pi \int_0^6 r (36 - r^2) \, dr = 648\pi,$$

so $\bar{z} = \frac{1}{648\pi} \int_0^6 \int_{r^2}^{36} \int_0^{2\pi} rz \, d\theta \, dz \, dr = \frac{1}{648} \int_0^6 r (1296 - r^4) \, dr = 24$. The center of mass is $(0, 0, 24)$.

58 The mass is $\frac{1}{3}$ times the area of the base (4) times the height (4), so is $\frac{16}{3}$. The plane intersects the x -axis at $x = 4$, the y -axis at $y = 2$, and the z -axis at $z = 4$, so by symmetry, $\bar{x} = \bar{z}$. Then $\bar{x} = \frac{1}{16/3} \int_0^2 \int_0^{4-2y} \int_0^{4-x-2y} x \, dz \, dx \, dy = \frac{3}{16} \int_0^2 \int_0^{4-2y} (4x - x^2 - 2xy) \, dx \, dy = \frac{1}{32} \int_0^2 (4-2y)^3 \, dy = 1$. $\bar{y} = \frac{1}{16/3} \int_0^2 \int_0^{4-2y} \int_0^{4-x-2y} y \, dz \, dx \, dy = \frac{3}{16} \int_0^2 \int_0^{4-2y} (4y - xy - 2y^2) \, dx \, dy = \frac{3}{16} \int_0^2 (8y - 8y^2 + 2y^3) \, dy = \frac{1}{2}$, so the center of mass is $\left(1, \frac{1}{2}, 1\right)$.

59 Use spherical coordinates. The mass is $\int_0^{\pi/2} \int_0^{16} \int_0^{2\pi} \left(1 + \frac{\rho}{4}\right) \rho^2 \sin \varphi \, d\theta \, d\rho \, d\varphi = 2\pi \int_0^{\pi/2} \int_0^{16} \left(\rho^2 + \frac{\rho^3}{4}\right) \sin \varphi \, d\rho \, d\varphi = \frac{32768\pi}{3} \int_0^{\pi/2} \sin \varphi \, d\varphi = \frac{32768\pi}{3}$. By symmetry of the region and of the density function around the z axis, we have $\bar{x} = \bar{y} = 0$.

$$\begin{aligned} \bar{z} &= \frac{1}{32768\pi/3} \int_0^{\pi/2} \int_0^{16} \int_0^{2\pi} \rho \cos \varphi \left(1 + \frac{\rho}{4}\right) \rho^2 \sin \varphi \, d\theta \, d\rho \, d\varphi \\ &= \frac{3}{32768} \int_0^{\pi/2} \int_0^{16} \left(\rho^3 + \frac{\rho^4}{4}\right) \cos \varphi \sin \varphi \, d\rho \, d\varphi = \frac{3}{32768} \int_0^{\pi/2} \frac{16384 \cdot 21}{5} \cos \varphi \sin \varphi \, d\varphi = \frac{63}{10}, \end{aligned}$$

so the center of mass is $\left(0, 0, \frac{63}{10}\right)$.

60 The mass is $\int_0^2 \int_0^2 \int_0^2 (1+x+y+z) \, dz \, dy \, dx = 32$. By symmetry, $\bar{x} = \bar{y} = \bar{z}$, and $\bar{x} = \frac{1}{32} \int_0^2 \int_0^2 \int_0^2 x(1+x+y+z) \, dz \, dy \, dx = \frac{13}{12}$, so the center of mass is $\left(\frac{13}{12}, \frac{13}{12}, \frac{13}{12}\right)$.

61 Place the vertex at the origin with the paraboloid opening upwards along the positive z -axis. Then the equation of the paraboloid is $z = \frac{h}{R^2} r^2$. Its mass is $\int_0^R \int_0^{2\pi} \int_{hr^2/R^2}^h r \, dz \, d\theta \, dr = 2\pi \int_0^R \left(r - \frac{h}{R^2} r^2\right) dr = \frac{1}{2} \pi h R^2$. Then $\bar{z} = \frac{1}{\pi h R^2/2} \int_0^R \int_0^{2\pi} \int_{hr^2/R^2}^h r z \, dz \, d\theta \, dr = \frac{2}{h R^2} \int_0^R r h^2 \frac{R^4 - r^4}{R^4} dr = \frac{2}{3} h$, so that the center of mass is $\frac{1}{3}$ of the way from the base to the vertex.

62 Place one vertex at the origin and another at $(s, 0)$ so that the third is at $\left(\frac{s}{2}, \frac{s\sqrt{3}}{2}\right)$. Then the equations of the two sides of the triangle to that third vertex are $y = x\sqrt{3}$ and $y = -\sqrt{3}(x-s)$. The area of the triangle is $\frac{s^2\sqrt{3}}{4}$ (one half the base times the height), so $\bar{y} = \frac{1}{s^2\sqrt{3}/4} \int_0^{s/2} \int_0^{\sqrt{3}x} y \, dy \, dx + \int_{s/2}^s \int_0^{-\sqrt{3}(x-s)} y \, dy \, dx = \frac{2}{s^2\sqrt{3}} \left(\int_0^{s/2} 3x^2 \, dx + \int_{s/2}^s 3(x-s)^2 \, dx \right) = \frac{6}{s^2\sqrt{3}} \left(\frac{s^3}{24} + \frac{s^3}{24} \right) = \frac{\sqrt{3}}{6} s$, which is one third the height of the triangle.

63 The area of the sector is $1/8$ the area of the entire circle, so it is $\frac{1}{8} \pi a^2$. We can calculate $M_y = \iint_R x \, dA = \int_0^{\pi/4} \int_0^a r^2 \cos \theta \, dr \, d\theta = \int_0^{\pi/4} \frac{r^3}{3} \Big|_0^a \cos \theta \, d\theta = \frac{a^3}{3} (\sin \theta) \Big|_0^{\pi/4} = \frac{a^3\sqrt{2}}{6}$. Thus $\bar{x} = \frac{M_y}{m} = \frac{4\sqrt{2}a}{3\pi}$. Also, we have $M_y = \iint_R y \, dA = \int_0^{\pi/4} \int_0^a r^2 \sin \theta \, dr \, d\theta = \int_0^{\pi/4} \frac{r^3}{3} \Big|_0^a \sin \theta \, d\theta = \frac{a^3}{3} (-\cos \theta) \Big|_0^{\pi/4} = \frac{a^3(2-\sqrt{2})}{6}$. Thus $\bar{y} = \frac{M_x}{m} = \frac{4a(2-\sqrt{2})}{3\pi}$.

64 By symmetry, we have $\bar{x} = \bar{y} = 0$. To calculate the mass, note that $x^2 + y^2 + z^2 = a^2$ and $x^2 + y^2 = z^2$, so $2z^2 = a^2$, and $z = \frac{a}{\sqrt{2}}$. The mass is given by $\int_0^{2\pi} \int_0^{a/\sqrt{2}} \int_r^{\sqrt{a^2-r^2}} r \, dz \, dr \, d\theta$. This is equal to $2\pi \int_0^{a/\sqrt{2}} \left(\sqrt{a^2-r^2} - r \right) r \, dr = 2\pi \left(-\frac{1}{3} (a^2-r^2)^{3/2} - \frac{r^3}{3} \right) \Big|_r^{a/\sqrt{2}} = \frac{\pi a^3(2-\sqrt{2})}{3}$. A similar calculation shows that $M_{xy} = \int_0^{2\pi} \int_0^{a/\sqrt{2}} \int_r^{\sqrt{a^2-r^2}} z \cdot r \, dz \, dr \, d\theta = \frac{a^4\pi}{8}$. We therefore have $\bar{z} = \frac{M_{xy}}{m} = \frac{3a(2+\sqrt{2})}{16}$.

65 The equation of the cone in cylindrical coordinates is $z = 2 - \frac{1}{2}r$. The volume of the cone is one third the area of the base times the height, or $\frac{32\pi}{3}$.

a. The volume of a slice of $\frac{\pi}{4}$ radians is $\int_0^4 \int_0^{\pi/4} \int_0^{2-r/2} r \, dz \, d\theta \, dr = \frac{\pi}{4} \int_0^4 r \left(2 - \frac{r}{2}\right) dr = \frac{4\pi}{3}$. The volume of the wedge is in fact one eighth the volume of the entire cone ($\frac{\pi}{4}$ radians is an eighth-circle).

b. The volume of a slice of Q radians is $\int_0^4 \int_0^Q \int_0^{2-r/2} r \, dz \, d\theta \, dr = Q \int_0^4 r \left(2 - \frac{r}{2}\right) dr = \frac{16}{3}Q$. Geometrically, Q radians is $\frac{Q}{2\pi}$ of a circle, and indeed the volume of the slice is $\frac{Q}{2\pi}$ times the volume of the cone.

66 Compute the volume of empty space, using spherical coordinates with the center of the tank at the origin. The water level is at $\varphi = \frac{\pi}{3}$ (Pythagorean theorem), so the volume of the empty spherical cap is $\int_0^{\pi/3} \int_{\sec \varphi/2}^1 \int_0^{2\pi} \rho^2 \sin \varphi \, d\theta \, d\rho \, d\varphi = \frac{2\pi}{3} \int_0^{\pi/3} \left(1 - \frac{\sec^3 \varphi}{8}\right) \sin \varphi \, d\varphi = \frac{5\pi}{24}$ cubic feet. The total volume of the sphere is $\frac{4\pi}{3}$ cubic feet.

a. The volume of water is $\frac{4\pi}{3} - \frac{5\pi}{24} = \frac{27\pi}{24} = \frac{9\pi}{8}$ cubic feet and it weighs $1.125 \cdot 62.5 \cdot \pi \approx 220.893$ pounds.

b. The amount of water that must be added to fill the tank is the volume of empty space, or $\frac{5\pi}{24}$ cubic feet.

67 The transformation just switches the coordinates, so the image is again the unit square.

68 $T = \{(x, y) : 0 \leq v \leq 1, 0 \leq u \leq 1\} = \{(x, y) : 0 \leq -x \leq 1, 0 \leq y \leq 1\} = \{(x, y) : -1 \leq x \leq 0, 0 \leq y \leq 1\}$ so that T is the unit square in the second quadrant with one vertex at the origin.

69 As (u, v) goes from $(0, 0)$ to $(1, 0)$, (x, y) goes from $(0, 0)$ to $(3, 1)$; as (u, v) goes from $(1, 0)$ to $(1, 1)$, (x, y) goes from $(3, 1)$ to $(4, 4)$; as (u, v) goes from $(1, 1)$ to $(0, 1)$, (x, y) goes from $(4, 4)$ to $(1, 3)$. Thus the image of S is the parallelogram with vertices $(0, 0)$, $(3, 1)$, $(4, 4)$, and $(1, 3)$.

70 The transformation leaves the x coordinate alone and stretches and translates the y coordinate. So the new region is the rectangle with vertices $(0, 2)$, $(1, 2)$, $(1, 4)$, $(0, 4)$.

71 $J(u, v) = \begin{vmatrix} 4 & -1 \\ -2 & 3 \end{vmatrix} = 10.$

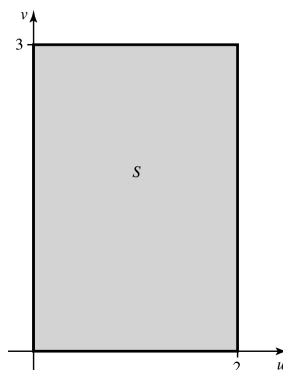
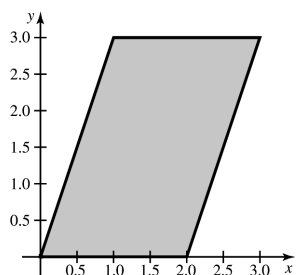
72 $J(u, v) = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2.$

73 $J(u, v) = \begin{vmatrix} 3 & 0 \\ 0 & 2 \end{vmatrix} = 6.$

74 $J(u, v) = \begin{vmatrix} 2u & -2v \\ 2v & 2u \end{vmatrix} = 4(u^2 + v^2).$

75

a.



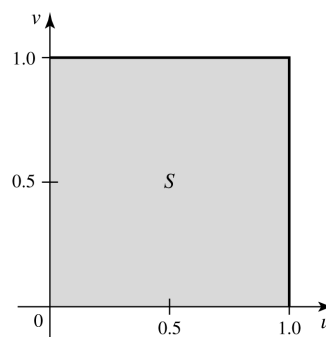
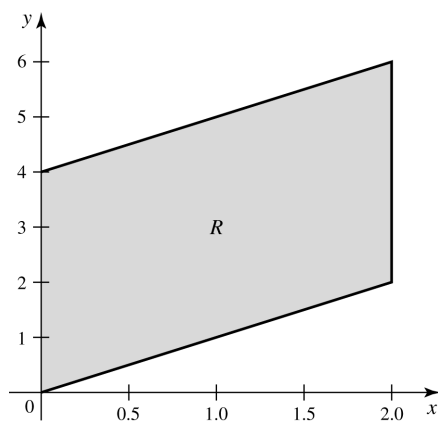
$$\text{b. } S = \left\{ (u, v) : \frac{v}{3} \leq u + \frac{v}{3} \leq \frac{v+6}{3}, 0 \leq v \leq 3 \right\} = \{(u, v) : 0 \leq u \leq 2, 0 \leq v \leq 3\}.$$

$$\text{c. } J(u, v) = \begin{vmatrix} 1 & \frac{1}{3} \\ 0 & 1 \end{vmatrix} = 1$$

$$\text{d. } \iint_R xy^2 dA = \int_0^2 \int_0^3 \left(u + \frac{v}{3}\right) v^2 dv du = \frac{63}{2}.$$

76

a.



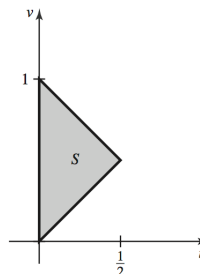
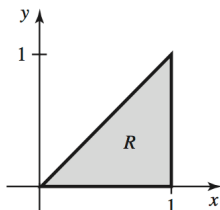
$$\text{b. } S = \{(u, v) : 0 \leq 2u \leq 2, 2u \leq 4v + 2u \leq 2u + 4\} = \{(u, v) : 0 \leq u \leq 1, 0 \leq v \leq 1\}.$$

$$\text{c. } J(u, v) = \begin{vmatrix} 2 & 0 \\ 2 & 4 \end{vmatrix} = 8.$$

$$\text{d. } \iint_R 3xy^2 dA = \int_0^1 \int_0^1 24(2u)(4v + 2u)^2 dv du = 192 \int_0^1 \int_0^1 u(2v + u)^2 dv du = 304.$$

77

a.



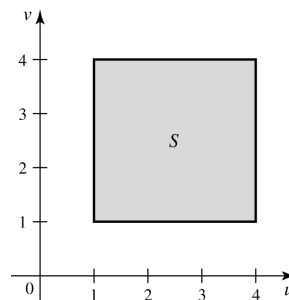
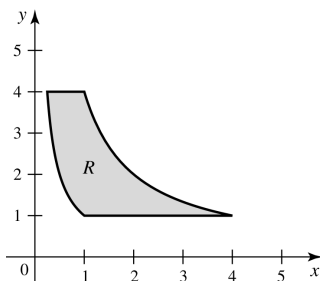
b. $S = \{(u, v) : u \leq v \leq 1 - u, 0 \leq u \leq \frac{1}{2}\}.$

c. $J(u, v) = \begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} = 2.$

d. The given integral is equal to $\int_0^{1/2} \int_u^{1-u} (2u+1)(2u)^9 \cdot 2 \, dv \, du = 1024 \int_0^{1/2} (2u+1)u^9 (1-u-u) \, du = 1024 \int_0^{1/2} (u^9 - 4u^{11}) \, du = 1024 \left(\frac{u^{10}}{10} - \frac{4u^{12}}{12} \right) = \frac{1}{60}.$

78

a.

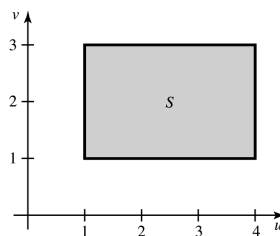
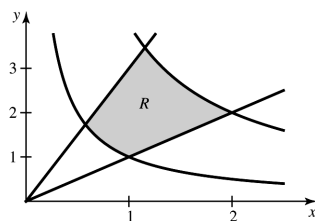


b. $xy = 1$ becomes $u = 1$ while $xy = 4$ becomes $u = 4$; $y = 1$ becomes $v = 1$; $y = 4$ becomes $v = 4$. So $S = \{(u, v) : 1 \leq u \leq 4, 1 \leq v \leq 4\}.$

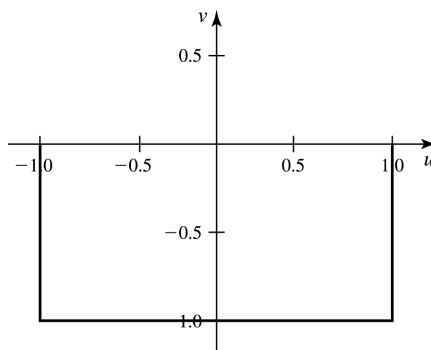
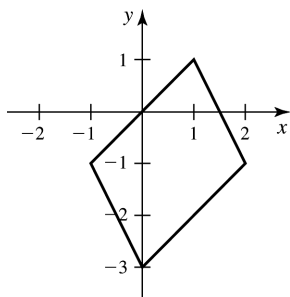
c. $J(u, v) = \begin{vmatrix} v^{-1} & -uv^{-2} \\ 0 & 1 \end{vmatrix} = v^{-1}.$

d. $\iint_R xy^2 \, dA = \int_1^4 \int_1^4 \frac{u}{v} v^2 v^{-1} \, dv \, du = \int_1^4 \int_1^4 u \, dv \, du = \frac{45}{2}.$

79 Use the transformation $u = xy$; $v = \frac{y}{x}$ so that $x = \sqrt{\frac{u}{v}}$, $y = \sqrt{uv}$. Then $xy = u$ and $\frac{y}{x} = v$, so the new region S is $\{(u, v) : 1 \leq u \leq 4, 1 \leq v \leq 3\}.$ The Jacobian of this transformation is $J(u, v) = \begin{vmatrix} \frac{u^{-1/2}v^{-1/2}}{2} & -\frac{u^{1/2}v^{-3/2}}{2} \\ \frac{v^{1/2}u^{-1/2}}{2} & \frac{u^{1/2}v^{-1/2}}{2} \end{vmatrix} = \frac{1}{2v}$ so that $\iint_R y^4 \, dA = \frac{1}{2} \int_1^4 \int_1^3 u^2 v \, dv \, du = 42.$



80 Let $x = u - v$, $y = u + 2v$; then $y = x$ becomes $v = 0$, $y = x - 3$ becomes $v = -1$, $y = -2x + 3$ becomes $u = 1$, and $y = -2x - 3$ becomes $u = -1$. So the new region of integration is $S = \{(u, v) : -1 \leq u \leq 1, -1 \leq v \leq 0\}$. The Jacobian is $J(u, v) = \begin{vmatrix} 1 & -1 \\ 1 & 2 \end{vmatrix} = 3$, so that $\iint_R (y^2 + xy - 2x^2) dA = 3 \int_{-1}^1 \int_{-1}^0 ((u + 2v)^2 + (u - v)(u + 2v) - 2(u - v)^2) dv du = 3 \int_{-1}^1 \int_{-1}^0 9uv dv du = 0$.



81 Use the transformation $u = x + 2y$, $v = x - z$, $w = 2y - z$; solving for x, y, z gives $x = \frac{u + v - w}{2}$, $y = \frac{u - v + w}{4}$, $z = \frac{u - v - w}{2}$.

The new range of integration becomes $1 \leq u \leq 2$, $0 \leq v \leq 2$, $0 \leq w \leq 3$. The Jacobian is $J(u, v, w) = \begin{vmatrix} \frac{1}{2} & \frac{1}{4} & -\frac{1}{2} \\ \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{vmatrix} = \frac{1}{4}$ so that $\iiint_D yz dV = \frac{1}{32} \int_1^2 \int_0^2 \int_0^3 (u - v + w)(u - v - w) dw dv du = -\frac{7}{16}$.

82 Use the transformation $u = y - 2x$, $v = z - 3y$, $w = z - 4x$; solving for x, y, z gives $x = \frac{-3u - v + w}{2}$, $y = -2u - v + w$, $z = -6u - 2v + 3w$. The new range of integration becomes $0 \leq u \leq 1$, $0 \leq v \leq 1$, $0 \leq w \leq 3$.

The Jacobian is $J(u, v, w) = \begin{vmatrix} -\frac{3}{2} & -\frac{1}{2} & \frac{1}{2} \\ -2 & -1 & 1 \\ -6 & -2 & 3 \end{vmatrix} = \frac{1}{2}$ so that

$$\iiint_D x dV = \frac{1}{4} \int_0^1 \int_0^1 \int_0^3 (-3u - v + w) dw dv du = -\frac{3}{8}.$$

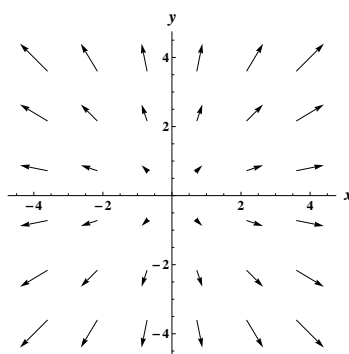
Chapter 17

Vector Calculus

17.1 Vector Fields

17.1.1 A vector field describes the motion of the air as a vector at each point in the room.

17.1.2



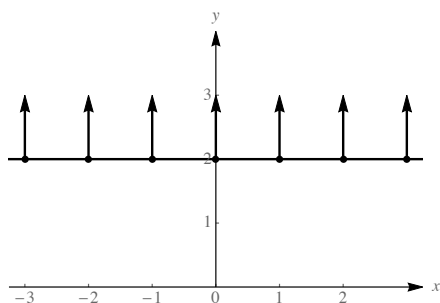
17.1.3 At selected points (a, b) , plot the vector $\langle f(a, b), g(a, b) \rangle$.

17.1.4 The gradient of a function at a point is a vector describing the direction in which the value of the function is increasing most rapidly. The collection of these vectors over all points is a vector field.

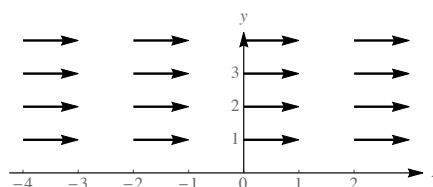
17.1.5 The gradient field gives, at each point, the direction in which the temperature is increasing most rapidly and the amount of increase.

17.1.6 The length of $\mathbf{F}$ is $\left| \frac{\sqrt{2} \langle x, y \rangle}{\sqrt{x^2 + y^2}} \right| = \frac{\sqrt{2}}{\sqrt{x^2 + y^2}} |\langle x, y \rangle| = \frac{\sqrt{2}}{\sqrt{x^2 + y^2}} \sqrt{x^2 + y^2} = \sqrt{2}$.

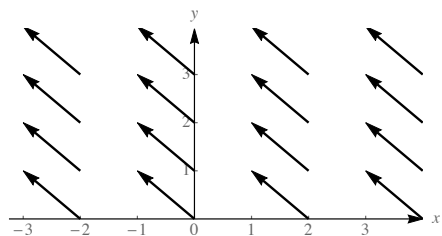
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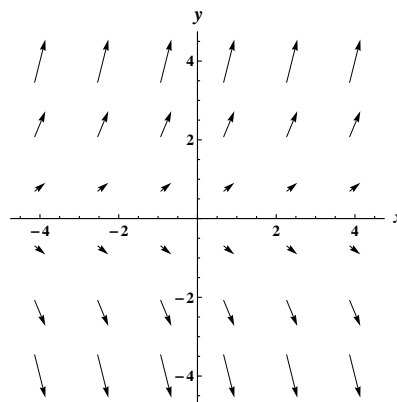
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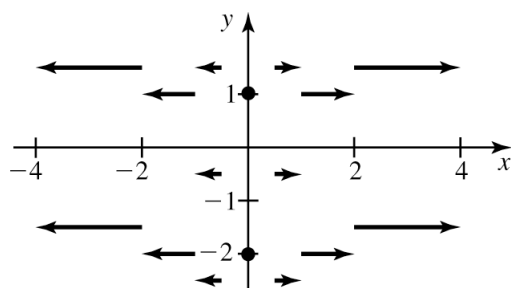
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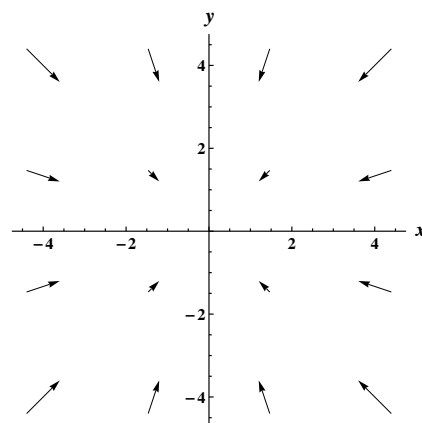
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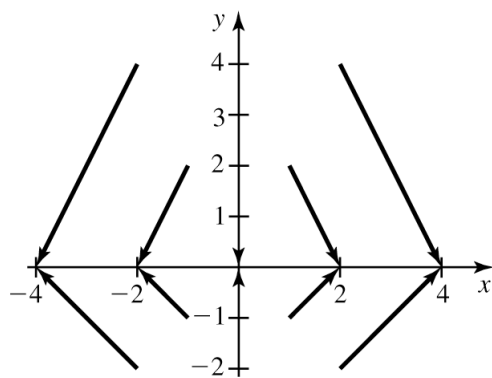
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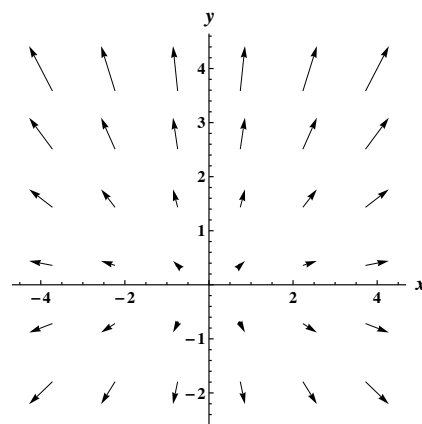
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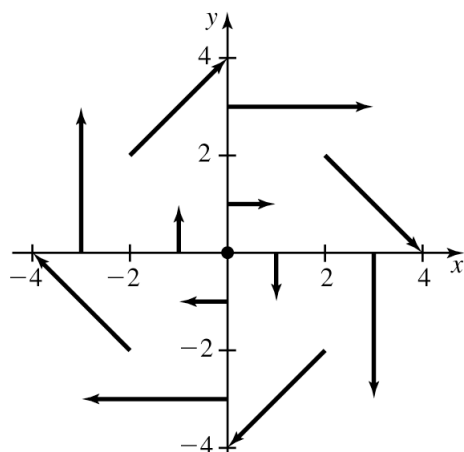
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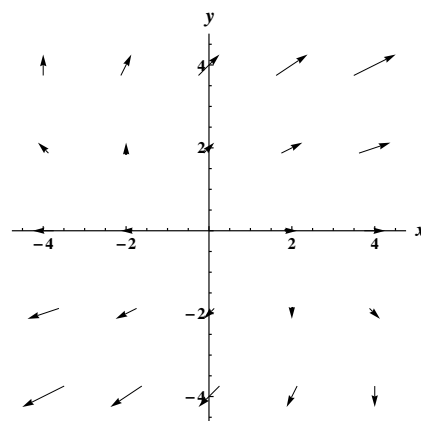
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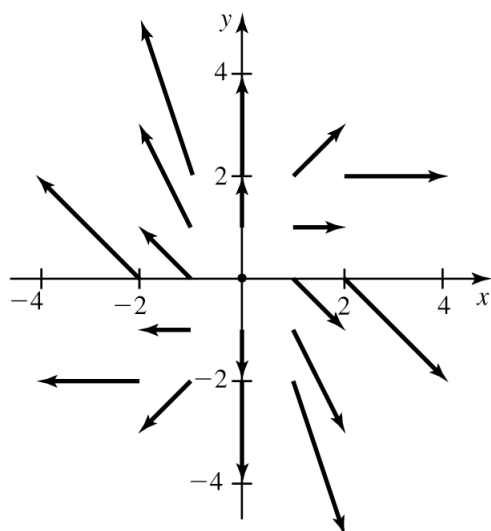
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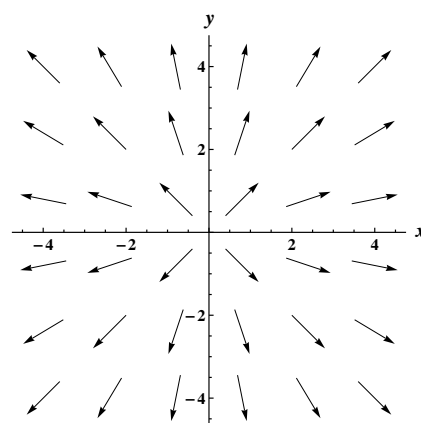
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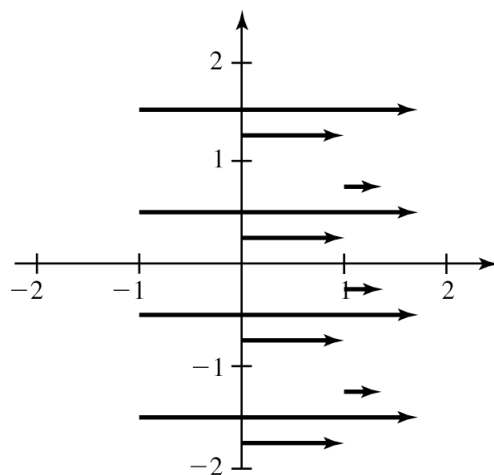
17.1.17



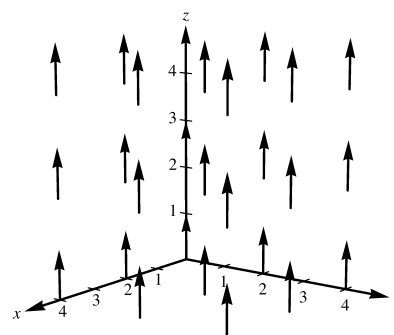
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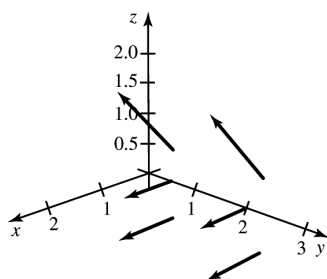
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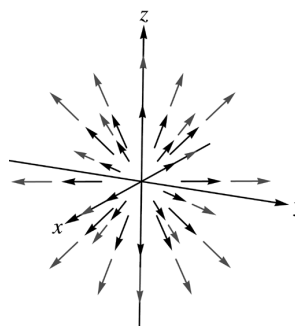
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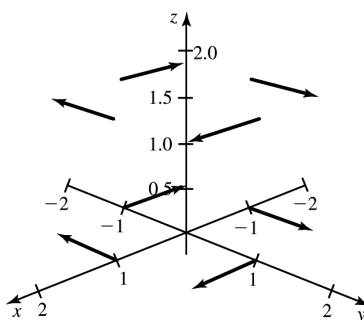
17.1.21



17.1.22



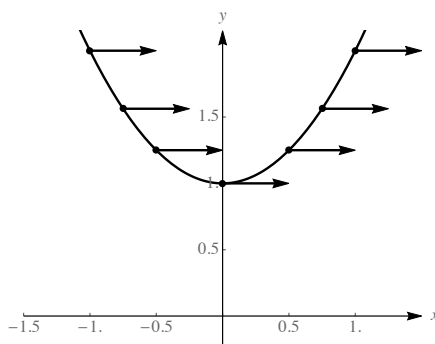
17.1.23



17.1.24 (a) corresponds to (D), since (D) has zero x component, and the y component increases as y does. (b) corresponds to (C), since the x component appears to be zero along the line $y = x$. (c) corresponds to (B) since the y component is zero on the x -axis and the x component is zero on the y axis. Finally, (d) corresponds to (A) since the x component is zero on the x -axis and the y component is zero on the y -axis.

17.1.25

- Let $g(x, y) = y - x^2$. Then $\nabla g = \langle -2x, 1 \rangle$. This is orthogonal to $\mathbf{F}$ when $\nabla g \cdot \mathbf{F} = 0$, or when $\frac{1}{2}(-2x) + 0 = 0$, or $x = 0$. So the point on C is $(0, 1)$.
- ∇g is never parallel to $\mathbf{F}$, because no nonzero multiple of $\langle \frac{1}{2}, 0 \rangle$ can have a second component of 1.
-

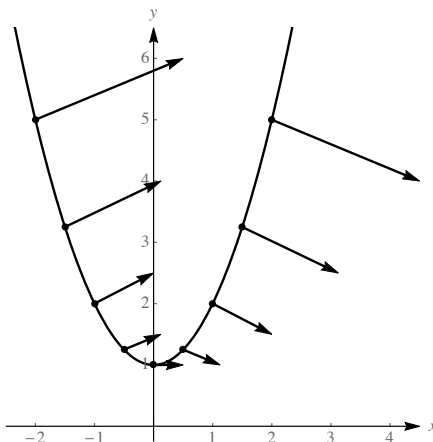


17.1.26

- Let $g(x, y) = y - x^2$. Then $\nabla g = \langle -2x, 1 \rangle$. This is perpendicular to $\mathbf{F}$ when $\nabla g \cdot \mathbf{F} = 0$, or $\frac{y}{2}(-2x) + \frac{-x}{2}(1) = 0$, or $-xy - \frac{x}{2} = 0$, so for $x = 0$ and $y = -\frac{1}{2}$. However, on our curve, $y \geq 1$, so we must have $x = 0$, in which case $y = 1$. So $\mathbf{F}$ is tangent to C at $(0, 1)$.

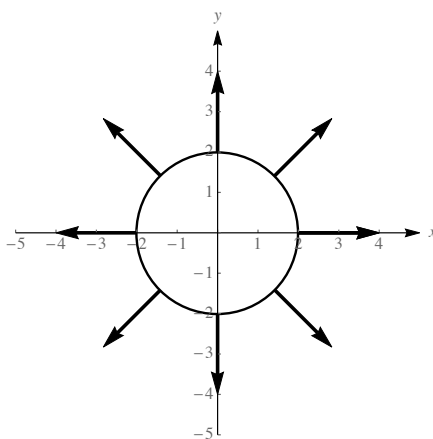
- b. If $\mathbf{F} = k\nabla g$, then $y = -4kx$ and $x = -2k$. Substituting into the equation for C , we have $2x^2 - x^2 = 1$, so $x = \pm 1$ and $y = 2$. So $\mathbf{F}$ is normal to C at the points $(\pm 1, 2)$.

c.



17.1.27

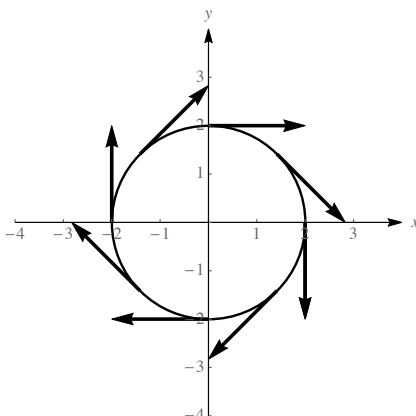
- a. Let $g(x, y) = x^2 + y^2$. Then $\nabla g = \langle 2x, 2y \rangle$. This is perpendicular to $\mathbf{F}$ when $\nabla g \cdot \mathbf{F} = 0$, or $2x^2 + 2y^2 = 0$, which doesn't occur on the circle C .
- b. If $\mathbf{F} = k\nabla g$, then $\langle x, y \rangle = k \langle 2x, 2y \rangle$, which occurs for all points on the circle.
- c.



17.1.28

- a. Let $g(x, y) = x^2 + y^2$. Then $\nabla g = \langle 2x, 2y \rangle$. This is perpendicular to $\mathbf{F}$ when $\nabla g \cdot \mathbf{F} = 0$, or $2xy - 2xy = 0$, which occurs for all points on the circle.
- b. If $\mathbf{F} = k\nabla g$, then $\langle y, -x \rangle = k \langle 2x, 2y \rangle$, or $y = 2kx$ and $x = -2ky$. But then $y = 2kx = 2k(-2ky) = -4k^2y$, which doesn't occur. There are no points where $\mathbf{F}$ is normal to the curve.

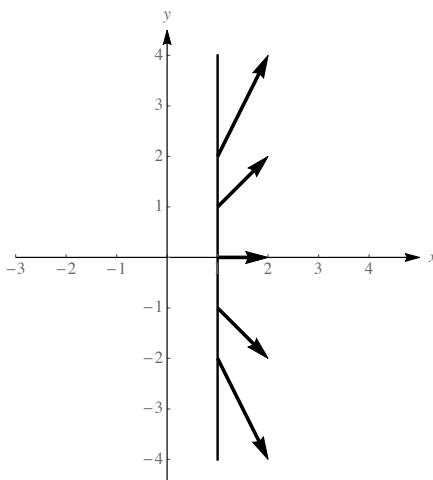
c.

**17.1.29**

a. Let $g(x, y) = x$. Then $\nabla g = \langle 1, 0 \rangle$. This is perpendicular to $\mathbf{F}$ for no points on C .

b. If $\mathbf{F} = k\nabla g$, then $y = 0$, so the point on C is $(1, 0)$.

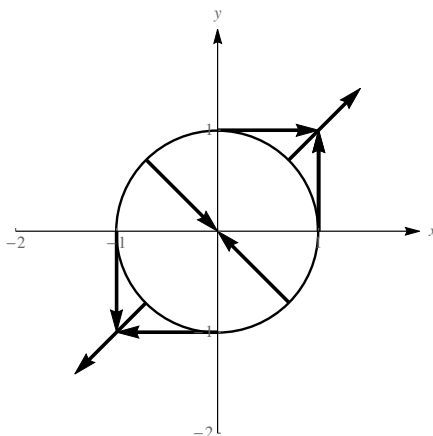
c.

**17.1.30**

a. Let $g(x, y) = x^2 + y^2$. Then $\nabla g = \langle 2x, 2y \rangle$. This is perpendicular to $\mathbf{F}$ if $2xy + 2yx = 4xy = 0$, which occurs if $x = 0$ or $y = 0$. So the corresponding points on C are $(\pm 1, 0)$ and $(0, \pm 1)$.

b. If $\mathbf{F} = k\nabla g$, then $\langle y, x \rangle = k \langle 2x, 2y \rangle$, so $y = 2kx$ and $x = 2ky$. Then $y = 2kx = 2k(2ky)$, so $4k^2 = 1$ and $k = \pm \frac{1}{2}$. Then either $x = y$ or $x = -y$, so the points are $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$, $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$, $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$, $\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$.

c.



17.1.31 For example, $\mathbf{F} = \langle -y, x \rangle$ or $\mathbf{F} = \langle -1, 1 \rangle$.

17.1.32 For example, $\mathbf{F} = \langle y, 0 \rangle$.

17.1.33 For example, $\mathbf{F} = \frac{1}{\sqrt{x^2+y^2}} \langle x, y \rangle = \frac{\mathbf{r}}{|\mathbf{r}|}$, $\mathbf{F}(0,0) = \mathbf{0}$.

17.1.34 For example, $\mathbf{F} = \langle -y, x \rangle$. The magnitude is $\sqrt{x^2 + y^2}$.

17.1.35 $\nabla\varphi = \langle \varphi_x, \varphi_y \rangle = \langle 2xy - y^2, x^2 - 2xy \rangle$.

17.1.36 $\nabla\varphi = \langle \varphi_x, \varphi_y \rangle = \langle \frac{y}{2\sqrt{xy}}, \frac{x}{2\sqrt{xy}} \rangle = \frac{1}{2\sqrt{xy}} \langle y, x \rangle$.

17.1.37 $\nabla\varphi = \langle \varphi_x, \varphi_y \rangle = \langle 1/y, -x/y^2 \rangle$.

17.1.38 $\nabla\varphi = \langle \frac{1}{1+(x^2/y^2)} \cdot \frac{1}{y}, \frac{1}{1+(x^2/y^2)} \cdot \frac{-x}{y^2} \rangle = \langle \frac{y}{y^2+x^2}, \frac{-x}{y^2+x^2} \rangle = \frac{1}{x^2+y^2} \langle y, -x \rangle$.

17.1.39 $\nabla\varphi = \langle x, y, z \rangle = \mathbf{r}$.

17.1.40 $\nabla\varphi = \langle \frac{2x}{1+x^2+y^2+z^2}, \frac{2y}{1+x^2+y^2+z^2}, \frac{2z}{1+x^2+y^2+z^2} \rangle = \frac{2\mathbf{r}}{|\mathbf{r}|^2+1}$.

17.1.41 $\nabla\varphi = -(x^2 + y^2 + z^2)^{-3/2} \langle x, y, z \rangle = -\frac{\mathbf{r}}{|\mathbf{r}|^3}$.

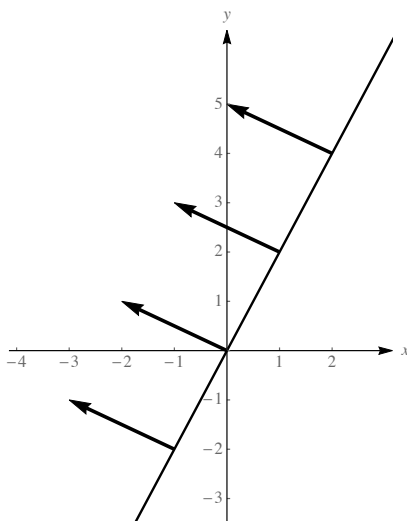
17.1.42 $\nabla\varphi = \langle e^{-z} \cos(x+y), e^{-z} \cos(x+y), -e^{-z} \sin(x+y) \rangle$.

17.1.43

a. $\mathbf{F} = \nabla\varphi = \langle -2, 1 \rangle$.

b. $\mathbf{F}(-1, -2) = \mathbf{F}(0, 0) = \mathbf{F}(1, 2) = \mathbf{F}(2, 4) = \langle -2, 1 \rangle$.

c.

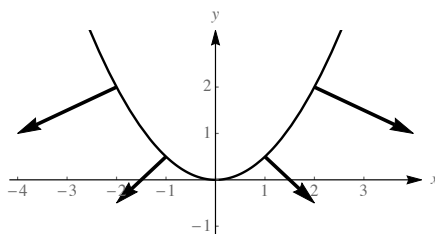


17.1.44

a. $\mathbf{F} = \nabla\varphi = \langle x, -1 \rangle$.

b. $\mathbf{F}(-2, 2) = \langle -2, -1 \rangle$, $\mathbf{F}(-1, 1/2) = \langle -1, -1 \rangle$, $\mathbf{F}(1, 1/2) = \langle 1, -1 \rangle$, $\mathbf{F}(2, 2) = \langle 2, -1 \rangle$.

c.

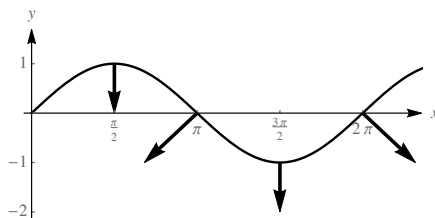


17.1.45

a. $\mathbf{F} = \nabla\varphi = \langle \cos x, -1 \rangle$.

b. $\mathbf{F}(\pi/2, 1) = \langle 0, -1 \rangle$, $\mathbf{F}(\pi, 0) = \langle -1, -1 \rangle$, $\mathbf{F}(3\pi/2, -1) = \langle 0, -1 \rangle$, $\mathbf{F}(2\pi, 0) = \langle 1, -1 \rangle$.

c.

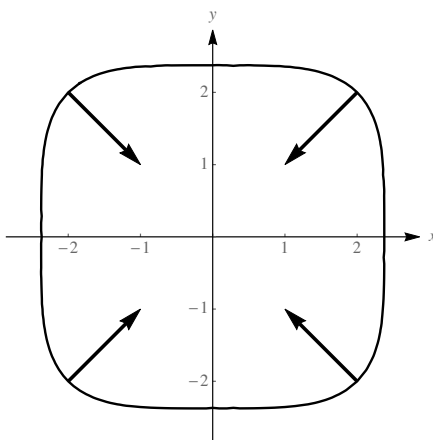


17.1.46

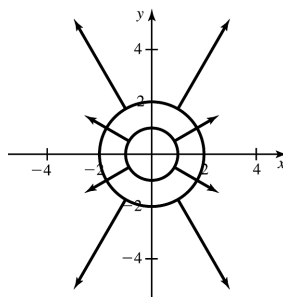
a. $\mathbf{F} = \nabla\varphi = \langle -x^3/8, -y^3/8 \rangle$.

b. $\mathbf{F}(2, 2) = \langle -1, -1 \rangle$, $\mathbf{F}(-2, 2) = \langle 1, -1 \rangle$, $\mathbf{F}(-2, -2) = \langle 1, 1 \rangle$, $\mathbf{F}(2, -2) = \langle -1, 1 \rangle$.

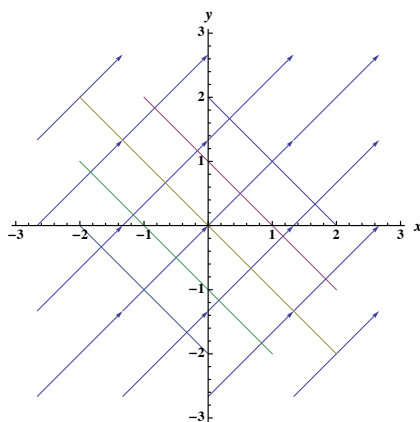
c.



17.1.47 The gradient field is $\langle \varphi_x, \varphi_y \rangle = \langle 2x, 2y \rangle$.



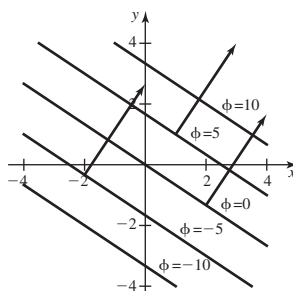
17.1.48 The gradient field is $\langle \varphi_x, \varphi_y \rangle = \langle 1, 1 \rangle$.



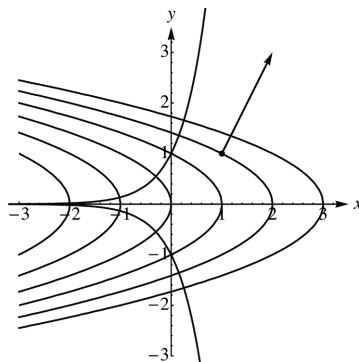
17.1.49

- The gradient field is $\langle 2, 3 \rangle$.
- The equipotential curve at $(1, 1)$ is $2x + 3y = 5$, which is a line of slope $-\frac{2}{3}$ so has a tangent vector at (x, y) parallel to $\langle 1, -\frac{2}{3} \rangle$. But $\langle 2, 3 \rangle \cdot \langle 1, -\frac{2}{3} \rangle = 0$, so the gradient field is normal to the equipotential line through $(1, 1)$.
- The equipotential curve at any point is a line of slope $-\frac{2}{3}$ and thus has a tangent vector at (x, y) parallel to $\langle x, -\frac{2}{3}y \rangle$. The same argument as in part (b) shows that this is normal to the gradient field.

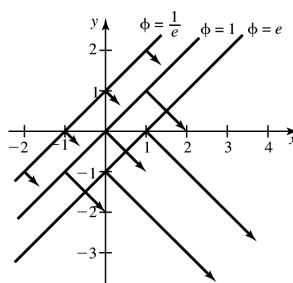
d.

**17.1.50**

- The gradient field is $\langle 1, 2y \rangle$.
- At $(1, 1)$, the tangent vector is parallel to $\langle -\varphi_y(1, 1), \varphi_x(1, 1) \rangle = \langle -2, 1 \rangle$, which is normal to the gradient at $(1, 1)$ (which is $\langle 1, 2 \rangle$).
- At (x, y) , the tangent vector is parallel to $\langle -2y, 1 \rangle$, and $\langle 1, 2y \rangle \cdot \langle -2y, 1 \rangle = 0$, so the gradient is everywhere normal to the equipotential curves.
-

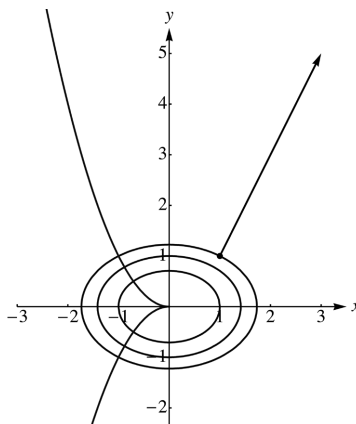
**17.1.51**

- The gradient field is $\langle e^{x-y}, -e^{x-y} \rangle$.
- At $(1, 1)$, the tangent vector is parallel to $\langle -\varphi_y(1, 1), \varphi_x(1, 1) \rangle = \langle 1, 1 \rangle$, which is normal to the gradient $\langle 1, -1 \rangle$ at $(1, 1)$.
- At (x, y) , the tangent vector is parallel to $\langle e^{x-y}, e^{x-y} \rangle$, and $\langle e^{x-y}, -e^{x-y} \rangle \cdot \langle e^{x-y}, e^{x-y} \rangle = 0$, so the gradient is everywhere normal to the equipotential curves.
-



17.1.52

- The gradient field is $\langle 2x, 4y \rangle$.
- At $(1, 1)$, the tangent vector is parallel to $\langle -\varphi_y(1, 1), \varphi_x(1, 1) \rangle = \langle -4, 2 \rangle$, which is normal to the gradient $\langle 2, 4 \rangle$ at $(1, 1)$.
- At (x, y) , the tangent vector is parallel to $\langle -4y, 2x \rangle$, which is normal to the gradient field.
-



17.1.53

- True. $(\varphi_1)_x = (\varphi_2)_x = 3x^2$, and $(\varphi_1)_y = (\varphi_2)_y = 1$.
- False. It is constant in magnitude (magnitude 1) but not direction.
- True. For example, it points outwards along the line $y = x$ but horizontally along the line $x = 0$.

17.1.54

- $V(x, y) = k(x^2 + y^2)^{-1/2}$, so $\mathbf{E} = -\nabla V = -\langle V_x, V_y \rangle = k(x^2 + y^2)^{-3/2} \langle x, y \rangle$.
- From the above formula, the field is a varying multiple of $\langle x, y \rangle$, which is a radial field pointing away from the origin. The radial component of $\mathbf{E}$ is thus $|\mathbf{E}| = k(x^2 + y^2)^{-3/2} |\langle x, y \rangle| = k(x^2 + y^2)^{-3/2} (x^2 + y^2)^{1/2} = \frac{k}{r^2}$.
- The equipotential curves are curves of the form $\frac{k}{\sqrt{x^2 + y^2}} = C$ so that $\sqrt{x^2 + y^2} = \frac{k}{C}$ and the equipotential curves are circles. Thus the tangent vectors to the equipotential curves are proportional to $\langle -y, x \rangle$ and thus are normal to $\mathbf{E}$, which is proportional to $\langle x, y \rangle$.

17.1.55

- $V_x = \frac{c\sqrt{x^2 + y^2}}{r_0} \cdot \left(-\frac{1}{2}r_0(x^2 + y^2)\right)$ and similarly, $V_y = -\frac{cy}{x^2 + y^2}$, so that $\mathbf{E} = -\nabla V = \frac{c}{x^2 + y^2} \langle x, y \rangle = \frac{c}{|\mathbf{r}|^2} \mathbf{r} = \frac{c}{|\mathbf{r}|} \cdot \frac{\mathbf{r}}{|\mathbf{r}|}$.
- From the above formula, the field is a varying multiple of $\langle x, y \rangle$, which is a radial field pointing away from the origin. The radial component of $\mathbf{E}$ is thus $|\mathbf{E}| = \frac{c}{x^2 + y^2} \sqrt{x^2 + y^2} = \frac{c}{\sqrt{x^2 + y^2}}$.

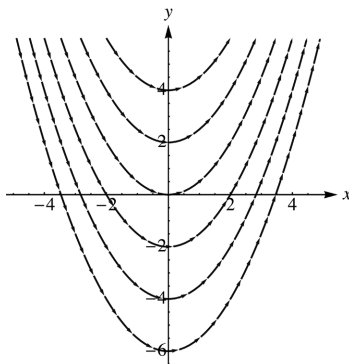
- c. The equipotential curves are curves of the form $c \ln \left(\frac{r_0}{\sqrt{x^2 + y^2}} \right) = K$, so are solutions to $\frac{r_0}{\sqrt{x^2 + y^2}} = e^{cK} = C$ and thus of the form $\sqrt{x^2 + y^2} = K_0$ for some constant K_0 . Hence the equipotential curves are circles, so have tangent vectors proportional to $\langle -y, x \rangle$; these are clearly normal to $\mathbf{E}$, which is proportional to $\langle x, y \rangle$.

17.1.56

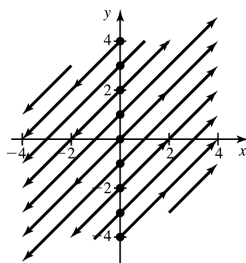
- a. $U_x = \left(GMm (x^2 + y^2 + z^2)^{-1/2} \right)_x = -\frac{1}{2} GMm (x^2 + y^2 + z^2)^{-3/2} (2x) = -GMmx (x^2 + y^2 + z^2)^{-3/2}$ and similarly for U_y and U_z . Thus $\mathbf{F} = -\nabla U = GMm (x^2 + y^2 + z^2)^{-3/2} \langle x, y, z \rangle$.
- b. From the above formula the field is a varying multiple of $\langle x, y, z \rangle$, which is a radial field pointing away from the origin. The radial component of $\mathbf{F}$ is thus $|\mathbf{F}| = GMm (x^2 + y^2 + z^2)^{-3/2} \sqrt{x^2 + y^2 + z^2} = \frac{GMm}{r^2}$.
- c. The equipotential surfaces are solutions to $GMm \sqrt{x^2 + y^2 + z^2} = K$ and so are spheres. The tangent plane at (x_0, y_0, z_0) is $U_x(x_0, y_0, z_0)(x - x_0) + U_y(x_0, y_0, z_0)(y - y_0) + U_z(x_0, y_0, z_0)(z - z_0)$ and so a normal to the plane is $\langle U_x, U_y, U_z \rangle$, which is proportional to $\mathbf{F}$.

17.1.57 The flow curve $y(x)$ of the vector field $\mathbf{F}$ at (x, y) is defined to be a continuous curve through (x, y) that is aligned with the vector field, i.e. whose tangent at (x, y) is given by $\mathbf{F}(x, y) = \langle f(x, y), g(x, y) \rangle$. The slope of the tangent line is then $\frac{g(x, y)}{f(x, y)}$, so this is $y'(x)$.

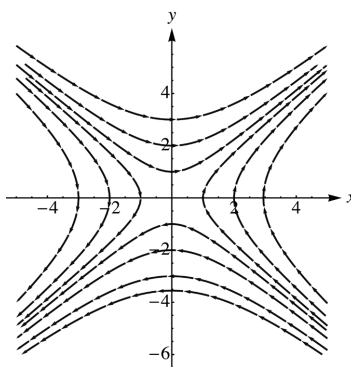
17.1.58 The streamlines satisfy $y'(x) = x$, so that $y(x) = \frac{1}{2}x^2 + C$.



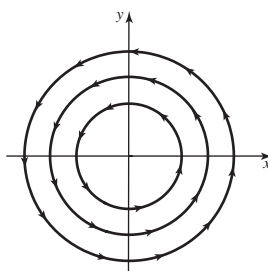
17.1.59 The streamlines satisfy $y'(x) = 1$, so that $y(x) = x + C$.



17.1.60 The streamlines satisfy $y'(x) = \frac{x}{y}$. But also $\frac{d}{dx}(y^2) = 2yy'(x)$, so that $y'(x) = \frac{\frac{d}{dx}(y^2)}{2y}$ and thus $\frac{d}{dx}(y^2) = 2x$. Thus $y^2 = x^2 + C$ and the streamlines are the hyperbolas $x^2 - y^2 = K$.



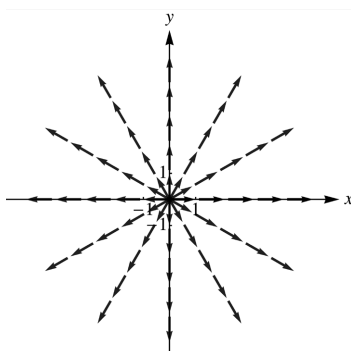
17.1.61 The streamlines satisfy $y'(x) = -\frac{x}{y}$. Because $\frac{d}{dx}(y^2) = 2yy'(x)$, we have $y'(x) = \frac{\frac{d}{dx}(y^2)}{2y}$ and thus $\frac{d}{dx}(y^2) = -2x$. Thus $y^2 = -x^2 + C$ and the streamlines are the circles $x^2 + y^2 = C$.



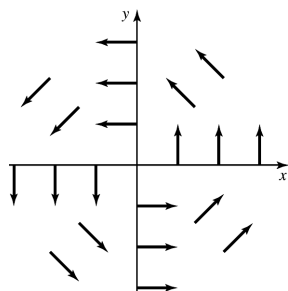
17.1.62 $\mathbf{u}_r$ is the unit vector forming an angle of θ with $\mathbf{i}$, so it is of unit length and proportional to $\langle \cos \theta, \sin \theta \rangle$. But $\cos^2(\theta) + \sin^2(\theta) = 1$, so this is in fact the unit vector. Thus $\mathbf{u}_r = \cos(\theta)\mathbf{i} + \sin(\theta)\mathbf{j}$. Similarly, $\mathbf{u}_\theta$ forms an angle of $\theta + \frac{\pi}{2}$ with $\mathbf{i}$, so it is the unit vector $\langle \cos(\theta + \frac{\pi}{2}), \sin(\theta + \frac{\pi}{2}) \rangle = \langle -\sin(\theta), \cos(\theta) \rangle$. The other two formulas can be found by solving these for $\mathbf{i}, \mathbf{j}$ as linear equations. For example, $\mathbf{u}_r = \cos(\theta)\mathbf{i} + \sin(\theta)\mathbf{j}$, $\mathbf{u}_\theta = -\sin(\theta)\mathbf{i} + \cos(\theta)\mathbf{j}$. Multiply the first equation by $\sin(\theta)$ and the second by $\cos(\theta)$ and add to obtain $\mathbf{u}_r \sin(\theta) + \mathbf{u}_\theta \cos(\theta) = \sin^2(\theta)\mathbf{j} + \cos^2(\theta)\mathbf{j} = \mathbf{j}$.

17.1.63 For $\theta = 0$, $\mathbf{u}_r$ is coincident with $\mathbf{i}$, and $\mathbf{u}_\theta$ with $\mathbf{j}$. From the formula $\mathbf{u}_r = \cos(0)\mathbf{i} + \sin(0)\mathbf{j} = \mathbf{i}$, $\mathbf{u}_\theta = -\sin(0)\mathbf{i} + \cos(0)\mathbf{j} = \mathbf{j}$. For $\theta = \frac{\pi}{2}$, the picture implies that we should have $\mathbf{u}_r = \mathbf{j}$, $\mathbf{u}_\theta = -\mathbf{i}$. From the formulas, $\mathbf{u}_r = \cos(\frac{\pi}{2})\mathbf{i} + \sin(\frac{\pi}{2})\mathbf{j} = \mathbf{j}$, $\mathbf{u}_\theta = -\sin(\frac{\pi}{2})\mathbf{i} + \cos(\frac{\pi}{2})\mathbf{j} = -\mathbf{i}$. For $\theta = \pi$, $\mathbf{u}_r$ is coincident with $-\mathbf{i}$, and $\mathbf{u}_\theta$ with $-\mathbf{j}$. From the formula $\mathbf{u}_r = \cos(\pi)\mathbf{i} + \sin(\pi)\mathbf{j} = -\mathbf{i}$, $\mathbf{u}_\theta = -\sin(\pi)\mathbf{i} + \cos(\pi)\mathbf{j} = -\mathbf{j}$. For $\theta = \frac{3\pi}{2}$, the picture implies that we should have $\mathbf{u}_r = -\mathbf{j}$, $\mathbf{u}_\theta = \mathbf{i}$. From the formulas, $\mathbf{u}_r = \cos(\frac{3\pi}{2})\mathbf{i} + \sin(\frac{3\pi}{2})\mathbf{j} = -\mathbf{j}$, $\mathbf{u}_\theta = -\sin(\frac{3\pi}{2})\mathbf{i} + \cos(\frac{3\pi}{2})\mathbf{j} = \mathbf{i}$.

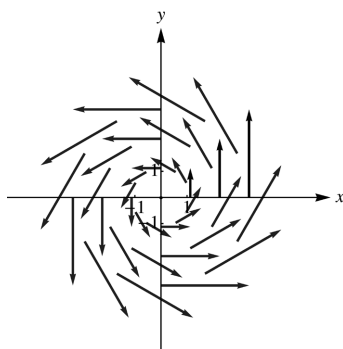
17.1.64 $\mathbf{F}(r, \theta) = \mathbf{u}_r = \cos(\theta)\mathbf{i} + \sin(\theta)\mathbf{j} = \frac{x}{\sqrt{x^2 + y^2}}\mathbf{i} + \frac{y}{\sqrt{x^2 + y^2}}\mathbf{j} = \frac{1}{\sqrt{x^2 + y^2}}(x\mathbf{i} + y\mathbf{j})$.



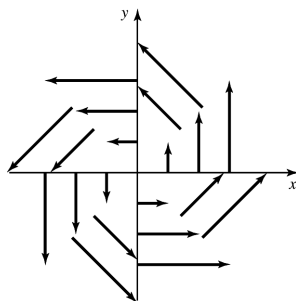
$$17.1.65 \quad \mathbf{F}(r, \theta) = \mathbf{u}_\theta = -\sin(\theta) \mathbf{i} + \cos(\theta) \mathbf{j} = -\frac{y}{\sqrt{x^2 + y^2}} \mathbf{i} + \frac{x}{\sqrt{x^2 + y^2}} \mathbf{j} = \frac{1}{\sqrt{x^2 + y^2}} (-y\mathbf{i} + x\mathbf{j}) \quad .$$



$$17.1.66 \quad \mathbf{F}(r, \theta) = r \mathbf{u}_\theta = \sqrt{x^2 + y^2} (-\sin(\theta) \mathbf{i} + \cos(\theta) \mathbf{j}) = \sqrt{x^2 + y^2} \frac{1}{\sqrt{x^2 + y^2}} (-y\mathbf{i} + x\mathbf{j}) = \langle -y, x \rangle \quad .$$



$$17.1.67 \quad \mathbf{F}(x, y) = -y(\mathbf{u}_r \cos(\theta) - \mathbf{u}_\theta \sin(\theta)) + x(\mathbf{u}_r \sin(\theta) + \mathbf{u}_\theta \cos(\theta)) = -r \sin(\theta)(\cos(\theta) \mathbf{u}_r - \sin(\theta) \mathbf{u}_\theta) + r \cos(\theta)(\sin(\theta) \cos(\theta) \mathbf{u}_r + \cos(\theta) \mathbf{u}_\theta) = r(\sin^2(\theta) + \cos^2(\theta)) \mathbf{u}_\theta = r \mathbf{u}_\theta.$$



17.2 Line Integrals

17.2.1 A single-variable integral integrates along a segment while a line integral integrates along an arbitrary curve.

$$17.2.2 \quad |\mathbf{r}'(t)| = \sqrt{(\mathbf{r}'_x)^2 + (\mathbf{r}'_y)^2} = \sqrt{1 + 4t^2}$$

$$17.2.3 \quad \mathbf{r}'(t) = \langle -\sin t, 1 \rangle, \text{ so } |\mathbf{r}'(t)| = \sqrt{1 + \sin^2 t}. \text{ The line integral is } \int_{\pi/2}^{\pi} \frac{\cos t \sqrt{1 + \sin^2 t}}{t} dt.$$

$$17.2.4 \quad \mathbf{r}(t) = \langle \sin \pi t, t \rangle \text{ for } 0 \leq t \leq 3.$$

17.2.5 A direction vector for the line is $\langle 5 - 1, 4 - 2, 0 - 3 \rangle = \langle 4, 2, -3 \rangle$. The line segment is given by $\mathbf{r}(t) = \langle 1 + 4t, 2 + 2t, 3 - 3t \rangle$ for $0 \leq t \leq 1$.

17.2.6 $\mathbf{r}(t) = \langle \cos t, \sin t \rangle$ for $0 \leq t \leq \pi/2$.

17.2.7 $\mathbf{r}(t) = \langle t^2 + 1, t \rangle$ for $2 \leq t \leq 4$.

17.2.8 $\mathbf{F} = \langle \cos t - \sin t, \sin t - \cos t \rangle$.

17.2.9

a. $\mathbf{F} \cdot \mathbf{T} = \langle t, 2t^3 \rangle \cdot \langle 1, 3t^2 \rangle = t + 6t^5$. The integral is given by $\int_0^2 (t + 6t^5) dt$.

b. $\int_0^2 (t + 6t^5) dt = \left(\frac{t^2}{2} + t^6 \right) \Big|_0^2 = 2 + 64 = 66$.

17.2.10

a. Let $\mathbf{r}(t) = \langle \cos t, \sin t \rangle$ and $\mathbf{F} = \langle f, g \rangle = \langle 1, x \rangle = \langle 1, \cos t \rangle$. Then

$$\int_C \mathbf{F} \cdot \mathbf{n} ds = \int_0^{2\pi} (f(t)y'(t) - g(t)x'(t)) dt = \int_0^{2\pi} (\cos t + \cos t \sin t) dt.$$

b. $\int_0^{2\pi} (\cos t + \cos t \sin t) dt = \left(\sin t + \frac{\sin^2 t}{2} \right) \Big|_0^{2\pi} = 0$.

17.2.11 $\int_C \mathbf{F} \cdot d\mathbf{r}$ and $\int_C f dx + g dy + h dz$.

17.2.12 By reversing the orientation of C , we obtain the curve $-C$ from B to A . The value of the integral with the opposite orientation is the opposite of the value of the integral, so $\int_{-C} f ds = -10$.

17.2.13 $\int_{C_1} f ds + \int_{C_2} f ds = \int_C f ds = 10$, so $\int_{C_2} f ds = 7$.

17.2.14

- Positive.
- 0.
- 0.
- Negative.
- 0.
- Positive.
- Negative.
- 0.

17.2.15 Take the line integral of $\mathbf{F} \cdot \mathbf{T}$ along the curve using arc length parameterization.

17.2.16 The flux measures the degree to which the vector field is outwards normal to the curve C as C is traversed with a particular orientation.

17.2.17 $\mathbf{r}(s)$ is an arc length parameterization, so we have $\int_C xy \, ds = \int_0^{2\pi} \cos(s) \sin(s) \, ds = 0$.

17.2.18 With $\mathbf{r}(s) = \langle \frac{s}{\sqrt{2}}, \frac{s}{\sqrt{2}} \rangle$, $|\mathbf{r}'(t)| = 1$, so that we have $\int_C (x^2 - 2y^2) \, ds = \int_0^4 \left(\frac{s^2}{2} - 2 \frac{s^2}{2} \right) ds = -\int_0^4 \frac{s^2}{2} ds = -\frac{s^3}{6} \Big|_0^4 = -\frac{32}{3}$.

17.2.19 $|\mathbf{r}'(t)| = |\langle 3, 4 \rangle| = \sqrt{9+16} = 5$, so $\int_C (2x+y) \, ds = \int_0^2 (6t+4t)5 \, dt = 25t^2 \Big|_0^2 = 100$.

17.2.20 $|\mathbf{r}'(t)| = |\langle 3t^2, 4 \rangle| = \sqrt{9t^4+16}$, so $\int_C x \, ds = \int_0^1 t^3 \sqrt{9t^4+16} \, dt = \frac{1}{36} \int_{16}^{25} u^{1/2} \, du = \frac{u^{3/2}}{54} \Big|_{16}^{25} = \frac{125-64}{54} = \frac{61}{54}$.

17.2.21 $|\mathbf{r}'(t)| = |\langle -2 \sin t, 2 \cos t \rangle| = 2$, so $\int_C xy^3 \, ds = \int_0^{\pi/2} 32 \cos t \sin^3 t \, dt = 32 \int_0^1 u^3 \, du = 8u^4 \Big|_0^1 = 8$.

17.2.22 $|\mathbf{r}'(t)| = |\langle \cos t, 1 \rangle| = \sqrt{\cos^2 t + 1}$, so

$$\int_C 3x \cos y \, ds = \int_0^{\pi/2} 3 \sin t \cos t \sqrt{\cos^2 t + 1} \, dt = -\frac{3}{2} \int_2^1 \sqrt{u} \, du = u^{3/2} \Big|_1^2 = 2^{3/2} - 1 = 2\sqrt{2} - 1.$$

17.2.23 $|\mathbf{r}'(t)| = |\langle -3 \sin t, 3 \cos t, 4 \rangle| = \sqrt{25} = 5$, so

$$\int_C (y-z) \, ds = \int_0^{2\pi} 5(3 \sin t - 4t) \, dt = 5(-3 \cos t - 2t^2) \Big|_0^{2\pi} = 5(-3 - 8\pi^2 + 3) = -40\pi^2.$$

17.2.24 $|\mathbf{r}'(t)| = \sqrt{0+9\sin^2 t+9\cos^2 t} = 3$, so $\int_C (x-y+2z) \, ds = 3 \int_0^{2\pi} (1-3\cos(t)+6\sin t) \, dt = 6\pi$.

17.2.25 $\mathbf{r}(t) = \langle 4 \cos t, 4 \sin t \rangle$, $0 \leq t \leq 2\pi$. $|\mathbf{r}'(t)| = \sqrt{(-4 \sin t)^2 + (4 \cos t)^2} = 4$, so

$$\int_C (x^2 + y^2) \, ds = \int_0^{2\pi} 4(16 \cos^2 t + 16 \sin^2 t) \, dt = \int_0^{2\pi} 64 \, dt = 128\pi.$$

17.2.26 $\mathbf{r}(t) = \langle 5t, 5t \rangle$, $0 \leq t \leq 1$. $|\mathbf{r}'(t)| = \sqrt{(5)^2 + (5)^2} = 5\sqrt{2}$, so

$$\int_C (x^2 + y^2) \, ds = \int_0^1 50t^2 \cdot 5\sqrt{2} \, dt = 250\sqrt{2} \int_0^1 t^2 \, dt = \frac{250\sqrt{2}}{3}.$$

17.2.27 $\mathbf{r}(t) = \langle t, t \rangle$, $1 \leq t \leq 10$. $|\mathbf{r}'(t)| = \sqrt{(1)^2 + (1)^2} = \sqrt{2}$, so

$$\int_C \frac{x}{x^2+y^2} \, ds = \int_1^{10} \frac{t}{t^2+t^2} \cdot \sqrt{2} \, dt = \frac{\sqrt{2}}{2} \int_1^{10} \frac{1}{t} \, dt = \frac{\sqrt{2}}{2} \ln 10.$$

17.2.28 $\mathbf{r}(t) = \langle t, t^2 \rangle$, $0 \leq t \leq 1$. $|\mathbf{r}'(t)| = \sqrt{(1)^2 + (2t)^2} = \sqrt{1+4t^2}$, so

$$\int_C (xy)^{1/3} \, ds = \int_0^1 (t^3)^{1/3} \sqrt{1+4t^2} \, dt = \int_0^1 t \sqrt{1+4t^2} \, dt = \frac{5\sqrt{5}-1}{12}.$$

17.2.29 $\mathbf{r}(t) = \langle 2 \cos t, 4 \sin t \rangle$, $0 \leq t \leq \pi/2$.

$$|\mathbf{r}'(t)| = \sqrt{4 \sin^2 t + 16 \cos^2 t} = \sqrt{4 + 12 \cos^2 t} = 2\sqrt{1 + 3 \cos^2 t}.$$

$$\int_C xy \, ds = \int_0^{\pi/2} 16 \sin t \cos t \sqrt{1 + 3 \cos^2 t} \, dt.$$

Let $u = (1 + 3 \cos^2 t) \, dt$ so that $du = -6 \cos t \sin t \, dt$. Substituting gives

$$\frac{8}{3} \int_1^4 u^{1/2} \, du = \frac{16}{9} \left(u^{3/2} \right) \Big|_1^4 = \frac{16}{9} (8 - 1) = \frac{112}{9}.$$

17.2.30 $\mathbf{r}_1(t) = \langle t - 1, t \rangle$, $\mathbf{r}_2(t) = \langle t, 1 - t \rangle$, $0 \leq t \leq 1$. $|\mathbf{r}'_1(t)| = \sqrt{2}$, $|\mathbf{r}'_2(t)| = \sqrt{2}$.

$$\int_C (2x - 3y) \, ds = \int_0^1 (2(t - 1) - 3t) \sqrt{2} \, dt + \int_0^1 (2t - 3(1 - t)) \sqrt{2} \, dt = \sqrt{2} \int_0^1 (4t - 5) \, dt = -3\sqrt{2}.$$

17.2.31 $|\mathbf{r}'(t)| = \sqrt{4 \sin^2 t + 0 + 4 \cos^2 t} = 2$, so

$$\int_C (x + y + z) \, ds = 2 \int_0^\pi (2 \cos t + 2 \sin t) \, dt = 4 (\sin t - \cos t) \Big|_0^\pi = 8.$$

17.2.32 Let $\mathbf{r}(t) = \langle 2t + 1, 2t + 4, 2t + 1 \rangle$, $0 \leq t \leq 1$. Then $|\mathbf{r}'(t)| = \sqrt{12} = 2\sqrt{3}$, so $\int_C \frac{xy}{z} \, ds = 2\sqrt{3} \int_0^1 \frac{(2t + 1)(2t + 4)}{(2t + 1)} \, dt = 2\sqrt{3} \int_0^1 (2t + 4) \, dt = 10\sqrt{3}$.

17.2.33 For the first segment, $\mathbf{r}(t) = \langle 3t, 2t, 6t \rangle$ for $0 \leq t \leq 1$, so $\mathbf{r}'(t) = \langle 3, 2, 6 \rangle$, so $|\mathbf{r}'(t)| = 7$, so $\int_C dz \, ds = \int_0^1 7(18)t^2 \, dt = 42t^3 \Big|_0^1 = 42$.

For the second segment, $\mathbf{r}(t) = \langle 3 + 4t, 2 + 7t, 6 + 4t \rangle$ for $0 \leq t \leq 1$, so $\mathbf{r}'(t) = \langle 4, 7, 4 \rangle$, and $|\mathbf{r}'(t)| = 9$, so $\int_C xz \, ds = \int_0^1 9(3 + 4t)(6 + 4t) \, dt = 18 \int_0^1 (8t^2 + 18t + 9) \, dt = 18 \left(\frac{8}{3}t^3 + 9t^2 + 9t \right) \Big|_0^1 = 372$.

Therefore, for the union of the two segments, we have $\int_C xz \, ds = 42 + 372 = 414$.

17.2.34 $|\mathbf{r}'(t)| = \sqrt{9} = 3$, so $\int_C xe^{yz} \, ds = 3 \int_0^2 te^{-4t^2} \, dt = -\frac{3}{8}e^{-4t^2} \Big|_0^2 = \frac{3}{8} \left(1 - \frac{1}{e^{16}} \right)$.

17.2.35 Parameterize C by $\mathbf{r}(t) = \langle t, 2t^2 \rangle$ for $0 \leq t \leq 3$. Then $|\mathbf{r}'(t)| = \sqrt{1 + 16t^2}$ and $\int_C \rho \, ds = \int_0^3 (1 + 2t^3) \sqrt{1 + 16t^2} \, dt \approx 409.5$.

17.2.36 $\mathbf{r}'(\theta) = \langle -\sin \theta, \cos \theta \rangle$, so that $|\mathbf{r}'(\theta)| = 1$ and $\int_C \rho \, ds = \int_0^\pi \left(\frac{2\theta}{\pi} + 1 \right) d\theta = 2\pi$.

17.2.37 Let $\mathbf{r}(t) = \langle t + 1, 4t + 1 \rangle$, $0 \leq t \leq 1$. Then $|\mathbf{r}'(t)| = \sqrt{17}$ and $\int_C (x + 2y) \, ds = \int_0^1 ((t + 1) + 2(4t + 1)) \cdot \sqrt{17} \, dt = \sqrt{17} \int_0^1 (9t + 3) \, dt = \frac{15}{2}\sqrt{17}$. The length of the line segment is $\sqrt{17}$, so the average value is $\frac{15}{2}$.

17.2.38 Let $\mathbf{r}(t) = \langle \cos t, \sin t \rangle$, $0 \leq t \leq 2\pi$. Then $|\mathbf{r}'(t)| = 1$ and $\int_C (x e^y) ds = \int_0^{2\pi} \cos t e^{\sin t} dt = 0$. Thus, the average value is 0.

17.2.39 The length of the curve is the line integral of 1 along the curve. Then $\mathbf{r}'(t)$ simplifies to $\langle 5 \cos(t/4), -5 \sin(t/4), \frac{1}{2} \rangle$, and then

$$|\mathbf{r}'(t)| = \sqrt{25 + \frac{1}{4}} = \frac{1}{2}\sqrt{101}$$

so that the arc length is $\frac{1}{2} \int_0^2 \sqrt{101} dt = \sqrt{101}$.

17.2.40 The length of the curve is the line integral of 1 along the curve.

$|\mathbf{r}'(t)| = \sqrt{900 \cos^2(t) + 1600 \cos^2 t + 2500 \sin^2 t} = 50$ so that $\int_C 1 ds = 50 \int_0^{2\pi} dt = 100\pi$.

17.2.41 $\mathbf{r}'(t) = \langle 4, 2t \rangle$, and $\mathbf{F} \cdot \mathbf{r}'(t) = \langle 4t, t^2 \rangle \cdot \langle 4, 2t \rangle = 16t + 2t^3$, so that

$$\int_C \mathbf{F} \cdot \mathbf{T} ds = \int_0^1 \mathbf{F} \cdot \mathbf{r}'(t) dt = \int_0^1 (16t + 2t^3) dt = \frac{17}{2}.$$

17.2.42 $\mathbf{r}'(t) = \langle -4 \sin t, 4 \cos t \rangle$, so $\mathbf{F} \cdot \mathbf{r}'(t) = \langle -4 \sin t, 4 \cos t \rangle \cdot \langle -4 \sin t, 4 \cos t \rangle = 16$. Then $\int_C \mathbf{F} \cdot \mathbf{T} ds = \int_0^\pi \mathbf{F} \cdot \mathbf{r}'(t) dt = \int_0^\pi 16 dt = 16\pi$.

17.2.43 Let $\mathbf{r}(t) = \langle 4t+1, 9t+1 \rangle$, $0 \leq t \leq 1$; then $\mathbf{r}'(t) = \langle 4, 9 \rangle$ and $\mathbf{F} \cdot \mathbf{r}'(t) = \langle 9t+1, 4t+1 \rangle \cdot \langle 4, 9 \rangle = 72t+13$. Then $\int_C \mathbf{F} \cdot \mathbf{T} ds = \int_0^1 (72t+13) dt = 49$.

17.2.44 Let $\mathbf{r}(t) = \langle t, t^2 \rangle$, $0 \leq t \leq 1$; then $\mathbf{r}'(t) = \langle 1, 2t \rangle$ and $\mathbf{F} \cdot \mathbf{r}'(t) = \langle -t^2, t \rangle \cdot \langle 1, 2t \rangle = t^2$. Then $\int_C \mathbf{F} \cdot \mathbf{T} ds = \int_0^1 t^2 dt = \frac{1}{3}$.

17.2.45 $\mathbf{r}'(t) = \langle 2t, 6t \rangle$; then $\mathbf{F} \cdot \mathbf{r}'(t) = (10t^4)^{-3/2} \langle t^2, 3t^2 \rangle \cdot \langle 2t, 6t \rangle = \frac{20t^3}{10\sqrt{10}t^6} = \frac{2}{\sqrt{10}}t^{-3}$ and $\int_C \mathbf{F} \cdot \mathbf{T} ds = \frac{2}{\sqrt{10}} \int_1^2 t^{-3} dt = \frac{3}{40}\sqrt{10}$.

17.2.46 $\mathbf{r}'(t) = \langle 1, 4 \rangle$, so that $\mathbf{F} \cdot \mathbf{r}'(t) = \frac{1}{17t^2} \langle t, 4t \rangle \cdot \langle 1, 4 \rangle = \frac{1}{t}$. Then $\int_C \mathbf{F} \cdot \mathbf{T} ds = \int_1^{10} \frac{1}{t} dt = \ln(10)$.

17.2.47

- For C_1 , the vector field points “against” the curve for most of its length, and with larger magnitude, so the integral is negative.
- For C_2 , the vector field points with the curve for its entire length, so the integral is positive.

17.2.48

- For C_1 , the vector field points against the curve for its entire length, so the integral over C_1 is negative.
- For C_2 , the vector field points with the curve, so the integral over C_2 is positive.

17.2.49 $\mathbf{r}_1(t) = \langle 1-t, 2-2t \rangle$, $\mathbf{r}_2(t) = \langle 0, 4t \rangle$, $0 \leq t \leq 1$. Then $\mathbf{r}'_1(t) = \langle -1, -2 \rangle$, $\mathbf{r}'_2(t) = \langle 0, 4 \rangle$, so that $\int_C \mathbf{F} \cdot \mathbf{r}'(t) dt = \int_0^1 \langle 2-2t, 1-t \rangle \cdot \langle -1, -2 \rangle dt + \int_0^1 \langle 4t, 0 \rangle \cdot \langle 0, 4 \rangle dt = \int_0^1 0 dt = 0$.

17.2.50 $\mathbf{r}_1(t) = \langle t-1, 8t \rangle$, $\mathbf{r}_2(t) = \langle 2t, 8 \rangle$, $0 \leq t \leq 1$. Then $\mathbf{r}'_1(t) = \langle 1, 8 \rangle$, $\mathbf{r}'_2(t) = \langle 2, 0 \rangle$, so that $\int_C \mathbf{F} \cdot \mathbf{r}'(t) dt = \int_0^1 \langle t-1, 8t \rangle \cdot \langle 1, 8 \rangle dt + \int_0^1 \langle 2t, 8 \rangle \cdot \langle 2, 0 \rangle dt = \int_0^1 (69t-1) dt = \frac{67}{2}$.

17.2.51 $\mathbf{r}(t) = \langle 2t, 8t^2 \rangle$, $0 \leq t \leq 1$. Then $\mathbf{r}'(t) = \langle 2, 16t \rangle$, so $\int_C \mathbf{F} \cdot \mathbf{r}'(t) dt = \int_0^1 \langle 8t^2, 2t \rangle \cdot \langle 2, 16t \rangle dt = \int_0^1 48t^2 dt = 16$.

17.2.52 $\mathbf{r}(t) = \langle 2t+1, 8-4t \rangle$, $0 \leq t \leq 1$. Then $\mathbf{r}'(t) = \langle 2, -4 \rangle$, so $\int_C \mathbf{F} \cdot \mathbf{r}'(t) dt = \int_0^1 \langle 8-4t, -1-2t \rangle \cdot \langle 2, -4 \rangle dt = \int_0^1 20 dt = 20$.

17.2.53 $\mathbf{r}'(t) = \langle -4\sin t, 4\cos t, -4\sin t \rangle$. Thus we have $\int_C \mathbf{F} \cdot \mathbf{r}'(t) dt = \int_0^{2\pi} \langle 4\cos t, 4\sin t, 4\cos t \rangle \cdot \langle -4\sin t, 4\cos t, -4\sin t \rangle dt = \int_0^{2\pi} -16\sin t \cos(t) dt = 0$.

17.2.54 $\mathbf{r}'(t) = \langle -2\sin t, 2\cos t, \frac{1}{2\pi} \rangle$, so $\int_C \mathbf{F} \cdot \mathbf{r}'(t) dt = \int_0^{2\pi} \langle -2\sin t, 2\cos t, \frac{t}{2\pi} \rangle \cdot \langle -2\sin t, 2\cos t, \frac{1}{2\pi} \rangle dt = \int_0^{2\pi} \left(4 + \frac{t}{4\pi^2} \right) dt = \frac{1}{2} + 8\pi$.

17.2.55 Let $\mathbf{r}(t) = \langle t+1, t+1, t+1 \rangle$, $0 \leq t \leq 9$, so that $\mathbf{r}'(t) = \langle 1, 1, 1 \rangle$. Then $\int_C \mathbf{F} \cdot \mathbf{r}'(t) dt = \int_0^9 \frac{1}{(3(t+1)^2)^{3/2}} \langle t+1, t+1, t+1 \rangle \cdot \langle 1, 1, 1 \rangle dt = \frac{1}{\sqrt{3}} \int_0^9 \frac{1}{(t+1)^3} (t+1) dt = \frac{1}{\sqrt{3}} \int_0^9 \frac{1}{(t+1)^2} dt = \frac{3}{10}\sqrt{3}$.

17.2.56 Let $\mathbf{r}(t) = \langle 7t+1, 3t+1, t+1 \rangle$, $0 \leq t \leq 1$, so that $\mathbf{r}'(t) = \langle 7, 3, 1 \rangle$. Then $\int_C \mathbf{F} \cdot \mathbf{r}'(t) dt = \int_0^1 \frac{1}{(7t+1)^2 + (3t+1)^2 + (t+1)^2} \langle 7t+1, 3t+1, t+1 \rangle \cdot \langle 7, 3, 1 \rangle dt = \int_0^1 \frac{59t+11}{(7t+1)^2 + (3t+1)^2 + (t+1)^2} dt = \ln(2) + \frac{1}{2}\ln(7) = \ln(2\sqrt{7})$.

17.2.57

a. Looking at the vector field, it appears that the vector field points counterclockwise just as much as it points clockwise at the boundary of the region, so we would expect the circulation to be zero.

b. $\mathbf{r}'(t) = \langle -2\sin t, 2\cos t \rangle$, so $\int_C \mathbf{F} \cdot \mathbf{r}'(t) dt = \int_0^{2\pi} \langle 2(\sin t - \cos t), 2\cos t \rangle \cdot \langle -2\sin t, 2\cos t \rangle dt = 4 \int_0^{2\pi} (\cos^2 t + \sin t \cos t - \sin^2 t) dt = 0$.

17.2.58

a. The circulation appears to be negative.

b. On C , $\mathbf{F} = \frac{\langle 4\sin t, -4\cos t \rangle}{\sqrt{16\cos^2 t + 16\sin^2 t}} = \langle \sin t, -\cos t \rangle$, so $\mathbf{F} \cdot \mathbf{r}'(t) = \langle \sin t, -\cos t \rangle \cdot \langle -2\sin t, 4\cos t \rangle = -2\sin^2 t - 4\cos^2 t = -2 - 2\cos^2 t$. So the circulation is $\int_0^{2\pi} (-2 - 2\cos^2 t) dt = \int_0^{2\pi} (-2 - 1 - \cos 2t) dt = \int_0^{2\pi} (-3 - \cos 2t) dt = -6\pi$.

17.2.59

a. Looking at the vector field, the inward-pointing vectors (in quadrants II and IV) appear larger than the outward-pointing vectors (in quadrants I and III). Thus we would expect the flux to be negative.

b. $\mathbf{r}'(t) = \langle -2 \sin t, 2 \cos t \rangle$, so that $\int_C \mathbf{F} \cdot \mathbf{n} \, ds = \int_0^{2\pi} (2(\sin t - \cos t) \cdot 2 \cos t - 2 \cos t \cdot (-2 \sin t)) \, dt = -4 \int_0^{2\pi} (\cos^2 t - 2 \sin t \cos t) \, dt = -4\pi$.

17.2.60

a. The flux appears to be 0.

b. We have $\mathbf{r}(t) = \langle 2 \cos t, 4 \sin t \rangle$, and on C , $\mathbf{F} = \frac{\langle 4 \sin t, -4 \cos t \rangle}{\sqrt{16 \cos^2 t + 16 \sin^2 t}} = \langle \sin t, -\cos t \rangle$, so the flux is $\int_0^{2\pi} (4 \sin t \cos t - (-\cos t)(-2 \sin t)) \, dt = \int_0^{2\pi} 2 \sin t \cos t \, dt = \sin^2 t \Big|_0^{2\pi} = 0$.

17.2.61

a. True. This is the definition of an arc length parameterization.

b. True. Let $\mathbf{r}(t) = \langle \cos t, \sin t \rangle$. Then $\mathbf{r}'(t) = \langle -\sin t, \cos t \rangle$, and $\int_C \mathbf{F} \cdot \mathbf{r}'(t) \, dt = \int_0^{2\pi} \langle \sin t, \cos t \rangle \cdot \langle -\sin t, \cos t \rangle \, dt = \int_0^{2\pi} (\cos^2 t - \sin^2 t) \, dt = 0$. $\int_C \mathbf{F} \cdot \mathbf{n} \, ds = \int_0^{2\pi} (\sin t \cdot \cos t - \cos t(-\sin t)) \, dt = 2 \int_0^{2\pi} \sin t \cos t \, dt = 0$.

c. True.

d. True. It is the line integral $\int_C \mathbf{F} \cdot \mathbf{n} \, ds$.

17.2.62 The work done on either path is simply $\int_C \mathbf{F} \cdot \mathbf{r}'(t) \, dt$.

a. Here $\mathbf{r}(t) = \langle -t, 50 \rangle$, for $-100 \leq t \leq 100$, so $\mathbf{r}'(t) = \langle -1, 0 \rangle$ and $\int_C \mathbf{F} \cdot \mathbf{r}'(t) \, dt = \int_{-100}^{100} -150 \, dt = 30000$.

b. Here $\mathbf{r}(t) = \langle 100 \cos t, 100 \sin t \rangle$ and $\mathbf{r}'(t) = \langle -100 \sin t, 100 \cos t \rangle$, so we have $\int_C \mathbf{F} \cdot \mathbf{r}'(t) \, dt = \int_0^\pi -15000 \sin t \, dt = 30000$. The same amount of work is done along each path.

17.2.63

a. For the first path, the work done is $\int_C \mathbf{F} \cdot \mathbf{r}'(t) \, dt = \int_{-100}^{100} -141 \, dt = 28200$, and for the second path $\int_C \mathbf{F} \cdot \mathbf{r}'(t) \, dt = \int_0^\pi (-14100 \sin t - 5000 \cos(t)) \, dt = 28200$, so again they are equal.

b. For the first path, the work done is $\int_C \mathbf{F} \cdot \mathbf{r}'(t) \, dt = \int_{-100}^{100} -141 \, dt = 28200$, while for the second path, $\int_C \mathbf{F} \cdot \mathbf{r}'(t) \, dt = \int_0^\pi (-14100 \sin t - 5000 \cos t) \, dt = 28200$, so the amount of work is still equal along the two paths.

17.2.64

- a. Let $\mathbf{r}(t) = \langle \cos t, \sin t \rangle$ for $0 \leq t \leq 2\pi$ so that $|\mathbf{r}'(t)| = 1$. Then $\int_C f \, ds = \int_0^{2\pi} (\cos t + 2 \sin(t)) \, dt = 0$.
- b. Let $\mathbf{r}(t) = \langle \cos t, -\sin t \rangle$ for $0 \leq t \leq 2\pi$ so that $|\mathbf{r}'(t)| = 1$. Note that this parameterization traces out the unit circle, but clockwise. Then $\int_C f \, ds = \int_0^{2\pi} (\cos t - 2 \sin(t)) \, dt = 0$.
- c. The two integrals are equal.

17.2.65

- a. Let $\mathbf{r}(t) = \langle t, t^2 \rangle$ for $0 \leq t \leq 1$ so that $|\mathbf{r}'(t)| = \sqrt{4t^2 + 1}$, and $\int_C f \, ds = \int_0^1 t \sqrt{4t^2 + 1} \, dt = \frac{5\sqrt{5} - 1}{12}$.
- b. $\mathbf{r}(t) = \langle 1 - t, (1 - t)^2 \rangle$ for $0 \leq t \leq 1$. Then $|\mathbf{r}'(t)| = \sqrt{(-1)^2 + (-2(1 - t))^2} = \sqrt{4t^2 - 8t + 5}$ and $\int_C f \, ds = \int_0^1 (1 - t) \sqrt{4t^2 - 8t + 5} \, dt = \frac{5\sqrt{5} - 1}{12}$.
- c. The two integrals are equal.

17.2.66 Parameterize C_1 by $\mathbf{r}(t) = \langle 1 - t, t \rangle$ for $0 \leq t \leq 1$. Then $\mathbf{r}'(t) = \langle -1, 1 \rangle$, and $\int_{C_1} \mathbf{F} \cdot \mathbf{T} \, ds = \int_0^1 ((-t)(-1) + (1 - t)(1)) \, dt = \int_0^1 1 \, dt = 1$.

Parameterize C_2 by $\mathbf{r}(t) = \langle \cos t, \sin t \rangle$ for $0 \leq t \leq \frac{\pi}{2}$. Then $\mathbf{r}'(t) = \langle -\sin t, \cos t \rangle$, and $\int_{C_2} \mathbf{F} \cdot \mathbf{T} \, ds = \int_0^{\pi/2} ((-\sin t)(-\sin t) + (\cos t)(\cos t)) \, dt = \int_0^{\pi/2} 1 \, dt = \frac{\pi}{2}$.

Finally, parameterize C_3 by the two paths $\mathbf{r}_1(t) = \langle 1 - t, 0 \rangle$ and $\mathbf{r}_2(t) = \langle 0, t \rangle$ for $0 \leq t \leq 1$, so that $\mathbf{r}'_1(t) = \langle -1, 0 \rangle$ and $\mathbf{r}'_2(t) = \langle 0, 1 \rangle$. Then $\int_{C_3} \mathbf{F} \cdot \mathbf{T} \, ds = \int_0^1 ((0)(-1) + (1 - t)(0)) \, dt + \int_0^1 ((-t)(0) + (0)(1)) \, dt = 0$. None of the three path integrals are equal.

17.2.67 Using the same parameterizations as for the previous problem, we have $\int_{C_1} \mathbf{F} \cdot \mathbf{T} \, ds = \int_0^1 ((t)(-1) + (1 - t)(1)) \, dt = \int_0^1 (1 - 2t) \, dt = 0$, $\int_{C_2} \mathbf{F} \cdot \mathbf{T} \, ds = \int_0^{\pi/2} ((\sin t)(-\sin t) + (\cos t)(\cos t)) \, dt = \int_0^{\pi/2} (\cos^2 t - \sin^2 t) \, dt = 0$, $\int_{C_3} \mathbf{F} \cdot \mathbf{T} \, ds = \int_0^1 ((0)(-1) + (1 - t)(0)) \, dt + \int_0^1 (t \cdot 0 + 0 \cdot 1) \, dt = 0$. All three are equal to zero.

17.2.68 We have $dx = dt$, $dy = 2dt$, and $dz = 2t \, dt$. It follows that $\int_C x^2 \, dx + dy + y \, dz = \int_0^3 (t^2 + 2 + 4t^2) \, dt = \int_0^3 (5t^2 + 2) \, dt = \left(\frac{5}{3}t^3 + 2t \right) \Big|_0^3 = 51$.

17.2.69 We have $dx = 2 \, dt$, $dy = \cos t \, dt$, and $dz = -\sin t \, dt$. It follows that $\int_C x^3 y \, dx + xz \, dy + (x + y)^2 \, dz = \int_0^{4\pi} (16t^3 \sin t + 2t \cos^2 t - (2t + \sin t)^2 \sin t) \, dt = 8\pi(48 + 7\pi - 128\pi^2)$, where the integral was calculated with a computer algebra system.

17.2.70 $\mathbf{r}(t) = \langle 3t^2, t \rangle$ for $1 \leq t \leq 3$. Therefore, $\int_C \frac{x^2}{y^4} \, ds = \int_1^3 \frac{9t^4}{t^4} \sqrt{36t^2 + 1} \, dt \approx 216.8$, where the integral was calculated with a computer algebra system.

17.2.71 $\mathbf{r}(t) = \langle 4 \sin t, 4 \cos t \rangle$, for $0 \leq t \leq \pi/2$. Then $\int_C \frac{y}{\sqrt{x^2 + y^2}} dx - \frac{x}{\sqrt{x^2 + y^2}} dy = \int_0^{\pi/2} (4 \cos^2 t + 4 \sin^2 t) dt = 4 \cdot \frac{\pi}{2} = 2\pi$.

17.2.72 $\mathbf{r}(t) = \langle 1 + 2t, 2 + 6t, 4 + 9t \rangle$ for $0 \leq t \leq 1$. So $\int_C (x + y) dx + (x - y) dy + x dz = \int_0^1 2(3 + 8t) + 6(-1 - 4t) + 9(1 + 2t) dt = \int_0^1 (9 + 10t) dt = 14$.

17.2.73

a. $\mathbf{r}'(t) = \langle -2 \sin t, 2 \cos t \rangle$, so the flux along the quarter circle is $\int_C \mathbf{F} \cdot \mathbf{n} ds = \int_0^{\pi/2} ((2 \sin t)(2 \cos t) - (2 \cos t)(-2 \sin t)) dt = \int_0^{\pi/2} 8 \sin t \cos t dt = 4$.

b. With $\mathbf{r}'(t)$ as above, the flux is $\int_C \mathbf{F} \cdot \mathbf{n} ds = \int_{\pi/2}^{\pi} (8 \sin t \cos t) dt = -4$.

c. Both the normal vectors and the vector field $\mathbf{F}$ in the third quadrant are the negatives of their values in the first quadrant, so their dot product is the same. Thus the flux is identical.

d. Both the normal vectors and the vector field $\mathbf{F}$ in the fourth quadrant are the negatives of their values in the second quadrant, so their dot product is the same. Thus the flux is identical.

e. The total flux is $4 - 4 + 4 - 4 = 0$.

17.2.74 Letting $\mathbf{r}(t) = \langle \cos t, \sin t \rangle$ and $\mathbf{r}'(t) = \langle -\sin t, \cos t \rangle$ as usual, we have $\int_C \mathbf{F} \cdot \mathbf{r}'(t) dt = \int_0^{2\pi} (-b \sin^2 t + c \cos^2 t) dt = \pi(c - b)$, so that the circulation is zero only for $b = c$.

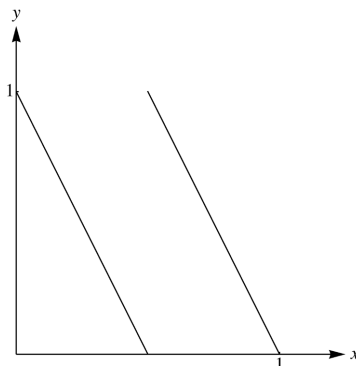
17.2.75 Let $\mathbf{r}(t) = \langle r \cos t, r \sin t \rangle$ for a circle of radius r , so that $\mathbf{r}'(t) = \langle -r \sin t, r \cos t \rangle$. Then $\int_C \mathbf{F} \cdot \mathbf{r}'(t) dt = \int_0^{2\pi} ((ar \cos t + br \sin t)(-r \sin t) + (cr \cos t + dr \sin t)(r \cos t)) dt = r^2 \int_0^{2\pi} (-a \sin t \cos t - b \sin^2 t + c \cos^2 t + d \sin t \cos t) dt = \pi(c - b)r^2$, so that the circulation is zero provided $b = c$.

17.2.76 Let $\mathbf{r}(t) = \langle \cos t, \sin t \rangle$; then the flux is $\int_C \mathbf{F} \cdot \mathbf{n} ds = \int_0^{2\pi} (a \cos t \cos t - d \sin t (-\sin t)) dt = (a + d)\pi$, so the flux is zero if $a = -d$.

17.2.77 Let $\mathbf{r}(t) = \langle r \cos t, r \sin t \rangle$ for a circle of radius r ; then the flux is $\int_C \mathbf{F} \cdot \mathbf{n} ds = \int_0^{2\pi} ((ar \cos t + br \sin t)(r \cos t) - (cr \cos t + dr \sin t)(-r \sin t)) dt = r^2 \int_0^{2\pi} (a \cos^2 t + b \sin t \cos t + c \sin t \cos t + d \sin^2 t) dt = r^2(a + d)\pi$, so the flux is zero provided that $a = -d$.

17.2.78

a.

b. The gradient is $-50\mathbf{i} - 25\mathbf{j}$.c. $\mathbf{F} = 50\mathbf{i} + 25\mathbf{j}$.

d. Parameterize the boundary C by $\mathbf{r}(t) = \langle 1, t \rangle$ for $0 \leq t \leq 1$. Then $\mathbf{r}'(t) = \langle 0, 1 \rangle$ and $\int_C \mathbf{F} \cdot \mathbf{n} \, ds = \int_0^1 (50 \cdot 1 - 25 \cdot 0) \, dt = 50$.

e. Parameterize the boundary C by $\mathbf{r}(t) = \langle 1 - t, 1 \rangle$ for $0 \leq t \leq 1$ (note: we do not use $\langle t, 1 \rangle$ because we need counterclockwise orientation). Then $\mathbf{r}'(t) = \langle -1, 0 \rangle$ and $\int_C \mathbf{F} \cdot \mathbf{n} \, ds = \int_0^1 (50 \cdot 0 - 25 \cdot (-1)) \, dt = 25$.

17.2.79 $\mathbf{r}'(t) = \langle 1, 1, 1 \rangle$ and $|\mathbf{r}'(t)| = t\sqrt{3}$, so the work is $\int_C \mathbf{F} \cdot \mathbf{T} \, ds = \int_1^a \frac{3t}{(t\sqrt{3})^p} dt = 3^{1-p/2} \int_1^a t^{1-p} dt$

a. For $p = 2$, we have $\int_1^a \frac{1}{t} dt = \ln a$.

b. The work is not finite.

c. For $p = 4$, we have $\frac{1}{3} \int_1^a t^{-3} dt = -\frac{1}{6t^2} \Big|_1^a = \frac{1}{6} - \frac{1}{6a^2}$.

d. As $a \rightarrow \infty$, the work approaches $\frac{1}{6}$.

e. For the general $p > 1$, the analysis above shows that the integral is (for $p \neq 2$) $3^{1-p/2} \frac{1}{2-p} t^{2-p} \Big|_1^a = \frac{3^{1-p/2}}{2-p} (-1 + a^{2-p})$ while for $p = 2$ the integral is (from part (a)) $\ln a$.

e. This approaches a limit only for $p > 2$ (when $2 - p < 0$). This limit is $\frac{3^{1-p/2}}{p-2}$.

17.2.80

a. $\int_C f \, dx = \int_a^b (f(t)x'(t) + 0(y'(t))) \, dt = \int_a^b f(t)x'(t) \, dt.$

b. $\int_C g \, dy = \int_a^b (0(x'(t)) + g(t)(y'(t))) \, dt = \int_a^b g(t)y'(t) \, dt.$

$$\text{c. } \int_C xy \, dx = \int_0^1 300t^2 \, dt = 100.$$

$$\text{d. } \int_C xy \, dy = \int_{-1}^1 t^3 \, dt = 0.$$

17.2.81 We use four line segment parameterizations for the rectangle, all for $0 \leq t \leq 1$: $\mathbf{r}_1(t) = \langle at, 0 \rangle$, so $\mathbf{r}'_1(t) = \langle a, 0 \rangle$. $\mathbf{r}_2(t) = \langle a, bt \rangle$, so $\mathbf{r}'_2(t) = \langle 0, b \rangle$. $\mathbf{r}_3(t) = \langle a - at, b \rangle$, so $\mathbf{r}'_3(t) = \langle -a, 0 \rangle$. $\mathbf{r}_4(t) = \langle 0, b - bt \rangle$, so $\mathbf{r}'_4(t) = \langle 0, -b \rangle$. Then

$$\int_C x \, dy = \int_0^1 (at \cdot 0 + a \cdot b + (a - at) \cdot 0 + 0 \cdot (-b)) \, dt = \int_0^1 ab \, dt = ab.$$

17.2.82 Parameterize C by $\mathbf{r}(t) = \langle a \cos t, a \sin t \rangle$, so that $\mathbf{r}'(t) = \langle -a \sin t, a \cos t \rangle$. Then

$$-\int_C y \, dx = -\int_0^{2\pi} (a \sin t \cdot (-a \sin t)) \, dt = a^2 \int_0^{2\pi} \sin^2 t \, dt = \pi a^2.$$

17.3 Conservative Vector Fields

17.3.1 A simple curve has no self-intersections; the initial and terminal points of a closed curve are identical.

17.3.2 A region is connected, roughly speaking, if it consists of one piece. A simply connected region has the property that every closed loop can be contracted to a point.

17.3.3 If $\mathbf{F} = \langle f, g \rangle$ is a vector field in $\mathbb{R}^2$ and if $\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}$, then $\mathbf{F}$ is conservative.

17.3.4 If $\mathbf{F} = \langle f, g, h \rangle$ is a vector field in $\mathbb{R}^3$ and if $\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}$ and $\frac{\partial f}{\partial z} = \frac{\partial h}{\partial x}$ and $\frac{\partial g}{\partial z} = \frac{\partial h}{\partial y}$, then $\mathbf{F}$ is conservative.

17.3.5 Integrate f with respect to x to get an answer where the “constant” is actually a function of y . Take the partial with respect to y and equate with g to compute the constant.

17.3.6 The integral is $\varphi(B) - \varphi(A)$ where $\nabla \varphi = \mathbf{F}$.

17.3.7 The integral is zero.

17.3.8

- There exists a potential function φ such that $\mathbf{F} = \nabla \varphi$.
- $\int_C \mathbf{F} \cdot d\mathbf{r} = \varphi(B) - \varphi(A)$ for all points A and B in R and all smooth oriented curves C from A to B (path independence).
- $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ on all simple smooth closed oriented curves C in R .

17.3.9 Conservative, because $1_y = 1_x = 0$.

17.3.10 Conservative, because $x_y = y_x = 0$.

17.3.11 Not Conservative, because $-y_y = -1$, but $x_x = 1$.

17.3.12 Not Conservative. $-y_y = -1$, but $(x + y)_x = 1$.

17.3.13 Conservative. $\frac{\partial}{\partial y} e^{-x} \cos(y) = -e^{-x} \sin(y) = \frac{\partial}{\partial x} e^{-x} \sin(y)$.

17.3.14 Not Conservative. $\frac{\partial}{\partial y}(2x^3 + xy^2) = 2xy$, and $\frac{\partial}{\partial x}(2y^3 - x^2y) = -2xy$.

17.3.15 Conservative. $f_y = z \cos xz = g_x$, $f_z = yz(-x \sin xz) + (\cos xz)y = -xyz \sin xz + y \cos xz = h_x$, and $g_z = x \cos xz = h_y$.

17.3.16 Not Conservative. Note that $f_z = -ye^{x-z}$ while $h_x = ye^{x-z}$.

17.3.17 Conservative. $\frac{\partial}{\partial y}(x) = \frac{\partial}{\partial x}(y) = 0$. A potential function is $\frac{x^2 + y^2}{2}$.

17.3.18 Conservative. $\frac{\partial}{\partial y}(-y) = \frac{\partial}{\partial x}(-x) = -1$. A potential function is $-xy$.

17.3.19 Not Conservative. $\frac{\partial}{\partial y}(x^3 - xy) = -x$, and $\frac{\partial}{\partial x}\left(\frac{x^2}{2} + y\right) = x$.

17.3.20 Conservative. $\frac{\partial}{\partial y}\left(\frac{x}{x^2 + y^2}\right) = \frac{-2xy}{(x^2 + y^2)^2} = \frac{\partial}{\partial x}\left(\frac{y}{x^2 + y^2}\right)$. Integrating $\frac{x}{x^2 + y^2}$ with respect to x , we see that a potential function is $\frac{1}{2}\ln(x^2 + y^2)$.

17.3.21 Conservative. $\frac{\partial}{\partial y}\left(\frac{x}{\sqrt{x^2 + y^2}}\right) = \frac{-xy}{(x^2 + y^2)^{3/2}} = \frac{\partial}{\partial x}\left(\frac{y}{\sqrt{x^2 + y^2}}\right)$. Integrating $\frac{x}{\sqrt{x^2 + y^2}}$ with respect to x , we obtain a potential function of $\sqrt{x^2 + y^2}$.

17.3.22 Not Conservative. $f_z = 0$ while $h_x = 1$.

17.3.23 Conservative, because the mixed partials are pairwise equal. A potential function is $xz + y$.

17.3.24 Conservative, because the mixed partials are pairwise equal. A potential function is xyz .

17.3.25 Not Conservative. $g_z = e^z$ but $h_y = -e^z$.

17.3.26 Not Conservative. $g_z = -1$ and $h_y = 1$.

17.3.27 Conservative, because the mixed partials are pairwise equal. To find the potential function, integrate $y + z$ with respect to x to get $x(y + z) + f(y, z)$; differentiating with respect to y gives $x + z = x + f_y(y, z)$ so that $f_y(y, z) = z$ and $f(y, z) = yz$. Thus a potential function is $xy + yz + xz$.

17.3.28 Conservative, because the mixed partials are pairwise equal. A potential function is $\frac{1}{2}\ln(x^2 + y^2 + z^2)$.

17.3.29 Conservative, because the mixed partials are pairwise equal. A potential function is $\sqrt{x^2 + y^2 + z^2}$.

17.3.30 Conservative, because the mixed partials are pairwise equal (and zero). A potential function is $\frac{1}{4}x^4 + y^2 - \frac{1}{4}z^4$.

17.3.31

a. $\nabla\varphi = \langle y, x \rangle$, so $\int_C \nabla\varphi \cdot \mathbf{r}'(t) dt = \int_0^\pi \langle \sin t, \cos t \rangle \cdot \langle -\sin t, \cos t \rangle dt = \int_0^\pi \cos(2t) dt = 0$.

b. Because $\nabla\varphi$ is obviously conservative, the integral is simply $\varphi(\cos(\pi), \sin(\pi)) - \varphi(\cos(0), \sin(0)) = 0$.

17.3.32

a. $\nabla\varphi = \langle 1, 3 \rangle$, so $\int_C \nabla\varphi \cdot \mathbf{r}'(t) dt = \int_0^2 \langle 1, 3 \rangle \cdot \langle -1, 1 \rangle dt = \int_0^2 2 dt = 4$.

b. The integral is $\varphi(0, 2) - \varphi(2, 0) = 6 - 2 = 4$. (Note that $\varphi(0, 2)$ is φ evaluated at the point where $t = 2$).

17.3.33

a. $\nabla\varphi = \langle x, y, z \rangle$, so $\int_C \nabla\varphi \cdot \mathbf{r}'(t) dt = \int_0^{2\pi} \langle \cos t, \sin t, \frac{t}{\pi} \rangle \cdot \langle -\sin t, \cos t, \frac{1}{\pi} \rangle dt = \int_0^{2\pi} \left(\frac{t}{\pi^2} \right) dt = 2$.

b. The integral is $\varphi(1, 0, 2) - \varphi(1, 0, 0) = \frac{5}{2} - \frac{1}{2} = 2$.

17.3.34

a. $\nabla\varphi = \langle y + z, x + z, x + y \rangle$, so $\int_C \nabla\varphi \cdot \mathbf{r}'(t) dt = \int_0^4 \langle 5t, 4t, 3t \rangle \cdot \langle 1, 2, 3 \rangle dt = \int_0^4 22t dt = 176$.

b. The integral is $\varphi(4, 8, 12) - \varphi(0, 0, 0) = 176 - 0 = 176$.

17.3.35 $\int_C \mathbf{F} \cdot d\mathbf{r} = \varphi(\mathbf{r}(2)) - \varphi(\mathbf{r}(1)) = \varphi(3, 6) - \varphi(1, 2) = 10 - 7 = 3$.

17.3.36 $\int_C \mathbf{F} \cdot \mathbf{T} ds = \varphi(6, 4) - \varphi(1, 2) = 20 - 7 = 13$.

17.3.37 $\int_C f dx + g dy = \varphi(1, 2) - \varphi(6, 4) + \varphi(3, 6) - \varphi(1, 2) = 7 - 20 + 10 - 7 = -10$.

17.3.38 $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$.

17.3.39 Write $\varphi(x, y) = x^2 + y^2$. Then using the Fundamental Theorem, this integral is equal to $\varphi(3, 4) - \varphi(0, 1) = 25 - 1 = 24$.

17.3.40 Write $\varphi(x, y, z) = x + y + z$. Then using the Fundamental Theorem, this integral is equal to $\varphi(3, 0, 7) - \varphi(1, -1, 2) = 10 - 2 = 8$.

17.3.41 Write $\varphi(x, y) = e^{-x} \cos(y)$. Then using the Fundamental Theorem, this integral is equal to $\varphi(\ln 2, 2\pi) - \varphi(0, 0) = e^{-\ln 2} \cos(2\pi) - e^0 \cos(0) = \frac{1}{2} - 1 = -\frac{1}{2}$.

17.3.42 Write $\varphi(x, y, z) = 1 + x^2 yz$. Then using the Fundamental Theorem, this integral is equal to $\varphi(\cos(8\pi), \sin(8\pi), 4\pi) - \varphi(\cos(0), \sin(0), 0) = \varphi(1, 0, 4\pi) - \varphi(1, 0, 0) = 1 - 1 = 0$.

17.3.43 Let $\varphi(x, y, z) = x \cos(z - 2y)$. The given integral is equal to $\varphi(\pi^2, \pi, \pi) - \varphi(0, 0, 0) = -\pi^2 - 0 = -\pi^2$.

17.3.44 Let $\varphi(x, y) = ye^x$. The given integral is equal to $\varphi(4, 9) - \varphi(0, 1) = 9e^4 - 1$.

17.3.45 Parameterize C by $\mathbf{r}(t) = \langle 4 \cos t, 4 \sin t \rangle$, $0 \leq t \leq 2\pi$. Then $\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \langle 4 \cos t, 4 \sin t \rangle \cdot \langle -4 \sin t, 4 \cos t \rangle dt = \int_0^{2\pi} 0 dt = 0$.

17.3.46 Parameterize C by $\mathbf{r}(t) = \langle 8 \cos t, 8 \sin t \rangle$, $0 \leq t \leq 2\pi$. Then $\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \langle 8 \sin t, 8 \cos t \rangle \cdot \langle -8 \sin t, 8 \cos t \rangle dt = 8 \int_0^{2\pi} (\cos^2 t - \sin^2 t) dt = 0$.

17.3.47 Parameterize C by three paths, all for $0 \leq t \leq 1$: $\mathbf{r}_1(t) = \langle t, t-1 \rangle$, so $\mathbf{r}'_1(t) = \langle 1, 1 \rangle$. $\mathbf{r}_2(t) = \langle 1-t, t \rangle$, so $\mathbf{r}'_2(t) = \langle -1, 1 \rangle$. $\mathbf{r}_3(t) = \langle 0, 1-2t \rangle$, so $\mathbf{r}'_3(t) = \langle 0, -2 \rangle$. Then $\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 \langle t, t-1 \rangle \cdot \langle 1, 1 \rangle dt + \int_0^1 \langle 1-t, t \rangle \cdot \langle -1, 1 \rangle dt + \int_0^1 \langle 0, 1-2t \rangle \cdot \langle 0, -2 \rangle dt = \int_0^1 (t + (t-1) + (t-1) + t - 2(1-2t)) dt = \int_0^1 (8t-4) dt = 0$.

17.3.48 Parameterize C by $\mathbf{r}(t) = \langle 3 \cos t, 3 \sin t \rangle$, $0 \leq t \leq 2\pi$. Then $\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \langle 3 \sin t, -3 \cos t \rangle \cdot \langle -3 \sin t, 3 \cos t \rangle dt = -9 \int_0^{2\pi} 1 dt = -18\pi$. This integral is not zero because the vector field $\langle y, -x \rangle$ is not conservative: $\frac{\partial}{\partial y}(y) = 1$, while $\frac{\partial}{\partial x}(-x) = -1$.

17.3.49 Using the given parameterization,

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \langle \cos t, \sin t, 2 \rangle \cdot \langle -\sin t, \cos t, 0 \rangle dt = \int_0^{2\pi} 0 dt = 0.$$

17.3.50 Using the given parameterization,

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \langle \sin t - \cos t, 0, \cos t - \sin t \rangle \cdot \langle -\sin t, \cos t, -\sin t \rangle dt = \int_0^{2\pi} 0 dt = 0.$$

17.3.51 $\oint_{C_1} \mathbf{F} \cdot d\mathbf{r} = 2 - 7 = -5$.

17.3.52 $\oint_{C_2} \mathbf{F} \cdot d\mathbf{r} = 9 - 0 = 9$.

17.3.53 $\mathbf{F}$ is a conservative vector field; a potential function can be found by integrating x^2 with respect to y to obtain $x^2y + f(x, z)$; differentiate with respect to z to get $f_z(x, z) = 2xz$, so that $f(x, z) = xz^2 + g(x)$. Thus the potential function is $x^2y + xz^2 + g(x)$; differentiating with respect to x gives $2xy + z^2 + g_x(x) = 2xy + z^2$, so that $g_x(x) = 0$ and we may take $g(x) = 0$. So if $\varphi = x^2y + xz^2$, then $\nabla\varphi = \mathbf{F}$ and thus the integral is zero because both sine and cosine, and thus φ , have the same values at the two endpoints of C .

17.3.54 If $x = x(t)$, $y = y(t)$, then $\oint_C e^{-x}(\cos y dx + \sin y dy) = \oint_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F} = \langle e^{-x} \cos y, e^{-x} \sin y \rangle$.

This is a conservative vector field, because $\frac{\partial}{\partial y}(e^{-x} \cos y) = -e^{-x} \sin y = \frac{\partial}{\partial x}(e^{-x} \sin y)$, so that the integral around the closed curve is zero.

17.3.55 The value of the integral is $\sin xy \Big|_{(0,0)}^{(2,\pi/4)} = \sin \pi/2 - \sin 0 = 1$.

17.3.56 Let $\varphi(x, y) = \frac{x^4}{4} + \frac{y^4}{4}$. The value of the integral is $\varphi(2, 0) - \varphi(1, 1) = 4 - \frac{1}{2} = \frac{7}{2}$.

17.3.57

a. False. Parametrize the curve by $x = 4 \cos t + 1$, $y = 4 \sin t$, for $0 \leq t \leq 2\pi$. Then $\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \langle -4 \sin t, 4 \cos t + 1 \rangle \cdot \langle -4 \sin t, 4 \cos t \rangle dt = \int_0^{2\pi} (16 \sin^2 t + 16 \cos^2 t + 4 \cos t) dt = 32\pi \neq 0$.

b. True. This is because $\mathbf{F}$ is conservative.

c. True. If the vector field is $\langle a, b \rangle$, then a potential function is $ax + by$.

d. True. This is because $\frac{\partial}{\partial y}f(x) = \frac{\partial}{\partial x}g(y) = 0$.

e. True. This follows from the definitions.

17.3.58 $\oint_C ds$ is the length of the curve C , which is 2π . The other two integrals are zero, because they are the same as integrating the conservative vector fields $\langle 1, 0, 0 \rangle$, and $\langle 0, 1, 0 \rangle$ respectively around a closed curve.

17.3.59 This is a conservative vector field with potential function $\varphi(x, y) = \frac{1}{2}x^2 + 2y$, so the work is $\varphi(2, 4) - \varphi(0, 0) = 10$.

17.3.60 This is a conservative vector field with potential function $\varphi(x, y) = \frac{1}{2}(x^2 + y^2)$, so the work is $\varphi(3, -6) - \varphi(1, 1) = \frac{45}{2} - 1 = \frac{43}{2}$.

17.3.61 This is a conservative vector field with potential function $\varphi(x, y, z) = \frac{1}{2}(x^2 + y^2 + z^2)$, so the work is $\varphi(2, 4, 6) - \varphi(1, 2, 1) = 28 - 3 = 25$.

17.3.62 This is not a conservative vector field, because for example $\frac{\partial}{\partial z}(e^{x+y}) = 0$, while $\frac{\partial}{\partial x}(ze^{x+y}) = ze^{x+y}$. Parameterize C by $\langle -t, 2t, -4t \rangle$, $0 \leq t \leq 1$; then $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 \langle e^t, e^t, -4te^t \rangle \cdot \langle -1, 2, -4 \rangle dt = \int_0^1 (e^t + 16te^t) dt = e + 15$.

17.3.63

a. Negative.

b. Positive.

c. No. The value of the integral is not zero, so $\mathbf{F}$ is not conservative.

17.3.64 No.

17.3.65 $\mathbf{F} = \langle a, b, c \rangle$ is a conservative force field with potential function $\varphi(x, y, z) = ax + by + cz$, so the work done is $\varphi(B) - \varphi(A) = \mathbf{F} \cdot B - \mathbf{F} \cdot A = \mathbf{F} \cdot (B - A) = \mathbf{F} \cdot \overrightarrow{AB}$.

17.3.66

a. The acceleration is the time derivative of velocity, so Newton's second law says that $m \frac{d\mathbf{v}}{dt} = m\mathbf{a} = \mathbf{F} = -\nabla\varphi$

b. By the product rule, $\frac{d}{dt}(\mathbf{v} \cdot \mathbf{v}) = \frac{d\mathbf{v}}{dt} \cdot \mathbf{v} + \mathbf{v} \cdot \frac{d\mathbf{v}}{dt} = 2 \frac{d\mathbf{v}}{dt} \cdot \mathbf{v}$, and the desired equation follows.

c. Multiplying part (a) by $\mathbf{v} = \mathbf{r}'$ and using (b), we have (letting C be a path from A to B) $m \frac{d\mathbf{v}}{dt} \cdot \mathbf{v} = -\nabla\varphi \cdot \mathbf{r}' = \frac{1}{2}m \frac{d}{dt}(\mathbf{v} \cdot \mathbf{v})$.

Thus, $\int_C -\nabla\varphi \cdot \mathbf{r}' = \frac{1}{2}m \int_A^B \frac{d}{dt}(\mathbf{v}(t) \cdot \mathbf{v}(t)) dt$, so $\varphi(A) - \varphi(B) = \frac{1}{2}m(|\mathbf{v}|^2)|_A^B$. The last equality follows because the integrand is a conservative vector field. Thus $\frac{1}{2}m|\mathbf{v}(B)|^2 - \frac{1}{2}m|\mathbf{v}(A)|^2 = \varphi(A) - \varphi(B)$, so $\frac{1}{2}m|\mathbf{v}(B)|^2 + \varphi(B) = \frac{1}{2}m|\mathbf{v}(A)|^2 + \varphi(A)$.

17.3.67

- a. Away from the origin (where the denominator of the force field equation is undefined), the force field is conservative because, for example,

$$\frac{\partial}{\partial y} GMm \frac{x}{(x^2 + y^2 + z^2)^{3/2}} = GMm \frac{2xy}{(x^2 + y^2 + z^2)^{5/2}} = \frac{\partial}{\partial x} GMm \frac{y}{(x^2 + y^2 + z^2)^{3/2}}.$$

- b. A potential function for the force field is $\varphi(x, y, z) = GMm (x^2 + y^2 + z^2)^{-1/2} = GMm \frac{1}{|\mathbf{r}|}$.
- c. The work done in moving the point from A to B , because the force field is conservative, is $\varphi(B) - \varphi(A) = GMm \left(\frac{1}{|B|} - \frac{1}{|A|} \right) = GMm \left(\frac{1}{r_2} - \frac{1}{r_1} \right)$.
- d. Because the field is conservative, the work done does not depend on the path.

17.3.68 This vector field is $\mathbf{F} = \langle x, y, z \rangle (x^2 + y^2 + z^2)^{-p/2}$, so away from the origin is conservative with potential function $\varphi(x, y, z) = \frac{1}{2-p} (x^2 + y^2 + z^2)^{1-p/2}$ as long as $p \neq 2$. When $p = 2$, the potential function is $\varphi = \frac{1}{2} \ln(x^2 + y^2 + z^2)$. The field is conservative at the origin if it is defined and if its potential function is defined, i.e. if both $-\frac{p}{2}$ and $1 - \frac{p}{2}$ are nonnegative, which happens only if $p \leq 0$.

17.3.69

- a. This field is

$$\mathbf{F} = \langle -y, x \rangle (x^2 + y^2)^{-p/2},$$

and we have

$$\begin{aligned} \frac{\partial}{\partial y} \left(-y (x^2 + y^2)^{-p/2} \right) &= - (x^2 + y^2)^{-p/2} + py^2 \frac{(x^2 + y^2)^{-p/2}}{x^2 + y^2} \\ &= - (x^2 + y^2)^{-p/2} + py^2 (x^2 + y^2)^{-1-p/2} = \frac{-x^2 + (p-1)y^2}{(x^2 + y^2)^{1+p/2}}. \end{aligned}$$

$$\begin{aligned} \frac{\partial}{\partial x} \left(x (x^2 + y^2)^{-p/2} \right) &= (x^2 + y^2)^{-p/2} - px^2 \frac{(x^2 + y^2)^{-p/2}}{x^2 + y^2} \\ &= (x^2 + y^2)^{-p/2} - px^2 (x^2 + y^2)^{-1-p/2} = \frac{-(p-1)x^2 + y^2}{(x^2 + y^2)^{1+p/2}}. \end{aligned}$$

For the force field to be conservative, these two would have to be equal. However, their difference is $\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 2 (x^2 + y^2)^{-p/2} - p (x^2 + y^2) (x^2 + y^2)^{-1-p/2} = 2 (x^2 + y^2)^{-p/2} - p (x^2 + y^2)^{-p/2} = (2-p) (x^2 + y^2)^{-p/2}$ which is in general nonzero.

- b. From the above formula, if $p = 2$, then the mixed partials are equal, so that $\mathbf{F}$ is conservative.
- c. For $p = 2$, $\mathbf{F} = \frac{1}{x^2 + y^2} \langle -y, x \rangle$. Integrating the x component of $\mathbf{F}$ with respect to x gives $\varphi = \tan^{-1} \left(\frac{y}{x} \right)$.

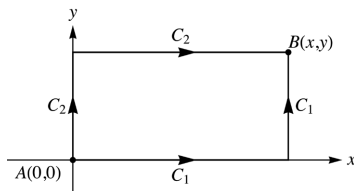
17.3.70

- a. Because $\frac{\partial}{\partial y} (ax + by) = b$ and $\frac{\partial}{\partial x} (cx + dy) = c$, the field is conservative when $b = c$.

b. Because $\frac{\partial}{\partial y}(ax^2 - by^2) = -2by$ and $\frac{\partial}{\partial x}(cxy) = cy$, the field is conservative when $c = -2b$.

17.3.71

a.



b. Parameterize C_1 by two paths: $\mathbf{r}_1(t) = \langle t, 0 \rangle$, $0 \leq t \leq x$, and $\mathbf{r}_2(t) = \langle x, t \rangle$, $0 \leq t \leq y$. Then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^x \langle 2t, -t \rangle \cdot \langle 1, 0 \rangle dt + \int_0^y \langle 2x - t, -x + 2t \rangle \cdot \langle 0, 1 \rangle dt = \int_0^x 2t dt + \int_0^y (2t - x) dt = x^2 + y^2 - xy.$$

c. Parameterize C_2 by the paths $\mathbf{r}_1(t) = \langle 0, t \rangle$, $0 \leq t \leq y$, and $\mathbf{r}_2(t) = \langle t, y \rangle$, $0 \leq t \leq x$. Then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^y \langle -t, 2t \rangle \cdot \langle 0, 1 \rangle dt + \int_0^x \langle 2t - y, -t + 2y \rangle \cdot \langle 1, 0 \rangle dt = \int_0^y 2t dt + \int_0^x (2t - y) dt = x^2 + y^2 - xy.$$

17.3.72 Using problem 71, we have the same paths $\mathbf{r}_1$, $\mathbf{r}_2$, and $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^x \langle 0, -t \rangle \cdot \langle 1, 0 \rangle dt + \int_0^y \langle -t, -x \rangle \cdot \langle 0, 1 \rangle dt = \int_0^x 0 dt + \int_0^y (-x) dt = -xy$.

17.3.73 Using problem 71 and the same paths $\mathbf{r}_1$, $\mathbf{r}_2$, we have $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^x \langle t, 0 \rangle \cdot \langle 1, 0 \rangle dt + \int_0^y \langle x, t \rangle \cdot \langle 0, 1 \rangle dt = \int_0^x t dt + \int_0^y t dt = \frac{1}{2}(x^2 + y^2)$.

17.3.74 Using problem 71 and the same paths $\mathbf{r}_1$, $\mathbf{r}_2$, note that $\mathbf{F} = \frac{\mathbf{r}}{|\mathbf{r}|} = \langle x, y \rangle (x^2 + y^2)^{-1/2}$, and $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^x \langle 1, 0 \rangle \cdot \langle 1, 0 \rangle dt + \int_0^y \left\langle \frac{x}{\sqrt{x^2 + t^2}}, \frac{t}{\sqrt{x^2 + t^2}} \right\rangle \cdot \langle 0, 1 \rangle dt = \int_0^x 1 dt + \int_0^y \frac{t}{\sqrt{x^2 + t^2}} dt = x + \sqrt{x^2 + y^2} - x = \sqrt{x^2 + y^2}$.

17.3.75 Using problem 71 and the same paths $\mathbf{r}_1$, $\mathbf{r}_2$, we have $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^x \langle 2t^3, 0 \rangle \cdot \langle 1, 0 \rangle dt + \int_0^y \langle 2x^3 + xt^2, 2t^3 + x^2t \rangle \cdot \langle 0, 1 \rangle dt = \int_0^x 2t^3 dt + \int_0^y (2t^3 + x^2t) dt = \frac{1}{2}(x^4 + x^2y^2 + y^4)$.

17.4 Green's Theorem

17.4.1 As with the Fundamental Theorem of Calculus, it allows evaluation of the integral of a derivative by looking at the value of the underlying function on the boundary of a region (or, in the case of the Fundamental Theorem, an interval).

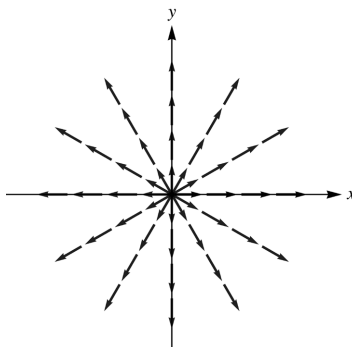
17.4.2 The line integral for flux corresponds to the double integral of the divergence; the line integral for circulation to the double integral of the curl.

17.4.3 The area is $\frac{1}{2} \oint_C (x dy - y dx)$ where C is the boundary of the region.

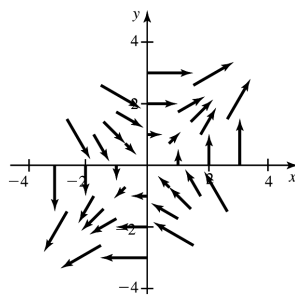
17.4.4 Because the curl being zero is an equivalent condition to the field being conservative.

17.4.5 Because the flux is the integrand in Green's theorem, so the integral vanishes.

17.4.6 A conservative vector field such as $\langle x, y \rangle$ will have zero curl:



17.4.7



17.4.8 A conservative and a source-free field each have functions (a potential function in the case of a conservative field; a stream function in the case of a source-free field) that closely reflect the vector field. The properties of the partials of these functions are such that the curl (or divergence, for a source-free field) vanish.

17.4.9

- a. The two-dimensional curl is $\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} = 0 - 0 = 0$.
- b. The two-dimensional divergence is $\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} = 1 + 1 = 2$.
- c. The vector field is irrotational because its curl is zero.
- d. The vector field is not source free because the divergence is not zero.

17.4.10

- a. The two-dimensional curl is $\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} = -1 - 1 = -2$.
- b. The two-dimensional divergence is $\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} = 0 + 0 = 0$.
- c. The vector field is not irrotational because its curl is not zero.
- d. The vector field is source free because the divergence is zero.

17.4.11

- a. The two-dimensional curl is $\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} = -3 - 1 = -4$.
- b. The two-dimensional divergence is $\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} = 0 + 0 = 0$.
- c. The vector field is not irrotational because its curl is not zero.
- d. The vector field is source free because the divergence is zero.

17.4.12

- a. The two-dimensional curl is $\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} = -2y - 2x$
- b. The two-dimensional divergence is $\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} = 2x + 2y - 2x - 2y = 0$.
- c. The vector field is not irrotational because its curl is not zero.
- d. The vector field is source free because the divergence is zero.

17.4.13

- a. The two-dimensional curl is $\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} = y^2 + 4x^3 - 4x^3 = y^2$.
- b. The two-dimensional divergence is $\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} = 12x^2y + 2xy$.
- c. The vector field is not irrotational because its curl is not zero.
- d. The vector field is not source free because the divergence is not zero.

17.4.14

- a. The two-dimensional curl is $\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} = 12y - 1$.
- b. The two-dimensional divergence is $\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} = 12x^2 + 12x$.
- c. The vector field is not irrotational because its curl is not zero.
- d. The vector field is not source free because the divergence is not zero.

17.4.15

- a. The two-dimensional curl is $\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} = 1 - 0 = 1$, so $\mathbf{F}$ is not irrotational.
- b. $\mathbf{r}_1(t) = \langle t, t^2 \rangle$ for $0 \leq t \leq 1$ and $\mathbf{r}_2(t) = \langle 1 - t, 1 - t \rangle$ for $0 \leq t \leq 1$.
- c.
$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_{C_1} \mathbf{F} \cdot d\mathbf{r} + \oint_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_0^1 (2t^2 + 1) dt + \int_0^1 (t - 2) dt = \left(\frac{2t^3}{3} + t \right) \Big|_0^1 + \left(\frac{t^2}{2} - 2t \right) \Big|_0^1 = \frac{5}{3} - \frac{3}{2} = \frac{1}{6}.$$

$$\int_0^1 \int_{x^2}^x 1 dy dx = \int_0^1 (x - x^2) dx = \left(\frac{x^2}{2} - \frac{x^3}{3} \right) \Big|_0^1 = \frac{1}{6}.$$
- d. The two-dimensional divergence is $0 + 0 = 0$, $\iint_R 0 dA = 0$.

17.4.16

- a. The two-dimensional curl is $\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} = 0 - 0 = 0$.
- b. The two-dimensional divergence is $1 + 0 = 1$. The vector field is not source free.
- c. $\mathbf{r}_1(t) = \langle t, 2t^2 - 2t \rangle$ for $0 \leq t \leq 1$ and $\mathbf{r}_2(t) = \langle 1 - t, 0 \rangle$ for $0 \leq t \leq 1$.
- d.
$$\oint_C \mathbf{F} \cdot \mathbf{n} \, ds = \oint_{C_1} \mathbf{F} \cdot \mathbf{n} \, ds + \oint_{C_2} \mathbf{F} \cdot \mathbf{n} \, ds = \int_0^1 (4t^2 - 2t - 1) \, dt + \int_0^1 dt = \left(\frac{4}{3}t^3 - t^2 - t \right) \Big|_0^1 + 1 = -\frac{2}{3} + 1 = \frac{1}{3}.$$
$$\int_0^1 \int_{2x^2-2x}^0 dy \, dx = 2 \int_0^1 (x - x^2) \, dx = 2 \left(\frac{x^2}{2} - \frac{x^3}{3} \right) \Big|_0^1 = \frac{1}{3}.$$

17.4.17

- a. The curl is $\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} = -2 - 2 = -4$.
- b.
$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^\pi \langle 0, -2t \rangle \cdot \langle 1, 0 \rangle \, dt + \int_\pi^0 \langle 2 \sin t, -2t \rangle \cdot \langle 1, \cos t \rangle \, dt = \int_\pi^0 (2 \sin t - 2t \cos t) \, dt = -8.$$
$$\iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dA = \iint_R (-4) \, dA = -4 \int_0^\pi \sin x \, dx = -8.$$

17.4.18

- a. The curl is $\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} = 3 + 3 = 6$.
- b.
$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 \langle 0, 3t \rangle \cdot \langle 1, 0 \rangle \, dt + \int_0^1 \langle -6t, 3 - 3t \rangle \cdot \langle -1, 2 \rangle \, dt + \int_0^1 \langle -3(2 - 2t), 0 \rangle \cdot \langle 0, -2 \rangle \, dt =$$
$$\int_0^1 (6t + 6 - 6t) \, dt = 6.$$
$$\iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dA = \iint_R 6 \, dA = 6 \int_0^1 (2 - 2x) \, dx = 6.$$

17.4.19

- a. The curl is $\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} = 2x + 2x = 4x$.
- b.
$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^2 \langle 0, t^2 \rangle \cdot \langle 1, 0 \rangle \, dt + \int_2^0 \langle -2t^2(2-t), t^2 \rangle \cdot \langle 1, 2-2t \rangle \, dt = \int_2^0 (-2t^2(2-t) + t^2(2-2t)) \, dt =$$
$$\int_2^0 (-4t^2 + 2t^3 + 2t^2 - 2t^3) \, dt = \int_2^0 -2t^2 \, dt = -\frac{2t^3}{3} \Big|_2^0 = \frac{16}{3}.$$
$$\iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dA = \iint_R 4x \, dA = \int_0^2 \int_0^{2x-x^2} 4x \, dy \, dx = \int_0^2 (8x^2 - 4x^3) \, dx = \left(\frac{8x^3}{3} - x^4 \right) \Big|_0^2 =$$
$$\frac{64}{3} - 16 = \frac{16}{3}.$$

17.4.20

- a. The curl is $\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} = 2x$.

$$\begin{aligned} \text{b. } \int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^{2\pi} \langle 0, \sin^2 t + \cos^2 t \rangle \cdot \langle -\sin t, \cos t \rangle dt = \int_0^{2\pi} \cos t dt = 0. \\ \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dA &= \iint_R (2x) dA = \int_0^{2\pi} \int_0^1 2r \cos \theta r dr d\theta = 0. \end{aligned}$$

17.4.21 Parameterize the boundary by $x = 5 \cos t$, $y = 5 \sin t$, $0 \leq t \leq 2\pi$. Then $dx = -5 \sin t dt$, $dy = 5 \cos t dt$, and the area is $\frac{1}{2} \oint_C x dy - y dx = \frac{1}{2} \int_0^{2\pi} ((5 \cos t) \cdot (5 \cos t) - (5 \sin t) \cdot (-5 \sin t)) dt = \frac{25}{2} \int_0^{2\pi} 1 dt = 25\pi$.

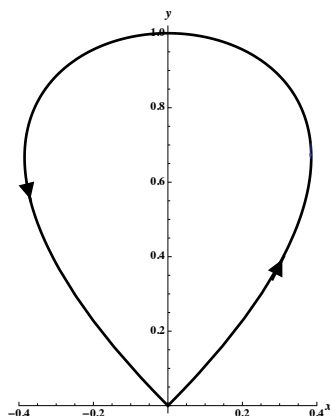
17.4.22 Parameterize the boundary by $x = 6 \cos t$, $y = 4 \sin t$, $0 \leq t \leq 2\pi$. Then $dx = -6 \sin t dt$, $dy = 4 \cos t dt$, and the area is $\frac{1}{2} \oint_C x dy - y dx = \frac{1}{2} \int_0^{2\pi} ((6 \cos t) \cdot (4 \cos t) - (4 \sin t) \cdot (-6 \sin t)) dt = 12 \int_0^{2\pi} 1 dt = 24\pi$.

17.4.23 Parameterize the boundary by $x = 4 \cos t$, $y = 4 \sin t$, $0 \leq t \leq 2\pi$. Then $dx = -4 \sin t dt$, $dy = 4 \cos t dt$, and the area is $\frac{1}{2} \oint_C x dy - y dx = \frac{1}{2} \int_0^{2\pi} ((4 \cos t) \cdot (4 \cos t) - (4 \sin t) \cdot (-4 \sin t)) dt = 8 \int_0^{2\pi} 1 dt = 16\pi$.

17.4.24 Note that C_1 can be parameterized by $x = -\frac{\sqrt{2}}{2}(1-t) + \frac{\sqrt{2}}{2}t$ and $y = \frac{\sqrt{2}}{2}$, $0 \leq t \leq 1$ while C_2 can be parametrized by $x = \cos t$ and $y = \sin t$ for $-\frac{\pi}{4} \leq t \leq \frac{\pi}{4}$. For C_1 we have $dy = 0$ and $dx = \sqrt{2} dt$. Thus $\frac{1}{2} \oint_{C_1} x dy - y dx = \frac{1}{2} \int_0^1 -\frac{\sqrt{2}}{2} \cdot \sqrt{2} dt = -\frac{1}{2}$. For C_2 , we have $dx = -\sin t dt$ and $dy = \cos t dt$, and we have $\frac{1}{2} \oint_{C_2} x dy - y dx = \frac{1}{2} \int_{-\pi/4}^{\pi/4} dt = \frac{\pi}{4}$. Thus the area is $\frac{\pi}{4} - \frac{1}{2}$. As a quick check, note that the region could be thought of as a quarter circle of radius one minus a triangle with area $\frac{1}{2}$.

17.4.25 Traverse the first path from -2 to 2 , then the second path back from 2 to -2 . The area is then $\frac{1}{2} \oint_C x dy - y dx = \frac{1}{2} \int_{-2}^2 (t \cdot (4t) - (2t^2) \cdot 1) dt + \frac{1}{2} \int_2^{-2} (t \cdot (-2t) - (12 - t^2) \cdot 1) dt = \frac{1}{2} \int_{-2}^2 2t^2 dt + \frac{1}{2} \int_2^{-2} (-2t^2 - 12 + t^2) dt = 32$.

17.4.26



We have $dx = (1 - t^2 + t \cdot (-2t)) dt = (1 - 3t^2) dt$ and $dy = -2t dt$. We parameterize the curve from $t = 1$ to $t = -1$ so that we traverse the region counterclockwise; then the area is $\frac{1}{2} \oint_C x dy - y dx = \frac{1}{2} \int_1^{-1} (t(1 - t^2)(-2t) - (1 - t^2)(1 - 3t^2)) dt = \frac{8}{15}$.

17.4.27

- a. The divergence is $\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} = 1 + 1 = 2$.
- b. $\int_C \mathbf{F} \cdot \mathbf{n} ds = 4 \int_0^{2\pi} (\cos t (\cos t) - \sin t (-\sin t)) dt = 8\pi$.
- $$\iint_R \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} \right) dA = \iint_R (2) dA = 2 \cdot 4\pi = 8\pi.$$

17.4.28

- a. The divergence is $\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} = 1 - 3 = -2$.
- b. Parameterize the triangle by the three paths: $\mathbf{r}_1(t) = \langle t, 2t \rangle$, $\mathbf{r}_2(t) = \langle 1-t, 2 \rangle$, and $\mathbf{r}_3(t) = \langle 0, 2-2t \rangle$, all for $0 \leq t \leq 1$. Then $\int_C \mathbf{F} \cdot \mathbf{n} ds = \int_0^1 (2t + 3(2t)) dt + \int_0^1 (0 - 6) dt + \int_0^1 (0 + 0) dt = \int_0^1 (8t - 6) dt = (4t^2 - 6t) \Big|_0^1 = -2$.
- $$\iint_R \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} \right) dA = \int_0^1 \int_{2x}^2 (-2) dy dx = \int_0^1 (-4 + 4x) dx = (-4x + 2x^2) \Big|_0^1 = -2.$$

17.4.29

- a. The divergence is $\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} = 2y + 0 = 2y$.
- b. Parameterize the region by $\mathbf{r}_1(t) = \langle t, 0 \rangle$ for $0 \leq t \leq 2$ and $\mathbf{r}_2(t) = \langle t, 2t - t^2 \rangle$, for $0 \leq t \leq 2$ traversed backwards (from $t = 2$ to $t = 0$.) Then $\int_C \mathbf{F} \cdot \mathbf{n} ds = \int_0^2 (0 - t^2) dt + \int_2^0 (2t(2t - t^2)(2 - 2t) - t^2) dt = \int_0^2 (-8t^2 + 12t^3 - 4t^4) dt = \left(-\frac{8t^3}{3} + 3t^4 - \frac{4t^5}{5} \right) \Big|_0^2 = \frac{16}{15}$.
- $$\iint_R \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} \right) dA = \int_0^2 \int_0^{2x-x^2} 2y dy dx = \int_0^2 (2x - x^2)^2 dx = \int_0^2 (4x^2 - 4x^3 + x^4) dx = \left(\frac{4x^3}{3} - x^4 + \frac{x^5}{5} \right) \Big|_0^2 = \frac{32}{3} - 16 + \frac{32}{5} = \frac{16}{15}.$$

17.4.30

- a. The divergence is $\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} = 2x$.
- b. $\int_C \mathbf{F} \cdot \mathbf{n} ds = \int_0^{2\pi} ((\sin^2 t + \cos^2 t) \cdot \cos t + 0 \cdot \sin t) dt = \int_0^{2\pi} \cos t dt = 0$.
- $$\iint_R \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} \right) dA = \iint_R 2x dA = \int_0^{2\pi} \int_0^1 2 \cos \theta \cdot r dr d\theta = 0.$$

$$\mathbf{17.4.31} \quad \oint_C \langle 3y + 1, 4x^2 + 3 \rangle \cdot d\mathbf{r} = \int_0^2 \int_0^4 (8x - 3) dx dy = \int_0^2 (4x^2 - 3x) \Big|_0^4 dx = \int_0^2 52 dx = 104.$$

$$\mathbf{17.4.32} \quad \oint_C \langle \sin y, x \rangle \cdot d\mathbf{r} = \int_0^{\pi/2} \int_0^x (1 - \cos y) dy dx = \int_0^{\pi/2} (y - \sin y) \Big|_0^x dx = \int_0^{\pi/2} (x - \sin x) dx = \left(\frac{x^2}{2} + \cos x \right) \Big|_0^{\pi/2} = \frac{\pi^2}{8} - 1.$$

$$\mathbf{17.4.33} \quad \oint_C x e^y dx + x dy = \int_0^2 \int_0^{x^2} (1 - x e^y) dy dx = \int_0^2 (y - x e^y) \Big|_0^{x^2} dx = \int_0^2 (x^2 + x - x e^{x^2}) dx = \left(\frac{x^3}{3} + \frac{x^2}{2} - \frac{1}{2} e^{x^2} \right) \Big|_0^2 = \frac{8}{3} + 2 - \frac{1}{2} e^4 + \frac{1}{2} = \frac{31 - 3e^4}{6}.$$

$$\mathbf{17.4.34} \quad \oint_C \frac{1}{1+y^2} dx + y dy = \int_0^1 \int_0^x \left(\frac{2y}{(1+y^2)^2} \right) dy dx = \int_0^1 \left(-\frac{1}{1+y^2} \right) \Big|_0^x dx = \int_0^1 \left(1 - \frac{1}{1+x^2} \right) dx = (x - \tan^{-1} x) \Big|_0^1 = 1 - \frac{\pi}{4} = \frac{4-\pi}{4}.$$

17.4.35 The line integral, using the flux form of Green's theorem, is equal to

$$\begin{aligned} \iint_R \left(\frac{\partial}{\partial x} (2x + e^{y^2}) + \frac{\partial}{\partial y} (4y^2 + e^{x^2}) \right) dA &= \iint_R (2 + 8y) dA = \int_0^1 \int_0^1 (2 + 8y) dx dy \\ &= \int_0^1 (2 + 8y) dy = 6. \end{aligned}$$

17.4.36 Using the flux form of Green's theorem, the integral is equal to

$$\iint_R \left(\frac{\partial}{\partial x} (2x - 3y) + \frac{\partial}{\partial y} (3x + 4y) \right) dA = \iint_R 6 dA = 6 \times \text{area of } R = 6\pi.$$

17.4.37 Using the flux form of Green's theorem, the integral is equal to

$$\iint_R \left(\frac{\partial}{\partial x} (0) + \frac{\partial}{\partial y} (xy) \right) dA = \int_0^2 \int_0^{4-2x} x dy dx = \int_0^2 (4x - 2x^2) dx = \frac{8}{3}.$$

17.4.38 Using the flux form of Green's theorem, the integral is equal to (note the leading minus sign to correct for the orientation)

$$\begin{aligned} - \iint_R \left(\frac{\partial}{\partial x} (x^2) + \frac{\partial}{\partial y} (2y^2) \right) dA &= - \int_{-1}^1 \int_0^{\sqrt{1-x^2}} (2x + 4y) dy dx \\ &= - \int_{-1}^1 (2xy + 2y^2) \Big|_{y=0}^{y=\sqrt{1-x^2}} dx = - \int_{-1}^1 (2x\sqrt{1-x^2} + 2 - 2x^2) dx = -\frac{8}{3}. \end{aligned}$$

17.4.39 Using the circulation form of Green's theorem, the integral is equal to

$$\iint_R \left(\frac{\partial}{\partial x} (4x + y^3) - \frac{\partial}{\partial y} (x^2 + y^2) \right) dA = \iint_R (4 - 2y) dA = \int_0^\pi \int_0^{\sin(x)} (4 - 2y) dy dx = 8 - \frac{\pi}{2}.$$

17.4.40 Using the flux form of Green's theorem, the integral is equal to

$$\begin{aligned} \iint_R \left(\frac{\partial}{\partial x} (e^{x-y}) + \frac{\partial}{\partial y} (e^{y-x}) \right) dA &= \iint_R (e^{x-y} + e^{y-x}) dA = \int_0^1 \int_0^x (e^{x-y} + e^{y-x}) dy dx \\ &= \int_0^1 (-e^{x-y} + e^{y-x}) \Big|_{y=0}^{y=x} dx = \int_0^1 (e^x - e^{-x}) dx = e + e^{-1} - 2. \end{aligned}$$

17.4.41

- a. Using Green's theorem, the circulation is $\iint_R \left(\frac{\partial}{\partial x} (y) - \frac{\partial}{\partial y} (x) \right) dA = \iint_R 0 dA = 0$,
- b. Using Green's theorem, the flux is $\iint_R \left(\frac{\partial}{\partial x} (x) + \frac{\partial}{\partial y} (y) \right) dA = \iint_R 2 dA = 2 \cdot \text{area of } R = 2 \cdot \frac{1}{2} (4\pi - \pi) = 3\pi$.

17.4.42

- a. Using Green's theorem, the circulation is $\iint_R \left(\frac{\partial}{\partial x} (x) - \frac{\partial}{\partial y} (-y) \right) dA = \iint_R 2 dA = 2 \cdot \text{area of } R = 2 \cdot (9\pi - \pi) = 16\pi$.
- b. Using Green's theorem, the flux is $\iint_R \left(\frac{\partial}{\partial x} (-y) + \frac{\partial}{\partial y} (x) \right) dA = 0$.

17.4.43

- a. Using Green's theorem, the circulation is $\iint_R \left(\frac{\partial}{\partial x} (x - 4y) - \frac{\partial}{\partial y} (2x + y) \right) dA = \iint_R (1 - 1) dA = 0$.
- b. Using Green's theorem, the flux is $\iint_R \left(\frac{\partial}{\partial x} (2x + y) + \frac{\partial}{\partial y} (x - 4y) \right) dA = \iint_R (-2) dA = -2 \cdot \text{area of } R = -2 \cdot \frac{1}{4} (16\pi - \pi) = -\frac{15}{2}\pi$.

17.4.44

- a. Using Green's theorem, the circulation is $\iint_R \left(\frac{\partial}{\partial x} (-x + 2y) - \frac{\partial}{\partial y} (x - y) \right) dA = \iint_R (0) dA = 0$.
- b. Using Green's theorem, the flux is $\iint_R \left(\frac{\partial}{\partial x} (x - y) + \frac{\partial}{\partial y} (-x + 2y) \right) dA = \iint_R 3 dA = 3 \cdot \text{area of } R = 6$.

17.4.45

- a. Because $\mathbf{F}$ is conservative, the circulation on the boundary of R is zero.
- b. $\mathbf{F} = (x^2 + y^2)^{-1/2} \langle x, y \rangle$, so the flux is $\iint_R \left(\frac{\partial}{\partial x} \left(\frac{x}{\sqrt{x^2 + y^2}} \right) + \frac{\partial}{\partial y} \left(\frac{y}{\sqrt{x^2 + y^2}} \right) \right) dA = \iint_R (x^2 + y^2)^{-1/2} dA = \int_0^\pi \int_1^3 \frac{1}{r} dr d\theta = \int_0^\pi \int_1^3 1 dr d\theta = 2\pi$.

17.4.46

- a. The circulation is $\iint_R \left(\frac{\partial}{\partial x} \left(\tan^{-1} \left(\frac{y}{x} \right) \right) - \frac{\partial}{\partial y} (\ln(x^2 + y^2)) \right) dA = \iint_R \left(\frac{-y}{x^2 + y^2} - \frac{2y}{x^2 + y^2} \right) dA = -3 \iint_R \frac{y}{x^2 + y^2} dA = -3 \int_0^{2\pi} \int_1^2 \frac{r \sin \theta}{r^2} r dr d\theta = -3 \int_0^{2\pi} \int_1^2 2 \sin \theta dr d\theta = -3 \int_0^{2\pi} \sin \theta d\theta = 0$.

b. The flux is
$$\iint_R \left(\frac{\partial}{\partial x} (\ln(x^2 + y^2)) + \frac{\partial}{\partial y} \left(\tan^{-1} \left(\frac{y}{x} \right) \right) \right) dA = \iint_R \frac{3x}{x^2 + y^2} dA =$$

$$\int_0^{2\pi} \int_1^2 \frac{3r \cos \theta}{r^2} r dr d\theta = \int_0^{2\pi} \int_1^2 3 \cos \theta dr d\theta = \int_0^{2\pi} 3 \cos(\theta) d\theta = 0.$$

17.4.47 Note that the region is the area between $x = y^2$ and $x = 2 - y^2$, which intersect at $y = \pm 1$.

a. The circulation is

$$\begin{aligned} \iint_R \left(\frac{\partial}{\partial x} (x^2 - y) - \frac{\partial}{\partial y} (x + y^2) \right) dA &= \iint_R (2x - 2y) dA \\ &= \int_{-1}^1 \int_{y^2}^{2-y^2} (2x - 2y) dx dy \\ &= \int_{-1}^1 (x^2 - 2xy) \Big|_{y^2}^{2-y^2} dy \\ &= \int_{-1}^1 ((2 - y^2)^2 - (4y - 2y^3) - (y^4 - 2y^3)) dy \\ &= \int_{-1}^1 (4 - 4y - 4y^2 + 4y^3) dy \\ &= \left(4y - 2y^2 - \frac{4y^3}{3} + y^4 \right) \Big|_{-1}^1 = \frac{16}{3}. \end{aligned}$$

b. The flux is
$$\iint_R \left(\frac{\partial}{\partial x} (x + y^2) + \frac{\partial}{\partial y} (x^2 - y) \right) dA = \iint_R 0 dA = 0.$$

17.4.48

a. The circulation is

$$\begin{aligned} \iint_R \left(\frac{\partial}{\partial x} (-\sin x) - \frac{\partial}{\partial y} (y \cos x) \right) dA &= 2 \iint_R (-\cos x) dA \\ &= 2 \int_0^{\pi/2} \int_0^{\pi/2} (-\cos x) dy dx = \int_0^{\pi/2} (-\pi \cos x) dx = -\pi. \end{aligned}$$

b. The flux is

$$\begin{aligned} \iint_R \left(\frac{\partial}{\partial x} (y \cos x) + \frac{\partial}{\partial y} (-\sin x) \right) dA &= \iint_R (-y \sin x) dA \\ &= \int_0^{\pi/2} \int_0^{\pi/2} (-y \sin x) dy dx = -\frac{1}{8} \pi^2. \end{aligned}$$

17.4.49

a. True. This is the definition of work along a path.

b. False. Divergence corresponds to flux, so if the divergence is zero throughout a region, the flux is zero across the boundary.

c. True. This follows from Green's theorem.

17.4.50 Let $\mathbf{F} = \langle 1, 0 \rangle$. By Green's theorem, $\oint_C 1 \, dx = \oint_C 1 \, dx + 0 \, dy = \iint_R \left(\frac{\partial 0}{\partial x} - \frac{\partial 1}{\partial y} \right) dA = \iint_R 0 \, dA = 0$. Similarly, if we let $\mathbf{F} = \langle 0, 1 \rangle$, we have $\oint_C 1 \, dy = \oint_C 0 \, dx + 1 \, dy = \iint_R \left(\frac{\partial 1}{\partial x} - \frac{\partial 0}{\partial y} \right) dA = \iint_R 0 \, dA = 0$.

17.4.51 Because $\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x} = 0$, the integral is zero (because $\mathbf{F}$ is conservative.)

17.4.52 Let $f(x, y) = 0$, $g(x, y) = xy^2 + y^4$; then

$$\iint_R (2xy + 4y^3) \, dA = \iint_R \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} \right) dA = \oint_C (0 \, dy - g \, dx) = - \oint_C (xy^2 + y^4) \, dx$$

by Green's theorem, where C is the boundary of the triangle. To evaluate this line integral, parameterize C by three paths, all for $0 \leq t \leq 1$: $\mathbf{r}_1(t) = \langle t, 0 \rangle$, so that $\mathbf{r}'_1(t) = \langle 1, 0 \rangle$. $\mathbf{r}_2(t) = \langle 1 - t, t \rangle$, so that $\mathbf{r}'_2(t) = \langle -1, 1 \rangle$. $\mathbf{r}_3(t) = \langle 0, 1 - t \rangle$, so that $\mathbf{r}'_3(t) = \langle 0, -1 \rangle$. Then $-\oint_C (xy^2 + y^4) \, dx = -\int_0^1 (t(0^2) - 0^4) \, dt - \int_0^1 t^2(1-t)(-1) \, dt - \int_0^1 (0(1-t)^2 + t^4 \cdot 0) \, dt = \int_0^1 (t^4 - t^3 + t^2) \, dt = \frac{17}{60}$.

17.4.53 By Green's theorem,

$$\begin{aligned} \oint_C xy^2 \, dx + (x^2y + 2x) \, dy &= \iint_R \left(\frac{\partial}{\partial x} (x^2y + 2x) - \frac{\partial}{\partial y} (xy^2) \right) dA \\ &= \iint_R (2xy + 2 - 2xy) \, dA = \iint_R 2 \, dA = 2 \cdot \text{area of } A. \end{aligned}$$

17.4.54 Using the circulation form of Green's theorem, the integral is

$$\oint_C ay \, dx + bx \, dy = \iint_R (b - a) \, dA = (b - a) \cdot \text{area of } A.$$

17.4.55

a. The divergence is $\frac{\partial}{\partial x} (4) + \frac{\partial}{\partial y} (2) = 0$.

b. $\psi = 4y - 2x$.

17.4.56

a. The divergence is $\frac{\partial}{\partial x} (y^2) + \frac{\partial}{\partial y} (x^2) = 0 + 0 = 0$.

b. $\psi = \frac{1}{3} (y^3 - x^3)$.

17.4.57

a. The divergence is $\frac{\partial}{\partial x} (-e^{-x} \sin y) + \frac{\partial}{\partial y} (e^{-x} \cos y) = e^{-x} \sin y - e^{-x} \sin y = 0$.

b. $\psi = e^{-x} \cos y$.

17.4.58

a. The divergence is $\frac{\partial}{\partial x} (x^2) + \frac{\partial}{\partial y} (-2xy) = 2x - 2x = 0$.

b. $\psi = x^2 y$.

17.4.59

a. The curl and divergence are $\text{curl } \mathbf{F} = \frac{\partial}{\partial x} (-e^x \sin y) - \frac{\partial}{\partial y} (e^x \cos y) = -e^x \sin y + e^x \sin y = 0$. $\text{div } \mathbf{F} = \frac{\partial}{\partial x} (e^x \cos y) + \frac{\partial}{\partial y} (-e^x \sin y) = e^x \cos y - e^x \cos y = 0$.

b. $\varphi(x, y) = e^x \cos(y)$. $\psi(x, y) = e^x \sin(y)$.

c. $\varphi_{xx} + \varphi_{yy} = \frac{\partial^2}{\partial x^2} (e^x \cos y) + \frac{\partial^2}{\partial y^2} (e^x \cos y) = e^x \cos y - e^x \cos y = 0$. $\psi_{xx} + \psi_{yy} = \frac{\partial^2}{\partial x^2} (e^x \sin y) + \frac{\partial^2}{\partial y^2} (e^x \sin y) = e^x \sin y - e^x \sin y = 0$.

17.4.60

a. The curl and divergence are $\text{curl } \mathbf{F} = \frac{\partial}{\partial x} (y^3 - 3x^2 y) - \frac{\partial}{\partial y} (x^3 - 3xy^2) = -6xy + 6xy = 0$.

$$\text{div } \mathbf{F} = \frac{\partial}{\partial x} (x^3 - 3xy^2) + \frac{\partial}{\partial y} (y^3 - 3x^2 y) = 3x^2 - 3y^2 + 3y^2 - 3x^2 = 0$$

b. $\varphi(x, y) = \frac{1}{4} (x^4 + y^4) - \frac{3}{2} x^2 y^2$. $\psi(x, y) = x^3 y - xy^3$.

c.

$$\begin{aligned} \varphi_{xx} + \varphi_{yy} &= \frac{\partial^2}{\partial x^2} \left(\frac{1}{4} (x^4 + y^4) - \frac{3}{2} x^2 y^2 \right) + \frac{\partial^2}{\partial y^2} \left(\frac{1}{4} (x^4 + y^4) - \frac{3}{2} x^2 y^2 \right) \\ &= \frac{\partial}{\partial x} (x^3 - 3xy^2) + \frac{\partial}{\partial y} (y^3 - 3x^2 y) = 3x^2 - 3y^2 + 3y^2 - 3x^2 = 0 \end{aligned}$$

$$\begin{aligned} \psi_{xx} + \psi_{yy} &= \frac{\partial^2}{\partial x^2} (x^3 y - xy^3) + \frac{\partial^2}{\partial y^2} (x^3 y - xy^3) \\ &= \frac{\partial}{\partial x} (3x^2 y - y^3) + \frac{\partial}{\partial y} (x^3 - 3xy^2) = 6xy - 6xy = 0. \end{aligned}$$

17.4.61

a. The curl and divergence are $\text{curl } \mathbf{F} = \frac{\partial}{\partial x} \left(\frac{1}{2} \ln(x^2 + y^2) \right) - \frac{\partial}{\partial y} \left(\tan^{-1} \left(\frac{y}{x} \right) \right) = 0$.

$$\text{div } \mathbf{F} = \frac{\partial}{\partial x} \left(\tan^{-1} \left(\frac{y}{x} \right) \right) + \frac{\partial}{\partial y} \left(\frac{1}{2} \ln(x^2 + y^2) \right) = 0.$$

b. $\varphi(x, y) = x \tan^{-1} \left(\frac{y}{x} \right) - y + \frac{1}{2} y \ln(x^2 + y^2)$. $\psi(x, y) = \int -\frac{1}{2} \ln(x^2 + y^2) dy = y \tan^{-1} \left(\frac{y}{x} \right) - \frac{x}{2} \ln(x^2 + y^2) + x$.

c.

$$\begin{aligned} \varphi_{xx} + \varphi_{yy} &= \frac{\partial^2}{\partial x^2} \left(x \tan^{-1} \left(\frac{y}{x} \right) - y + \frac{1}{2} y \ln(x^2 + y^2) \right) + \frac{\partial^2}{\partial y^2} \left(x \tan^{-1} \left(\frac{y}{x} \right) - y + \frac{1}{2} y \ln(x^2 + y^2) \right) \\ &= \frac{\partial}{\partial x} \left(\tan^{-1} \left(\frac{y}{x} \right) - \frac{xy}{x^2 + y^2} + \frac{xy}{x^2 + y^2} \right) + \frac{\partial}{\partial y} \left(\frac{x^2}{x^2 + y^2} - 1 + \frac{1}{2} \ln(x^2 + y^2) + \frac{y^2}{x^2 + y^2} \right) \\ &= \frac{\partial}{\partial x} \left(\tan^{-1} \left(\frac{y}{x} \right) \right) + \frac{\partial}{\partial y} \left(\frac{1}{2} \ln(x^2 + y^2) \right) = 0. \end{aligned}$$

$$\begin{aligned}
\psi_{xx} + \psi_{yy} &= \frac{\partial^2}{\partial x^2} \left(y \tan^{-1} \left(\frac{y}{x} \right) - \frac{x}{2} \ln(x^2 + y^2) + x \right) + \frac{\partial^2}{\partial y^2} \left(y \tan^{-1} \left(\frac{y}{x} \right) - \frac{x}{2} \ln(x^2 + y^2) + x \right) \\
&= \frac{\partial}{\partial x} \left(-\frac{y^2}{x^2 + y^2} - \frac{x^2}{x^2 + y^2} - \frac{1}{2} \ln(x^2 + y^2) + 1 \right) + \frac{\partial}{\partial y} \left(\tan^{-1} \left(\frac{y}{x} \right) + \frac{xy}{x^2 + y^2} - \frac{xy}{x^2 + y^2} \right) \\
&= \frac{\partial}{\partial x} \left(\frac{1}{2} \ln(x^2 + y^2) \right) + \frac{\partial}{\partial y} \left(\tan^{-1} \left(\frac{y}{x} \right) \right) = 0.
\end{aligned}$$

17.4.62

a. The curl and divergence are $\text{curl } \mathbf{F} = \frac{\partial}{\partial x} \left(\frac{y}{x^2 + y^2} \right) - \frac{\partial}{\partial y} \left(\frac{x}{x^2 + y^2} \right) = 0$, and

$$\text{div } \mathbf{F} = \frac{\partial}{\partial x} \left(\frac{x}{x^2 + y^2} \right) + \frac{\partial}{\partial y} \left(\frac{y}{x^2 + y^2} \right) = 0.$$

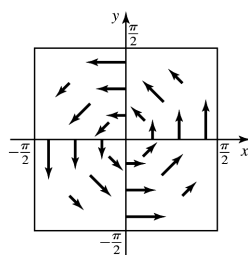
b. $\varphi(x, y) = \frac{1}{2} \ln(x^2 + y^2)$. $\psi(x, y) = \tan^{-1} \left(\frac{y}{x} \right)$.

$$\begin{aligned}
\text{c. } \varphi_{xx} + \varphi_{yy} &= \frac{\partial^2}{\partial x^2} \left(\frac{1}{2} \ln(x^2 + y^2) \right) + \frac{\partial^2}{\partial y^2} \left(\frac{1}{2} \ln(x^2 + y^2) \right) = \frac{\partial}{\partial x} \left(\frac{x}{x^2 + y^2} \right) + \frac{\partial}{\partial y} \left(\frac{y}{x^2 + y^2} \right) = \\
&= \frac{y^2 - x^2}{(x^2 + y^2)^2} + \frac{x^2 - y^2}{(x^2 + y^2)^2} = 0.
\end{aligned}$$

$$\begin{aligned}
\psi_{xx} + \psi_{yy} &= \frac{\partial^2}{\partial x^2} \left(\tan^{-1} \left(\frac{y}{x} \right) \right) + \frac{\partial^2}{\partial y^2} \left(\tan^{-1} \left(\frac{y}{x} \right) \right) = \frac{\partial}{\partial x} \left(\frac{-y}{x^2 + y^2} \right) + \frac{\partial}{\partial y} \left(\frac{x}{x^2 + y^2} \right) = \frac{2xy}{(x^2 + y^2)^2} - \\
&= \frac{2xy}{(x^2 + y^2)^2} = 0.
\end{aligned}$$

17.4.63

a. The velocity field is $\langle -4 \cos x \sin y, 4 \sin x \cos y \rangle$.



b. The field is source-free if its divergence is zero.

$$\text{div } \mathbf{F} = \frac{\partial}{\partial x} (-4 \cos x \sin y) + \frac{\partial}{\partial y} (4 \sin x \cos y) = 4 \sin x \sin y - 4 \sin x \sin y = 0,$$

so the field is source-free.

c. The field is irrotational if its curl is zero.

$$\text{curl } \mathbf{F} = \frac{\partial}{\partial x} (4 \sin x \cos y) - \frac{\partial}{\partial y} (-4 \cos x \sin y) = 4 \cos x \cos y + 4 \cos x \cos y = 8 \cos x \cos y,$$

so the field is not irrotational.

d. Since the field is source-free, it has zero flux across the boundary.

e. The circulation around the boundary of the rectangle is (by Green's theorem) given by

$$\iint_R 8 \cos x \cos y \, dA = \int_{-\pi/2}^{\pi/2} \int_{-\pi/2}^{\pi/2} 8 \cos x \cos y \, dy \, dx = \int_{-\pi/2}^{\pi/2} 16 \cos x \, dx = 32.$$

17.4.64 If $f(x)$ is continuous, then the circulation form of Green's theorem says that

$$\oint_C \frac{f(x)}{c} dy = \frac{1}{c} \iint_R \frac{df}{dx} dA.$$

The right side of this equation evaluates to

$$\frac{1}{c} \iint_R \frac{df}{dx} dA = \frac{1}{c} \int_a^b \int_0^c \frac{df}{dx} dy \, dx = \int_a^b \frac{df}{dx} dx.$$

To evaluate the left side, parameterize the boundary of R with four paths, each for $0 \leq t \leq 1$: $\mathbf{r}_1(t) = \langle a + (b-a)t, 0 \rangle$, so $\mathbf{r}'_1(t) = \langle b-a, 0 \rangle$. $\mathbf{r}_2(t) = \langle b, ct \rangle$, so $\mathbf{r}'_2(t) = \langle 0, c \rangle$. $\mathbf{r}_3(t) = \langle b + (a-b)t, c \rangle$, so $\mathbf{r}'_3(t) = \langle a-b, 0 \rangle$. $\mathbf{r}_4(t) = \langle a, c-ct \rangle$, so $\mathbf{r}'_4(t) = \langle 0, -c \rangle$. Then we evaluate $\mathbf{F} \cdot \mathbf{r}'_i$ for each i and add:

$$\frac{1}{c} \oint_C f(x) dy = \frac{1}{c} \int_0^1 (f(a + (b-a)t) \cdot 0 + f(b) \cdot c + f(b + (a-b)t) \cdot 0 + f(a) \cdot (-c)) dt = f(b) - f(a).$$

17.4.65 If $f(x)$ is continuous, then the flux form of Green's theorem says that

$$\oint_C \frac{f(x)}{c} dx = \frac{1}{c} \iint_R \frac{df}{dx} dA.$$

The right side of this equation evaluates to

$$\frac{1}{c} \iint_R \frac{df}{dx} dA = \frac{1}{c} \int_a^b \int_0^c \frac{df}{dx} dy \, dx = \int_a^b \frac{df}{dx} dx.$$

To evaluate the left side, parameterize the boundary of R with four paths, each for $0 \leq t \leq 1$: $\mathbf{r}_1(t) = \langle a + (b-a)t, 0 \rangle$, so $\mathbf{r}'_1(t) = \langle b-a, 0 \rangle$. $\mathbf{r}_2(t) = \langle b, ct \rangle$, so $\mathbf{r}'_2(t) = \langle 0, c \rangle$. $\mathbf{r}_3(t) = \langle b + (a-b)t, c \rangle$, so $\mathbf{r}'_3(t) = \langle a-b, 0 \rangle$. $\mathbf{r}_4(t) = \langle a, c-ct \rangle$, so $\mathbf{r}'_4(t) = \langle 0, -c \rangle$. Then we evaluate $\mathbf{F} \cdot \mathbf{r}'_i$ for each i and add:

$$\oint_C \frac{f(x)}{c} dx = \frac{1}{c} \int_0^1 (0 + f(b) \cdot c + 0 + f(a) \cdot (-c)) dt = f(b) - f(a)$$

so that

$$\int_a^b \frac{df}{dx} dx = f(b) - f(a).$$

17.4.66

a. The curl is $\frac{\partial}{\partial x} \left(\frac{x}{x^2 + y^2} \right) - \frac{\partial}{\partial y} \left(\frac{-y}{x^2 + y^2} \right) = 0$.

b. Take a line integral around the unit circle, parameterized as $\langle \cos t, \sin t \rangle$. The circulation is then

$$\oint_C \frac{-y}{x^2 + y^2} dx + \frac{x}{x^2 + y^2} dy = \int_0^{2\pi} (-\sin t (-\sin t) + \cos t \cos t) dt = \int_0^{2\pi} 1 dt = 2\pi.$$

c. The vector field is not defined everywhere in R ; specifically, it is undefined at the origin.

17.4.67

a. The divergence is $\frac{\partial}{\partial x} \left(\frac{x}{x^2 + y^2} \right) + \frac{\partial}{\partial y} \left(\frac{y}{x^2 + y^2} \right) = 0$.

b. Take a line integral around the unit circle, parameterized as $\langle \cos t, \sin t \rangle$. The flux is then

$$\oint_C \frac{x}{x^2 + y^2} dy + \frac{y}{x^2 + y^2} dx = \int_0^{2\pi} (\cos t \cos t - \sin t (-\sin t)) dt = \int_0^{2\pi} 1 dt = 2\pi.$$

c. The vector field is not defined everywhere in R ; specifically, it is undefined at the origin.

17.4.68

a. Green's theorem does not apply to a region including the origin because $\mathbf{F}$ is not defined at the origin.

$$\iint_R \left(\frac{\partial}{\partial x} \left(\frac{x}{\sqrt{x^2 + y^2}} \right) + \frac{\partial}{\partial y} \left(\frac{y}{\sqrt{x^2 + y^2}} \right) \right) dA = \iint_R (x^2 + y^2)^{-1/2} dA = \int_0^{2\pi} \int_0^1 \frac{1}{r} r dr d\theta = 2\pi.$$

$$\oint_C \frac{x}{\sqrt{x^2 + y^2}} dy - \frac{y}{\sqrt{x^2 + y^2}} dx = \int_0^{2\pi} (\cos t (\cos t) - \sin t (-\sin t)) dt = \int_0^{2\pi} 1 dt = 2\pi.$$

d. They do agree. Because Green's theorem does not apply, there is no particular reason why they should.

17.4.69 Because ψ is a stream function, $d\psi = \psi_x dx + \psi_y dy$, so the flux integral is

$$\int_C \mathbf{F} \cdot \mathbf{n} ds = \int_C f dy - g dx = \int_C \psi_y dy - \psi_x dx = \int_C d\psi = \psi(B) - \psi(A),$$

so that the integral is independent of the path.

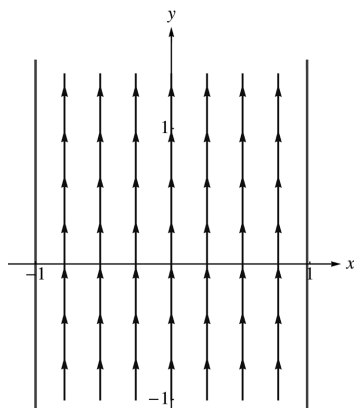
17.4.70 Showing that $\mathbf{F}$ is tangent to the level curves of the stream function is the same as showing that $\mathbf{F}$ is normal to the gradient of the stream function. But that gradient is $\langle \psi_x, \psi_y \rangle$, and

$$\mathbf{F} \cdot \langle \psi_x, \psi_y \rangle = \langle -\psi_y, \psi_x \rangle \cdot \langle \psi_x, \psi_y \rangle = 0.$$

17.4.71 Showing that the level curves of φ and ψ are orthogonal is equivalent to showing that the gradients of φ and ψ are orthogonal. But $\nabla\varphi \cdot \nabla\psi = \langle f, g \rangle \cdot \langle -g, f \rangle = 0$.

17.4.72

a. The stream function is found by taking $-\int (1 - x^2) dx = \frac{1}{3}x^3 - x$. A plot together with some streamlines is



b. The curl of $\mathbf{F}$ is $\frac{\partial}{\partial x}(1 - x^2) = -2x$, so the curl on $x = 0$ is 0; on $x = \frac{1}{4}$ it is $-\frac{1}{2}$; on $x = \frac{1}{2}$, it is -1 , and on $x = 1$, it is -2 .

c. The circulation is (by Green's theorem) $\iint_R (-2x) \, dA = \int_{-5}^5 \int_{-1}^1 (-2x) \, dx \, dy = \int_{-5}^5 0 \, dy = 0$.

d. The curl is positive for negative x and negative for positive x . These cancel, giving a net circulation of zero. This can easily be seen from the picture - any circulation resulting from the top boundary ($y = 1$) is cancelled by the circulation in the opposite direction resulting from the bottom boundary.

17.5 Divergence and Curl

17.5.1 The divergence is $\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}$.

17.5.2 The divergence measures the expansion or contraction of the vector field at each point.

17.5.3 It means that the field has no sources or sinks.

17.5.4 The curl is $\nabla \times \mathbf{F} = \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z} \right) \mathbf{i} + \left(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x} \right) \mathbf{j} + \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) \mathbf{k}$.

17.5.5 The curl indicates the axis and speed of rotation of a vector field at each point.

17.5.6 It means that the vector field is irrotational.

17.5.7 $\nabla \cdot (\nabla \times \mathbf{F}) = 0$; see Theorem 17.10.

17.5.8 Here $\mathbf{u}$ is a potential function, so $\nabla \times \nabla \mathbf{u}$ is the curl of a conservative vector field, which is 0.

17.5.9 $\frac{\partial}{\partial x}(2x) + \frac{\partial}{\partial y}(4y) + \frac{\partial}{\partial z}(-3z) = 3$.

17.5.10 $\frac{\partial}{\partial x}(-2y) + \frac{\partial}{\partial y}(3x) + \frac{\partial}{\partial z}(z) = 1$.

17.5.11 $\frac{\partial}{\partial x}(12x) + \frac{\partial}{\partial y}(-6y) + \frac{\partial}{\partial z}(-6z) = 0$.

17.5.12 $\frac{\partial}{\partial x}(x^2yz) + \frac{\partial}{\partial y}(-xy^2z) + \frac{\partial}{\partial z}(-xyz^2) = 2xyz - 2xyz - 2xyz = -2xyz$.

17.5.13 $\frac{\partial}{\partial x}(x^2 - y^2) + \frac{\partial}{\partial y}(y^2 - z^2) + \frac{\partial}{\partial z}(z^2 - x^2) = 2x + 2y + 2z$.

17.5.14 $\frac{\partial}{\partial x}(e^{y-x}) + \frac{\partial}{\partial y}(e^{z-y}) + \frac{\partial}{\partial z}(e^{x-z}) = -(e^{y-x} + e^{z-y} + e^{x-z})$.

17.5.15 $\frac{\partial}{\partial x} \left(\frac{x}{1+x^2+y^2} \right) + \frac{\partial}{\partial y} \left(\frac{y}{1+x^2+y^2} \right) + \frac{\partial}{\partial z} \left(\frac{z}{1+x^2+y^2} \right) = \frac{x^2+y^2+3}{(1+x^2+y^2)^2}$.

17.5.16 $\frac{\partial}{\partial x}(yz \sin(x)) + \frac{\partial}{\partial y}(xz \cos(y)) + \frac{\partial}{\partial z}(xy \cos(z)) = yz \cos(x) - xz \sin(y) - xy \sin(z)$.

17.5.17

$$\begin{aligned}
& \frac{\partial}{\partial x} \left(\frac{x}{x^2 + y^2 + z^2} \right) + \frac{\partial}{\partial y} \left(\frac{y}{x^2 + y^2 + z^2} \right) + \frac{\partial}{\partial z} \left(\frac{z}{x^2 + y^2 + z^2} \right) \\
&= \frac{1}{(x^2 + y^2 + z^2)^2} ((z^2 + y^2 - x^2) + (x^2 + z^2 - y^2) + (x^2 + y^2 - z^2)) \\
&= \frac{1}{(x^2 + y^2 + z^2)^2} (x^2 + y^2 + z^2) = \frac{1}{|\mathbf{r}|^2}.
\end{aligned}$$

17.5.18

$$\begin{aligned}
& \frac{\partial}{\partial x} \left(\frac{x}{(x^2 + y^2 + z^2)^{3/2}} \right) + \frac{\partial}{\partial y} \left(\frac{y}{(x^2 + y^2 + z^2)^{3/2}} \right) + \frac{\partial}{\partial z} \left(\frac{z}{(x^2 + y^2 + z^2)^{3/2}} \right) \\
&= \frac{1}{(x^2 + y^2 + z^2)^{5/2}} ((z^2 + y^2 - 2x^2) + (x^2 + z^2 - 2y^2) + (x^2 + y^2 - 2z^2)) = 0.
\end{aligned}$$

17.5.19

$$\begin{aligned}
& \frac{\partial}{\partial x} \left(\frac{x}{(x^2 + y^2 + z^2)^2} \right) + \frac{\partial}{\partial y} \left(\frac{y}{(x^2 + y^2 + z^2)^2} \right) + \frac{\partial}{\partial z} \left(\frac{z}{(x^2 + y^2 + z^2)^2} \right) \\
&= \frac{1}{(x^2 + y^2 + z^2)^3} ((z^2 + y^2 - 3x^2) + (x^2 + z^2 - 3y^2) + (x^2 + y^2 - 3z^2)) \\
&= \frac{-1}{(x^2 + y^2 + z^2)^3} (x^2 + y^2 + z^2) = \frac{-1}{|\mathbf{r}|^4}.
\end{aligned}$$

17.5.20

$$\begin{aligned}
& \frac{\partial}{\partial x} (x(x^2 + y^2 + z^2)) + \frac{\partial}{\partial y} (y(x^2 + y^2 + z^2)) + \frac{\partial}{\partial z} (z(x^2 + y^2 + z^2)) \\
&= (3x^2 + y^2 + z^2) + (x^2 + 3y^2 + z^2) + (x^2 + y^2 + 3z^2) = 5|\mathbf{r}|^2.
\end{aligned}$$

17.5.21

- a. At both P and Q , the arrows going away from the point are larger in both number and magnitude than those going in, so we would expect the divergence to be positive at both points.
- b. The divergence is $\frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(x+y) = 1 + 1 = 2$, so is positive everywhere.
- c. The arrows all point roughly away from the origin, so the flux is outward everywhere.
- d. The net flux across C should be positive.

17.5.22

- At P , the divergence should be positive, while at Q , the larger arrows point in towards Q , so the divergence should be negative.
- The divergence is $\frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y^2) = 1 + 2y$; at $P = (-1, 1)$, this is 3, while at $Q = (-1, -1)$, it is -1 .
- The flux is outward above the line $y = -1$ (approximately); below this line, the flux is inward across C .
- The size of the arrows pointing outward at the top of the circle seems to roughly equal those pointing inward at the bottom, so the remaining outward-pointing arrows result in a net positive flux across C .

17.5.23

- The axis of rotation is $\langle 1, 0, 0 \rangle$, the x -axis. $\nabla \times \mathbf{F} = \nabla \times \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ x & y & z \end{vmatrix} = \nabla \times (-z\mathbf{j} + y\mathbf{k}) = (1+1)\mathbf{i} + (0-0)\mathbf{j} + (0-0)\mathbf{k} = 2\mathbf{i}$. It is in the same direction as the axis of rotation.
- The magnitude of the curl is $|2\mathbf{i}| = 2$.

17.5.24

- The axis of rotation is $\langle 1, -1, 0 \rangle$. $\nabla \times \mathbf{F} = \nabla \times \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 0 \\ x & y & z \end{vmatrix} = \nabla \times (-z\mathbf{i} - z\mathbf{j} + (x+y)\mathbf{k}) = (1+1)\mathbf{i} + (-1-1)\mathbf{j} + (0-0)\mathbf{k} = 2\langle 1, -1, 0 \rangle$ and the curl is in the same direction as the axis of rotation.
- The magnitude of the curl is $2|\langle 1, -1, 0 \rangle| = 2\sqrt{2}$.

17.5.25

- The axis of rotation is $\langle 1, -1, 1 \rangle$. $\nabla \times \mathbf{F} = \nabla \times \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 1 \\ x & y & z \end{vmatrix} = \nabla \times (-(y+z)\mathbf{i} + (x-z)\mathbf{j} + (x+y)\mathbf{k}) = (1+1)\mathbf{i} + (-1-1)\mathbf{j} + (1+1)\mathbf{k} = 2\langle 1, -1, 1 \rangle$, and the curl is in the same direction as the axis of rotation.
- The magnitude of the curl is $2|\langle 1, -1, 1 \rangle| = 2\sqrt{3}$.

17.5.26

- The axis of rotation is $\langle 1, -2, -3 \rangle$. $\nabla \times \mathbf{F} = \nabla \times \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -2 & -3 \\ x & y & z \end{vmatrix} = \nabla \times ((3y-2z)\mathbf{i} + (-3x-z)\mathbf{j} + (2x+y)\mathbf{k}) = (1+1)\mathbf{i} + (-2-2)\mathbf{j} + (-3-3)\mathbf{k} = 2\langle 1, -2, -3 \rangle$, and the curl is in the same direction as the axis of rotation.
- The magnitude of the curl is $2|\langle 1, -2, -3 \rangle| = 2\sqrt{14}$.

17.5.27 $\nabla \times \langle x^2 - y^2, xy, z \rangle = (0-0)\mathbf{i} + (0-0)\mathbf{j} + (y+2y)\mathbf{k} = 3y\mathbf{k}$.

$$17.5.28 \quad \nabla \times \langle 0, z^2 - y^2, -yz \rangle = (-z - 2z)\mathbf{i} + (0 - 0)\mathbf{j} + (0 - 0)\mathbf{k} = -3z\mathbf{i}.$$

$$17.5.29 \quad \nabla \times \langle x^2 - z^2, 1, 2xz \rangle = (0 - 0)\mathbf{i} + (-2z - 2z)\mathbf{j} + (0 - 0)\mathbf{k} = -4z\mathbf{j}.$$

$$17.5.30 \quad \nabla \times \langle x, y, z \rangle = (0 - 0)\mathbf{i} + (0 - 0)\mathbf{j} + (0 - 0)\mathbf{k} = \mathbf{0}.$$

17.5.31

$$\begin{aligned} \nabla \times \frac{1}{(x^2 + y^2 + z^2)^{3/2}} \langle x, y, z \rangle \\ = \frac{1}{(x^2 + y^2 + z^2)^{5/2}} ((-3yz + 3yz)\mathbf{i} + (-3xz + 3xz)\mathbf{j} + (-3xy + 3xy)\mathbf{k}) = \mathbf{0}. \end{aligned}$$

17.5.32

$$\begin{aligned} \nabla \times \frac{1}{(x^2 + y^2 + z^2)^{1/2}} \langle x, y, z \rangle \\ = \frac{1}{(x^2 + y^2 + z^2)^{3/2}} ((-yz + yz)\mathbf{i} + (-xz + xz)\mathbf{j} + (-xy + xy)\mathbf{k}) = \mathbf{0}. \end{aligned}$$

17.5.33

$$\begin{aligned} \nabla \times \langle z^2 \sin y, xz^2 \cos y, 2xz \sin y \rangle \\ = (2xz \cos y - 2xz \cos y)\mathbf{i} + (2z \sin y - 2z \sin y)\mathbf{j} + (z^2 \cos y - z^2 \cos y)\mathbf{k} = \mathbf{0}. \end{aligned}$$

17.5.34

$$\begin{aligned} \nabla \times \langle 3xz^3e^{y^2}, 2xz^3e^{y^2}, 3xz^2e^{y^2} \rangle &= (6xyz^2e^{y^2} - 6xz^2e^{y^2})\mathbf{i} + (9xz^2e^{y^2} - 3z^2e^{y^2})\mathbf{j} + (2z^3e^{y^2} - 6xyz^3e^{y^2})\mathbf{k} \\ &= z^2e^{y^2} ((6xy - 6x)\mathbf{i} + (9x - 3)\mathbf{j} + (2z - 6xyz)\mathbf{k}). \end{aligned}$$

17.5.35 Simply compute it:

$$\begin{aligned} \left\langle \frac{\partial}{\partial x} \left(\frac{1}{(x^2 + y^2 + z^2)^{3/2}} \right), \frac{\partial}{\partial y} \left(\frac{1}{(x^2 + y^2 + z^2)^{3/2}} \right), \frac{\partial}{\partial z} \left(\frac{1}{(x^2 + y^2 + z^2)^{3/2}} \right) \right\rangle \\ = \left\langle \frac{-3x}{(x^2 + y^2 + z^2)^{5/2}}, \frac{-3y}{(x^2 + y^2 + z^2)^{5/2}}, \frac{-3z}{(x^2 + y^2 + z^2)^{5/2}} \right\rangle = \frac{-3\mathbf{r}}{|\mathbf{r}|^5}. \end{aligned}$$

17.5.36

$$\begin{aligned} \left\langle \frac{\partial}{\partial x} \left(\frac{1}{x^2 + y^2 + z^2} \right), \frac{\partial}{\partial y} \left(\frac{1}{x^2 + y^2 + z^2} \right), \frac{\partial}{\partial z} \left(\frac{1}{x^2 + y^2 + z^2} \right) \right\rangle \\ = \left\langle \frac{-2x}{(x^2 + y^2 + z^2)^2}, \frac{-2y}{(x^2 + y^2 + z^2)^2}, \frac{-2z}{(x^2 + y^2 + z^2)^2} \right\rangle = \frac{-2\mathbf{r}}{|\mathbf{r}|^4}. \end{aligned}$$

$$17.5.37 \quad \nabla \left(\frac{1}{|\mathbf{r}|^2} \right) = \frac{-2\mathbf{r}}{|\mathbf{r}|^4}, \text{ from Problem 36; applying Theorem 17.8 we have}$$

$$\nabla \cdot \nabla \left(\frac{1}{|\mathbf{r}|^2} \right) = -2\nabla \cdot \frac{\mathbf{r}}{|\mathbf{r}|^4} = -2\frac{3-4}{|\mathbf{r}|^4} = \frac{2}{|\mathbf{r}|^4}.$$

17.5.38

$$\begin{aligned} \nabla (\ln |\mathbf{r}|) &= \nabla \left(\ln \left(\sqrt{x^2 + y^2 + z^2} \right) \right) = \frac{1}{2} \nabla (\ln (x^2 + y^2 + z^2)) \\ &= \frac{1}{2(x^2 + y^2 + z^2)} \langle 2x, 2y, 2z \rangle = \frac{\mathbf{r}}{|\mathbf{r}|^2}. \end{aligned}$$

17.5.39

- False. For example, $\mathbf{F} = \langle y, z, x \rangle$ has zero divergence yet is not constant.
- False. For example, $\mathbf{F} = \langle x, y, z \rangle$ is a counterexample.
- False. For example, consider the vector field $\langle 0, 1 - x^2 \rangle$.
- False. For example, $\mathbf{F} = \langle x, 0, 0 \rangle$ has divergence 1.
- False. For example, the curl of $\langle z, -z, y \rangle$ is $\langle 2, 1, 0 \rangle$.

17.5.40

- $(\mathbf{F} \cdot \nabla)u = \left(\langle f, g, h \rangle \cdot \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \right) u = \left(f \frac{\partial}{\partial x} + g \frac{\partial}{\partial y} + h \frac{\partial}{\partial z} \right) u = f \frac{\partial u}{\partial x} + g \frac{\partial u}{\partial y} + h \frac{\partial u}{\partial z}.$
- Because $\mathbf{F} = \langle 1, 1, 1 \rangle$, $\mathbf{F} \cdot \nabla (xy^2z^3) = \frac{\partial}{\partial x} (xy^2z^3) + \frac{\partial}{\partial y} (xy^2z^3) + \frac{\partial}{\partial z} (xy^2z^3) = y^2z^3 + 2xyz^3 + 3xy^2z^2.$

17.5.41

- No; divergence is a concept that applies to vector fields.
- No; the gradient applies to functions.
- Yes; this is the divergence of the gradient and is thus a scalar function.
- No, because $\nabla \cdot \varphi$ does not make sense (part (a)).
- No; curl applies to vector fields, $\nabla \times \varphi$ does not make sense.
- No, because $\nabla \cdot \mathbf{F}$ is a function, so that applying $\nabla \cdot$ to it does not make sense.
- Yes, this is the curl of a vector field and is thus a vector field.
- No, because $\nabla \cdot \mathbf{F}$ is a function, not a vector field.
- Yes; this is the curl of the curl of a vector field and is thus a vector field.

17.5.42 Let $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$; then $\mathbf{F} = \mathbf{a} \times \mathbf{r} = (a_2z - a_3y)\mathbf{i} + (a_3x - a_1z)\mathbf{j} + (a_1y - a_2x)\mathbf{k}$, so that $\nabla \cdot \mathbf{F} = \frac{\partial}{\partial x} (a_2z - a_3y) + \frac{\partial}{\partial y} (a_3x - a_1z) + \frac{\partial}{\partial z} (a_1y - a_2x) = 0.$

17.5.43

- $\langle 0, 1, 0 \rangle \times \langle 0, 1, 1 \rangle = \langle 1, 0, 0 \rangle$, so $\mathbf{F}$ points in the positive x -direction. $\langle 0, 1, 0 \rangle \times \langle 1, 1, 0 \rangle = \langle 0, 0, -1 \rangle$, so $\mathbf{F}$ points in the negative z -direction. $\langle 0, 1, 0 \rangle \times \langle 0, 1, -1 \rangle = \langle -1, 0, 0 \rangle$, so $\mathbf{F}$ points in the negative x -direction. $\langle 0, 1, 0 \rangle \times \langle -1, 1, 0 \rangle = \langle 0, 0, 1 \rangle$, so $\mathbf{F}$ points in the positive z -direction.
- Note that these vectors circle the y -axis in the counterclockwise direction looking along $\mathbf{a}$ from head to tail.

17.5.44 Note that $\mathbf{a} \times \mathbf{r} = \langle 0, 1, 0 \rangle \times \langle x, y, z \rangle = \langle z, 0, -x \rangle$ is a rotational field whose vectors circle the y -axis in the counterclockwise direction looking along $\mathbf{a}$ from head to tail.

17.5.45 Let $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$; then $\mathbf{F} = \mathbf{a} \times \mathbf{r} = (a_2z - a_3y)\mathbf{i} + (a_3x - a_1z)\mathbf{j} + (a_1y - a_2x)\mathbf{k}$, so that $\nabla \times \mathbf{F} = \frac{\partial}{\partial x} (a_1 + a_1)\mathbf{i} + \frac{\partial}{\partial y} (a_2 + a_2)\mathbf{j} + \frac{\partial}{\partial z} (a_3 + a_3)\mathbf{k} = 2\mathbf{a}.$

17.5.46 The field switches from inward-pointing to outward-pointing at points where it is tangent to the circle $x^2 + y^2 = 2$, i.e. where it is orthogonal to the normal to the circle. The normal to the circle at (x, y) is a multiple of $\langle x, y \rangle$, so we want to find x, y so that $\langle x, y \rangle \cdot \langle x^2, y^2 \rangle = x^3 + y^2 = 0$ with $x^2 + y^2 = 2$. Thus $x^3 - x^2 + 2 = 0$. The solutions are $x = -1$ and $y = \pm 1$.

17.5.47 $\operatorname{div} \mathbf{F} = 2x + 2xyz + 2x = 2x(yz + 2)$; this function clearly achieves its maximum magnitude at $(-1, 1, 1)$, $(-1, -1, -1)$, $(1, 1, 1)$, and $(1, -1, -1)$, where its magnitude is 6.

17.5.48 For $\mathbf{F} = \langle z, 0, -y \rangle$, $\operatorname{curl} \mathbf{F} = \langle -1, 1, 0 \rangle$.

a. $\operatorname{scal}_{\langle 1, 0, 0 \rangle} \langle -1, 1, 0 \rangle = \langle -1, 1, 0 \rangle \cdot \langle 1, 0, 0 \rangle = -1$.

b. $\operatorname{scal}_{\langle 1/\sqrt{3}, -1/\sqrt{3}, 1/\sqrt{3} \rangle} \langle -1, 1, 0 \rangle = \langle -1, 1, 0 \rangle \cdot \left\langle \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right\rangle = -\frac{2}{\sqrt{3}}$.

c. We have $\operatorname{scal}_{\mathbf{n}} \langle -1, 1, 0 \rangle = \langle -1, 1, 0 \rangle \cdot \mathbf{n} = \sqrt{2} \cos \theta$, where θ is the angle between $\langle -1, 1, 0 \rangle$ and $\mathbf{n}$. This is maximized when $\mathbf{n}$ points in the same direction as $\operatorname{curl} \mathbf{F}$. So $\mathbf{n} = \frac{\langle -1, 1, 0 \rangle}{|\langle -1, 1, 0 \rangle|} = \left\langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \right\rangle$, in which case $\operatorname{scal}_{\mathbf{n}} \langle -1, 1, 0 \rangle = \sqrt{2}$.

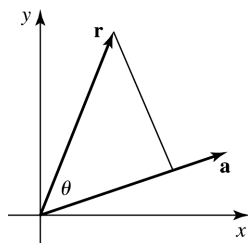
17.5.49 $\operatorname{curl} \mathbf{F} = \langle 0 + 2, 0 + 1, 0 - 1 \rangle = \langle 2, 1, -1 \rangle$. If $\mathbf{n} = \langle a, b, c \rangle$, then $\operatorname{curl} \mathbf{F} \cdot \mathbf{n} = 0$ when $2a + b - c = 0$ so that $c = 2a + b$; thus all such vectors are of the form $\langle a, b, 2a + b \rangle$, where a, b are real numbers.

17.5.50 $\mathbf{F} = \langle z, 0, 0 \rangle$, or $\mathbf{F} = \langle 0, 0, -x \rangle$, so it is not unique.

17.5.51 $\mathbf{F} = \frac{1}{2} \langle y^2 + z^2, 0, 0 \rangle$ or $\mathbf{F} = \langle 0, -xy, -xz \rangle$, so it is not unique.

17.5.52

a. Looking at the picture, it is clear that the distance from P to $\mathbf{a}$ is $|\mathbf{r}|$.



b. The velocity field is $\mathbf{a} \times \mathbf{r}$, so the speed, which is the magnitude of velocity, is $|\mathbf{a} \times \mathbf{r}|$. Now $|\mathbf{a} \times \mathbf{r}| = |\mathbf{a}| \cdot |\mathbf{r}| \cdot \sin(\theta) = R|\mathbf{a}| \sin(\theta)$, where $\mathbf{n}$ is a vector normal to the plane determined by $\mathbf{a}$ and $\mathbf{r}$. Thus the motion of the particle is always perpendicular to this plane, so it rotates about the axis $\mathbf{a}$. It is moving at a speed $R|\mathbf{a}| \sin(\theta)$ around a circle of radius $R \sin(\theta)$, so its angular speed is $\frac{R|\mathbf{a}| \sin(\theta)}{R \sin(\theta)} = |\mathbf{a}|$.

c. Because $|\nabla \times \mathbf{v}| = 2|\mathbf{a}|$, it follows from part (b) that $\omega = |\mathbf{a}| = \frac{1}{2} |\nabla \times \mathbf{v}|$.

17.5.53 The curl of this vector field is $\langle 0, 1, 0 \rangle$. The component of the curl along some unit vector $\mathbf{n}$ is $(\nabla \times \mathbf{F}) \cdot \mathbf{n}$.

a. $\langle 0, 1, 0 \rangle \cdot \langle 1, 0, 0 \rangle = 0$, so the wheel does not spin.

b. $\langle 0, 1, 0 \rangle \cdot \langle 0, 1, 0 \rangle = 1$, so the wheel spins clockwise (looking towards positive y).

c. $\langle 0, 1, 0 \rangle \cdot \langle 0, 0, 1 \rangle = 0$, so the wheel does not spin.

17.5.54 The curl of the vector field is $\nabla \times \mathbf{v} = \langle -2, 0, 2 \rangle$.

a. The wheel is placed with its axis in the direction $\langle 0, 0, 1 \rangle$, so the component of velocity in that direction is $\langle -2, 0, 2 \rangle \cdot \langle 0, 0, 1 \rangle = 2$, and $\omega = \frac{1}{2} \cdot 2 = 1$.

- b. The wheel is placed with its axis in the direction $\langle 0, 1, 0 \rangle$, so the component of velocity in that direction is $\langle -2, 0, 2 \rangle \cdot \langle 0, 1, 0 \rangle = 0$, and the wheel does not turn.
- c. The wheel is placed with its axis in the direction $\langle 1, 0, 0 \rangle$, so the component of velocity in that direction is $\langle -2, 0, 2 \rangle \cdot \langle 1, 0, 0 \rangle = -2$, and $\omega = \frac{1}{2} \cdot |-2| = 1$.

17.5.55 The curl of the vector field is $\nabla \times \mathbf{v} = \langle -20, 0, 0 \rangle$. Because the wheel is placed with its axis normal to the plane $x + y + z = 1$, its axis must point in the direction $\langle 1, 1, 1 \rangle$ (with unit vector $\frac{1}{\sqrt{3}} \langle 1, 1, 1 \rangle$). Thus, the component of velocity along that direction is $\frac{1}{\sqrt{3}} \langle -20, 0, 0 \rangle \cdot \langle 1, 1, 1 \rangle = \frac{-20}{\sqrt{3}}$ and then ω is the absolute value of one half of that amount, or $\omega = \frac{10}{\sqrt{3}}$ or $\frac{5}{\pi\sqrt{3}} \approx 0.9189$ revolutions per time unit.

17.5.56

$$\mathbf{F} = -100k \nabla e^{-\sqrt{x^2+y^2+z^2}} = \frac{100k e^{-\sqrt{x^2+y^2+z^2}}}{\sqrt{x^2+y^2+z^2}} \langle x, y, z \rangle.$$

Looking at the x component, its contribution to the divergence is

$$100k \frac{\partial}{\partial x} \left[\frac{x e^{-\sqrt{x^2+y^2+z^2}}}{\sqrt{x^2+y^2+z^2}} \right] = -100k \frac{\left(x^2 \sqrt{x^2+y^2+z^2} - y^2 - z^2 \right) e^{-\sqrt{x^2+y^2+z^2}}}{(x^2+y^2+z^2)^{3/2}}$$

and similarly for the y and z components. Thus the divergence is the sum of these three terms, which is

$$\begin{aligned} & -100k \frac{e^{-\sqrt{x^2+y^2+z^2}}}{(x^2+y^2+z^2)^{3/2}} \left((x^2+y^2+z^2)^{3/2} - 2(x^2+y^2+z^2) \right) \\ & = 100k \frac{e^{-\sqrt{x^2+y^2+z^2}}}{\sqrt{x^2+y^2+z^2}} \left(2 - \sqrt{x^2+y^2+z^2} \right). \end{aligned}$$

17.5.57 $\mathbf{F} = -100k \nabla e^{-x^2+y^2+z^2} = -200k e^{-x^2+y^2+z^2} \langle -x, y, z \rangle$, so the divergence is

$$\begin{aligned} & -200k \left(\frac{\partial}{\partial x} \left(-x e^{-x^2+y^2+z^2} \right) + \frac{\partial}{\partial y} \left(y e^{-x^2+y^2+z^2} \right) + \frac{\partial}{\partial z} \left(z e^{-x^2+y^2+z^2} \right) \right) \\ & = -200k \left(-e^{-x^2+y^2+z^2} + 2x^2 e^{-x^2+y^2+z^2} + e^{-x^2+y^2+z^2} + 2y^2 e^{-x^2+y^2+z^2} + e^{-x^2+y^2+z^2} + 2z^2 e^{-x^2+y^2+z^2} \right) \\ & = -200k \left(e^{-x^2+y^2+z^2} + 2(x^2+y^2+z^2) e^{-x^2+y^2+z^2} \right) \\ & = -200k (2x^2 + 2y^2 + 2z^2 + 1) e^{-x^2+y^2+z^2}. \end{aligned}$$

17.5.58 $\mathbf{F} = -100k \nabla \left(1 + \sqrt{1+x^2+y^2+z^2} \right) = -100k (x^2+y^2+z^2)^{-1/2} \langle x, y, z \rangle$, and thus the divergence is

$$\nabla \cdot \mathbf{F} = \frac{-200k}{\sqrt{x^2+y^2+z^2}}.$$

17.5.59

$$\begin{aligned} \text{a. } \mathbf{F} &= -\nabla \varphi = -GMm \left\langle \frac{\partial}{\partial x} \left[\frac{1}{\sqrt{x^2+y^2+z^2}} \right], \frac{\partial}{\partial y} \left[\frac{1}{\sqrt{x^2+y^2+z^2}} \right], \frac{\partial}{\partial z} \left[\frac{1}{\sqrt{x^2+y^2+z^2}} \right] \right\rangle = \\ &= -GMm (x^2+y^2+z^2)^{-3/2} \langle -x, -y, -z \rangle = GMm (x^2+y^2+z^2)^{-3/2} \langle x, y, z \rangle = GMm \frac{\mathbf{r}}{|\mathbf{r}|^3}. \end{aligned}$$

- b. $\frac{\partial}{\partial y} x (x^2 + y^2 + z^2)^{-3/2} = -3xy (x^2 + y^2 + z^2)^{-5/2}$. Applying this pattern in computing the curl gives $\nabla \times \mathbf{F} = GMm (x^2 + y^2 + z^2)^{-5/2} ((-3yz + 3yz)\mathbf{i} + (-3xz + 3xz)\mathbf{j} + (-3xy + 3xy)\mathbf{k}) = \mathbf{0}$, so the field is irrotational.

17.5.60 Note: this is identical to the previous problem except for the constant.

- a. $\mathbf{F} = -\nabla\varphi = -\frac{q}{4\pi\epsilon_0} \left\langle \frac{\partial}{\partial x} \left[\frac{1}{\sqrt{x^2 + y^2 + z^2}} \right], \frac{\partial}{\partial y} \left[\frac{1}{\sqrt{x^2 + y^2 + z^2}} \right], \frac{\partial}{\partial z} \left[\frac{1}{\sqrt{x^2 + y^2 + z^2}} \right] \right\rangle = -\frac{q}{4\pi\epsilon_0} (x^2 + y^2 + z^2)^{-3/2} \langle -x, -y, -z \rangle = \frac{q}{4\pi\epsilon_0} (x^2 + y^2 + z^2)^{-3/2} \langle x, y, z \rangle = \frac{q}{4\pi\epsilon_0} \frac{\mathbf{r}}{|\mathbf{r}|^3}$.
- b. $\frac{\partial}{\partial y} x (x^2 + y^2 + z^2)^{-3/2} = -3xy (x^2 + y^2 + z^2)^{-5/2}$. Applying this pattern in computing the curl gives $\nabla \times \mathbf{F} = \frac{q}{4\pi\epsilon_0} (x^2 + y^2 + z^2)^{-5/2} ((-3yz + 3yz)\mathbf{i} + (-3xz + 3xz)\mathbf{j} + (-3xy + 3xy)\mathbf{k}) = \mathbf{0}$, so the field is irrotational.

17.5.61 Using Exercise 40, we have

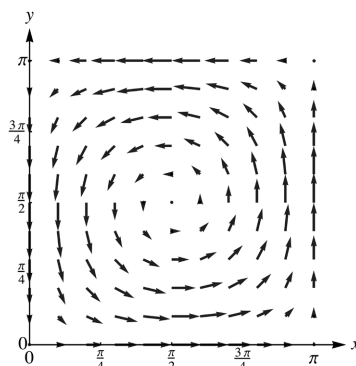
$$\rho \left(\left\langle \frac{\partial u}{\partial t}, \frac{\partial v}{\partial t}, \frac{\partial w}{\partial t} \right\rangle + \left(u \frac{\partial}{\partial t} + v \frac{\partial}{\partial t} + w \frac{\partial}{\partial t} \right) \langle u, v, w \rangle \right) = -\left\langle \frac{\partial p}{\partial t}, \frac{\partial p}{\partial t}, \frac{\partial p}{\partial t} \right\rangle + \mu \left(\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial t^2} \right) \langle u, v, w \rangle,$$

so that

$$\begin{aligned} \rho \left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right) &= -\frac{\partial p}{\partial x} + \mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) \\ \rho \left(\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} \right) &= -\frac{\partial p}{\partial y} + \mu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right) \\ \rho \left(\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} \right) &= -\frac{\partial p}{\partial z} + \mu \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right). \end{aligned}$$

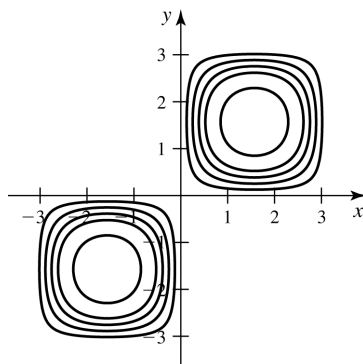
17.5.62

- a. $\nabla \times \langle 2, -3y, 5z \rangle = \mathbf{0}$ and $\nabla \times \langle y, x - z, -y \rangle = \mathbf{0}$ so they are both irrotational.
- b. If ψ is defined as stated, then $\nabla^2 \psi = \frac{\partial^2}{\partial x^2} \psi + \frac{\partial^2}{\partial y^2} \psi + \frac{\partial^2}{\partial z^2} \psi = -\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}$ while $\zeta = \mathbf{k} \cdot \nabla \times \langle u, v, 0 \rangle = \mathbf{k} \cdot \left(-\frac{\partial v}{\partial z} \mathbf{i} + \frac{\partial u}{\partial z} \mathbf{j} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \mathbf{k} \right) = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$ so that $\nabla^2 \psi = -\zeta$ as desired.
- c. $u = \frac{\partial \psi}{\partial y} = \sin(x) \cos(y)$ and $v = -\frac{\partial \psi}{\partial x} = -\cos(x) \sin(y)$. The velocity field looks like



- d. The vorticity function is $\zeta = -\nabla^2\psi = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \sin(x)\sin(y) + \sin(x)\sin(y) = 2\sin(x)\sin(y)$

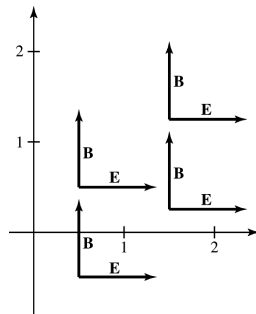
The diagram shows level curves for ζ at $\frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$ and $\frac{3}{2}$ (from outer to inner).



Using implicit differentiation (or from looking at the diagram), ζ achieves its maximum at $x = y = \frac{\pi}{2}$, where it has value 2, and its minimum on the boundary, where it is zero.

17.5.63

- a. We have $\nabla \times \mathbf{B} = -\frac{\partial}{\partial z}(A \sin(kz - \omega t))\mathbf{i} + 0\mathbf{j} + \frac{\partial}{\partial x}(A \sin(kz - \omega t))\mathbf{k} = -Ak \cos(kz - \omega t)\mathbf{i}$.
 $C \frac{\partial \mathbf{E}}{\partial t} = C \frac{\partial}{\partial t}(A \sin(kz - \omega t)\mathbf{i}) = -\omega CA \cos(kz - \omega t)\mathbf{i}$ so that the two are equal when $k = \omega C$, or
 $\omega = \frac{k}{C}$.
- b.



17.5.64 Let $\mathbf{V} = \langle xy, -\frac{1}{2}y^2, 0 \rangle$ and $\mathbf{W} = \langle 0, \frac{1}{2}y^2, 0 \rangle$. Then $\nabla \cdot \mathbf{V} = 0$ and $\nabla \times \mathbf{W} = \mathbf{0}$.

17.5.65 Let $\mathbf{F} = \langle f, g, h \rangle$ and $\mathbf{G} = \langle k, m, n \rangle$. Then

- a. $\nabla \cdot (\mathbf{F} + \mathbf{G}) = \nabla \cdot \langle f + k, g + m, h + n \rangle = \frac{\partial}{\partial x}(f + k) + \frac{\partial}{\partial y}(g + m) + \frac{\partial}{\partial z}(h + n) = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z} + \frac{\partial k}{\partial x} + \frac{\partial m}{\partial y} + \frac{\partial n}{\partial z} = \nabla \cdot \mathbf{F} + \nabla \cdot \mathbf{G}$.

b.

$$\begin{aligned} \nabla \times (\mathbf{F} + \mathbf{G}) &= \left(\frac{\partial}{\partial y}(h + n) - \frac{\partial}{\partial z}(g + m) \right) \mathbf{i} + \left(\frac{\partial}{\partial z}(f + k) - \frac{\partial}{\partial x}(h + n) \right) \mathbf{j} + \left(\frac{\partial}{\partial x}(g + m) - \frac{\partial}{\partial y}(f + k) \right) \mathbf{k} \\ &= \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z} \right) \mathbf{i} + \left(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x} \right) \mathbf{j} + \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) \mathbf{k} + \left(\frac{\partial n}{\partial y} - \frac{\partial m}{\partial z} \right) \mathbf{i} + \left(\frac{\partial k}{\partial z} - \frac{\partial n}{\partial x} \right) \mathbf{j} + \left(\frac{\partial m}{\partial x} - \frac{\partial k}{\partial y} \right) \mathbf{k} \\ &= \nabla \times \mathbf{F} + \nabla \times \mathbf{G}. \end{aligned}$$

$$c. \nabla \cdot (c\mathbf{F}) = \frac{\partial}{\partial x}(cf) + \frac{\partial}{\partial y}(cg) + \frac{\partial}{\partial z}(ch) = c \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z} \right) = c(\nabla \cdot \mathbf{F}).$$

$$d. \nabla \times (c\mathbf{F}) = \left(\frac{\partial}{\partial y}(ch) - \frac{\partial}{\partial z}(cg) \right) \mathbf{i} + \left(\frac{\partial}{\partial z}(cf) - \frac{\partial}{\partial x}(ch) \right) \mathbf{j} + \left(\frac{\partial}{\partial x}(cg) - \frac{\partial}{\partial y}(cf) \right) \mathbf{k} = \\ c \left[\left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z} \right) \mathbf{i} + \left(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x} \right) \mathbf{j} + \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) \mathbf{k} \right] = c(\nabla \times \mathbf{F}).$$

17.5.66 The statement is not true. The conditions imply that $\mathbf{F} - \mathbf{G}$ is irrotational and source-free, but this can happen with nonconstant vector fields. For example, if $\mathbf{F} = \langle x^2 - y^2, -2xy \rangle$ and $\mathbf{G} = \langle 2xy, x^2 - y^2 \rangle$, then $\mathbf{F} - \mathbf{G}$ is irrotational and source-free (i.e. has zero curl and zero divergence). Clearly the two vector fields do not differ by a constant.

$$17.5.67 \quad \nabla \cdot (\varphi \mathbf{F}) = \nabla \cdot \langle \varphi f, \varphi g, \varphi h \rangle = \frac{\partial}{\partial x}(\varphi f) + \frac{\partial}{\partial y}(\varphi g) + \frac{\partial}{\partial z}(\varphi h) = \varphi \frac{\partial f}{\partial x} + f \frac{\partial \varphi}{\partial x} + \varphi \frac{\partial g}{\partial y} + g \frac{\partial \varphi}{\partial y} + \varphi \frac{\partial h}{\partial z} + \\ h \frac{\partial \varphi}{\partial z} = \varphi \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z} \right) + \langle f, g, h \rangle \left\langle \frac{\partial \varphi}{\partial x}, \frac{\partial \varphi}{\partial y}, \frac{\partial \varphi}{\partial z} \right\rangle = \varphi \nabla \cdot \mathbf{F} + \nabla \varphi \cdot \mathbf{F}$$

$$17.5.68 \quad \nabla \times (\varphi \mathbf{F}) = \nabla \times \langle \varphi f, \varphi g, \varphi h \rangle = \left\langle \frac{\partial}{\partial y}(\varphi h) - \frac{\partial}{\partial z}(\varphi g), \frac{\partial}{\partial z}(\varphi f) - \frac{\partial}{\partial x}(\varphi h), \frac{\partial}{\partial x}(\varphi g) - \frac{\partial}{\partial y}(\varphi f) \right\rangle = \\ \left\langle \varphi \frac{\partial h}{\partial y} + h \frac{\partial \varphi}{\partial y} - \varphi \frac{\partial g}{\partial z} - g \frac{\partial \varphi}{\partial z}, \varphi \frac{\partial f}{\partial z} + f \frac{\partial \varphi}{\partial z} - \varphi \frac{\partial h}{\partial x} - h \frac{\partial \varphi}{\partial x}, \varphi \frac{\partial g}{\partial x} + g \frac{\partial \varphi}{\partial x} - \varphi \frac{\partial f}{\partial y} - f \frac{\partial \varphi}{\partial y} \right\rangle = \left\langle \varphi \frac{\partial h}{\partial y} - \varphi \frac{\partial g}{\partial z}, \varphi \frac{\partial f}{\partial z} - \varphi \frac{\partial h}{\partial x}, \varphi \frac{\partial g}{\partial x} - \varphi \frac{\partial f}{\partial y} \right\rangle + \\ \left\langle h \frac{\partial \varphi}{\partial y} - g \frac{\partial \varphi}{\partial z}, f \frac{\partial \varphi}{\partial z} - h \frac{\partial \varphi}{\partial x}, g \frac{\partial \varphi}{\partial x} - f \frac{\partial \varphi}{\partial y} \right\rangle = \varphi \nabla \times \mathbf{F} + \nabla \varphi \times \mathbf{F}$$

17.5.69 If $\mathbf{F} = \langle f, g, h \rangle$ and $\mathbf{G} = \langle k, m, n \rangle$, then

$$\mathbf{G} \cdot (\nabla \times \mathbf{F}) - \mathbf{F} \cdot (\nabla \times \mathbf{G}) = \langle k, m, n \rangle \cdot \left\langle \frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}, \frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}, \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right\rangle \\ - \langle f, g, h \rangle \cdot \left\langle \frac{\partial n}{\partial y} - \frac{\partial m}{\partial z}, \frac{\partial k}{\partial z} - \frac{\partial n}{\partial x}, \frac{\partial m}{\partial x} - \frac{\partial k}{\partial y} \right\rangle \\ = k \frac{\partial h}{\partial y} - k \frac{\partial g}{\partial z} + m \frac{\partial f}{\partial z} - m \frac{\partial h}{\partial x} + n \frac{\partial g}{\partial x} - n \frac{\partial f}{\partial y} \\ - f \frac{\partial n}{\partial y} + f \frac{\partial m}{\partial z} - g \frac{\partial k}{\partial z} + g \frac{\partial n}{\partial x} - h \frac{\partial m}{\partial x} + h \frac{\partial k}{\partial y} \\ = n \frac{\partial g}{\partial x} + g \frac{\partial n}{\partial x} - h \frac{\partial m}{\partial x} - m \frac{\partial h}{\partial x} + h \frac{\partial k}{\partial y} + k \frac{\partial h}{\partial y} - f \frac{\partial n}{\partial y} - n \frac{\partial f}{\partial y} \\ + f \frac{\partial m}{\partial z} + m \frac{\partial f}{\partial z} - g \frac{\partial k}{\partial z} - k \frac{\partial g}{\partial z} \\ = \frac{\partial}{\partial x}(gn - hm) + \frac{\partial}{\partial y}(hk - fn) + \frac{\partial}{\partial z}(fm - gk) = \nabla \cdot (\mathbf{F} \times \mathbf{G}).$$

$$17.5.70 \quad \text{First, } (\mathbf{G} \cdot \nabla) \mathbf{F} = \left(k \frac{\partial}{\partial x} + m \frac{\partial}{\partial y} + n \frac{\partial}{\partial z} \right) \langle f, g, h \rangle = \left\langle k \frac{\partial f}{\partial x} + m \frac{\partial f}{\partial y} + n \frac{\partial f}{\partial z}, k \frac{\partial g}{\partial x} + m \frac{\partial g}{\partial y} + n \frac{\partial g}{\partial z}, k \frac{\partial h}{\partial x} + \right. \\ \left. m \frac{\partial h}{\partial y} + n \frac{\partial h}{\partial z} \right\rangle \text{ and similarly for } (\mathbf{F} \cdot \nabla) \mathbf{G}. \text{ Next, } \mathbf{G}(\nabla \cdot \mathbf{F}) = \langle k, m, n \rangle \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z} \right) = \left\langle k \frac{\partial f}{\partial x} + k \frac{\partial g}{\partial y} + \right.$$

$k\frac{\partial h}{\partial z}, m\frac{\partial f}{\partial x} + m\frac{\partial g}{\partial y} + m\frac{\partial h}{\partial z}, n\frac{\partial f}{\partial x} + n\frac{\partial g}{\partial y} + n\frac{\partial h}{\partial z}\rangle$ and similarly for $\mathbf{F}(\nabla \cdot \mathbf{G})$. Thus

$$\begin{aligned} & (\mathbf{G} \cdot \nabla) \mathbf{F} - \mathbf{G}(\nabla \cdot \mathbf{F}) - (\mathbf{F} \cdot \nabla) \mathbf{G} + \mathbf{F}(\nabla \cdot \mathbf{G}) \\ &= \langle k\frac{\partial f}{\partial x} + m\frac{\partial f}{\partial y} + n\frac{\partial f}{\partial z}, k\frac{\partial g}{\partial x} + m\frac{\partial g}{\partial y} + n\frac{\partial g}{\partial z}, k\frac{\partial h}{\partial x} + m\frac{\partial h}{\partial y} + n\frac{\partial h}{\partial z} \rangle \\ &\quad - \langle k\frac{\partial f}{\partial x} + k\frac{\partial g}{\partial y} + k\frac{\partial h}{\partial z}, m\frac{\partial f}{\partial x} + m\frac{\partial g}{\partial y} + m\frac{\partial h}{\partial z}, n\frac{\partial f}{\partial x} + n\frac{\partial g}{\partial y} + n\frac{\partial h}{\partial z} \rangle \\ &\quad - \langle f\frac{\partial k}{\partial x} + g\frac{\partial k}{\partial y} + h\frac{\partial k}{\partial z}, f\frac{\partial m}{\partial x} + g\frac{\partial m}{\partial y} + h\frac{\partial m}{\partial z}, f\frac{\partial n}{\partial x} + g\frac{\partial n}{\partial y} + h\frac{\partial n}{\partial z} \rangle \\ &\quad + \langle f\frac{\partial k}{\partial x} + f\frac{\partial m}{\partial y} + f\frac{\partial n}{\partial z}, g\frac{\partial k}{\partial x} + g\frac{\partial m}{\partial y} + g\frac{\partial n}{\partial z}, h\frac{\partial k}{\partial x} + h\frac{\partial m}{\partial y} + h\frac{\partial n}{\partial z} \rangle \\ &= \langle \frac{\partial}{\partial y}(fm - gk) - \frac{\partial}{\partial z}(hk - fn), \frac{\partial}{\partial z}(gn - hm) - \frac{\partial}{\partial x}(fm - gk), \frac{\partial}{\partial x}(hk - fn) - \frac{\partial}{\partial y}(gn - hm) \rangle. \end{aligned}$$

But $\mathbf{F} \times \mathbf{G} = \langle gn - hm, hk - fn, fm - gk \rangle$, so the above expression is indeed equal to $\nabla \times (\mathbf{F} \times \mathbf{G})$.

17.5.71 Use the values of $(\mathbf{G} \cdot \nabla) \mathbf{F}$ and $(\mathbf{F} \cdot \nabla) \mathbf{G}$ from the previous problem. Then $\mathbf{G} \times (\nabla \times \mathbf{F}) = \langle k, m, n \rangle \times \langle \frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}, \frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}, \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \rangle = \langle m\frac{\partial g}{\partial x} - m\frac{\partial f}{\partial y} - n\frac{\partial f}{\partial z} + n\frac{\partial h}{\partial x}, n\frac{\partial h}{\partial y} - n\frac{\partial g}{\partial z} - k\frac{\partial g}{\partial x} + k\frac{\partial f}{\partial y}, k\frac{\partial f}{\partial z} - k\frac{\partial h}{\partial x} - m\frac{\partial h}{\partial y} + m\frac{\partial g}{\partial z} \rangle$ and similarly for $\mathbf{F} \times (\nabla \times \mathbf{G})$.

Thus

$$\begin{aligned} & (\mathbf{G} \cdot \nabla) \mathbf{F} + (\mathbf{F} \cdot \nabla) \mathbf{G} + \mathbf{G} \times (\nabla \times \mathbf{F}) + \mathbf{F} \times (\nabla \times \mathbf{G}) \\ &= \langle k\frac{\partial f}{\partial x} + m\frac{\partial f}{\partial y} + n\frac{\partial f}{\partial z}, k\frac{\partial g}{\partial x} + m\frac{\partial g}{\partial y} + n\frac{\partial g}{\partial z}, k\frac{\partial h}{\partial x} + m\frac{\partial h}{\partial y} + n\frac{\partial h}{\partial z} \rangle \\ &\quad + \langle f\frac{\partial k}{\partial x} + g\frac{\partial k}{\partial y} + h\frac{\partial k}{\partial z}, f\frac{\partial m}{\partial x} + g\frac{\partial m}{\partial y} + h\frac{\partial m}{\partial z}, f\frac{\partial n}{\partial x} + g\frac{\partial n}{\partial y} + h\frac{\partial n}{\partial z} \rangle \\ &\quad + \langle m\frac{\partial g}{\partial x} - m\frac{\partial f}{\partial y} - n\frac{\partial f}{\partial z} + n\frac{\partial h}{\partial x}, n\frac{\partial h}{\partial y} - n\frac{\partial g}{\partial z} - k\frac{\partial g}{\partial x} + k\frac{\partial f}{\partial y}, k\frac{\partial f}{\partial z} - k\frac{\partial h}{\partial x} - m\frac{\partial h}{\partial y} + m\frac{\partial g}{\partial z} \rangle \\ &\quad + \langle g\frac{\partial m}{\partial x} - g\frac{\partial k}{\partial y} - h\frac{\partial k}{\partial z} + h\frac{\partial n}{\partial x}, h\frac{\partial n}{\partial y} - h\frac{\partial m}{\partial z} - f\frac{\partial m}{\partial x} + f\frac{\partial k}{\partial y}, f\frac{\partial k}{\partial z} - f\frac{\partial n}{\partial x} - g\frac{\partial n}{\partial y} + g\frac{\partial m}{\partial z} \rangle \\ &= \langle \frac{\partial}{\partial x}(fk + gm + hn), \frac{\partial}{\partial y}(fk + gm + hn), \frac{\partial}{\partial z}(fk + gm + hn) \rangle = \nabla(\mathbf{F} \cdot \mathbf{G}). \end{aligned}$$

17.5.72

$$\begin{aligned} \nabla \times (\nabla \times \mathbf{F}) &= \nabla \times \langle \frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}, \frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}, \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \rangle \\ &= \langle \frac{\partial^2 g}{\partial y \partial x} - \frac{\partial^2 f}{\partial y^2} - \frac{\partial^2 f}{\partial z^2} + \frac{\partial^2 h}{\partial z \partial x}, \frac{\partial^2 h}{\partial z \partial y} - \frac{\partial^2 g}{\partial z^2} - \frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 f}{\partial x \partial y}, \frac{\partial^2 f}{\partial x \partial z} - \frac{\partial^2 h}{\partial x^2} - \frac{\partial^2 h}{\partial y^2} - \frac{\partial^2 g}{\partial y \partial z} \rangle \end{aligned}$$

and

$$\begin{aligned} \nabla(\nabla \cdot \mathbf{F}) - (\nabla \cdot \nabla) \mathbf{F} &= \nabla \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z} \right) - \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \langle f, g, h \rangle \\ &= \langle \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 g}{\partial x \partial y} + \frac{\partial^2 h}{\partial x \partial z}, \frac{\partial^2 f}{\partial y \partial x} + \frac{\partial^2 g}{\partial y^2} + \frac{\partial^2 h}{\partial y \partial z}, \frac{\partial^2 f}{\partial z \partial x} + \frac{\partial^2 g}{\partial z \partial y} + \frac{\partial^2 h}{\partial z^2} \rangle \\ &\quad - \langle \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}, \frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} + \frac{\partial^2 g}{\partial z^2}, \frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} + \frac{\partial^2 h}{\partial z^2} \rangle. \end{aligned}$$

The two expressions are equal after cancellations and noting that mixed partials are equal.

17.5.73

$$\begin{aligned}
& \nabla \cdot \frac{\langle x, y, z \rangle}{(x^2 + y^2 + z^2)^{p/2}} \\
&= \frac{(1-p)x^2 + y^2 + z^2}{(x^2 + y^2 + z^2)^{1+p/2}} + \frac{x^2 + (1-p)y^2 + z^2}{(x^2 + y^2 + z^2)^{1+p/2}} + \frac{x^2 + y^2 + (1-p)z^2}{(x^2 + y^2 + z^2)^{1+p/2}} \\
&= \frac{(3-p)(x^2 + y^2 + z^2)}{(x^2 + y^2 + z^2)^{1+p/2}} = \frac{3-p}{(x^2 + y^2 + z^2)^{p/2}} = \frac{3-p}{|\mathbf{r}|^p}
\end{aligned}$$

$$17.5.74 \quad \nabla \left(\frac{1}{|\mathbf{r}|^p} \right) = \nabla \left(\frac{1}{(x^2 + y^2 + z^2)^{p/2}} \right) = -\frac{p}{(x^2 + y^2 + z^2)^{1+p/2}} \langle x, y, z \rangle = -\frac{p\mathbf{r}}{|\mathbf{r}|^{p+2}}.$$

$$\begin{aligned}
17.5.75 \quad \nabla \cdot \nabla \left(\frac{1}{|\mathbf{r}|^p} \right) &= \nabla \cdot \left(-\frac{p\mathbf{r}}{|\mathbf{r}|^{p+2}} \right) \text{ by Exercise 72, and then by Exercise 71, } \nabla \cdot \left(-\frac{p\mathbf{r}}{|\mathbf{r}|^{p+2}} \right) = \\
&= -p \nabla \cdot \frac{\mathbf{r}}{|\mathbf{r}|^{p+2}} = \frac{-p(3-(p+2))}{|\mathbf{r}|^{p+2}} = \frac{p(p-1)}{|\mathbf{r}|^{p+2}}.
\end{aligned}$$

17.6 Surface Integrals

17.6.1 $\mathbf{r}(u, v) = \langle a \cos u, a \sin u, v \rangle$ where $0 \leq u \leq 2\pi$; $0 \leq v \leq h$.

17.6.2 $\mathbf{r}(u, v) = \left\langle \frac{av}{h} \cos u, \frac{av}{h} \sin u, v \right\rangle$ where $0 \leq u \leq 2\pi$; $0 \leq v \leq h$.

17.6.3 $\mathbf{r}(u, v) = \langle a \sin u \cos v, a \sin u \sin v, a \cos u \rangle$ where $0 \leq u \leq \pi$; $0 \leq v \leq 2\pi$.

17.6.4 A cone of height h and radius a has equation $a^2 z^2 = h^2 (x^2 + y^2)$; thus $z_x = \frac{h^2 x}{a^2 z}$ and similarly for z_y , so compute

$$\begin{aligned}
\iint_S f(x, y, z) \, dS &= \iint_R f\left(x, y, \frac{h^2}{a^2} \sqrt{x^2 + y^2}\right) \sqrt{\left(\frac{h^2 x}{a^2 z}\right)^2 + \left(\frac{h^2 y}{a^2 z}\right)^2 + 1} \, dA \\
&= \iint_R f\left(x, y, \frac{h^2}{a^2} \sqrt{x^2 + y^2}\right) \sqrt{1 + \frac{h^2}{a^2}} \, dA.
\end{aligned}$$

17.6.5 Use the parametric description from problem 3 and compute

$$\int_0^\pi \int_0^{2\pi} a^2 f(a \sin u \cos v, a \sin u \sin v, a \cos u) \sin u \, du \, dv.$$

17.6.6 It means that we can make a consistent choice of normal vectors such that when you walk along the surface, the direction of the normal vectors does not change discontinuously.

17.6.7 The usual orientation of a closed surface is that the normal vectors point outwards.

17.6.8 Because the vector field is vertical, the same amount of materials goes through the surface as through its projection on the xy -plane.

$$17.6.9 \quad \left\langle u, v, \frac{1}{3}(16 - 2u + 4v) \right\rangle, |u| < \infty, |v| < \infty.$$

$$17.6.10 \quad \langle 4 \sin u \cos v, 4 \sin u \sin v, 4 \cos u \rangle, 0 \leq u \leq \frac{\pi}{4}, 0 \leq v \leq 2\pi.$$

17.6.11 $\langle v \cos u, v \sin u, v \rangle$, $0 \leq u \leq 2\pi$, $2 \leq v \leq 8$.

17.6.12 $\langle \frac{v}{2} \cos u, \frac{v}{2} \sin u, v \rangle$, $0 \leq u \leq 2\pi$, $0 \leq v \leq 4$.

17.6.13 $\langle 3 \cos u, 3 \sin u, v \rangle$, $0 \leq u \leq \frac{\pi}{2}$, $0 \leq v \leq 3$.

17.6.14 $\langle v, 6 \cos u, 6 \sin u \rangle$, $0 \leq u \leq 2\pi$, $0 \leq v \leq 9$.

17.6.15 The segment of the plane $z = 2x + 3y - 1$ above $[1, 3] \times [2, 4]$.

17.6.16 The segment of the plane $z = 2 - y$ above $[0, 2] \times [0, 4]$.

17.6.17 The portion of the cone $z^2 = 16x^2 + 16y^2$ of height 12 and radius 3, where $y \geq 0$.

17.6.18 The cylinder $y^2 + z^2 = 36$ of radius 6 whose axis is the x -axis, for $0 \leq x \leq 2$.

17.6.19 Using the standard parametric description of the cylinder, we have $\mathbf{r}(u, v) = \langle 4 \cos u, 4 \sin u, v \rangle$ for $0 \leq v \leq 7$, $0 \leq u \leq \pi$. Then $|\mathbf{t}_u \times \mathbf{t}_v| = 4$ and the area is $\iint_S 1 \, dS = \iint_R 4 \, dA = \int_0^\pi \int_0^7 4 \, dv \, du = 28\pi$.

17.6.20 The plane has the parametric description $\mathbf{r}(u, v) = \langle u, v, 3 - u - 3v \rangle$, $0 \leq v \leq 1$, $0 \leq u \leq 3 - 3v$. Then $\mathbf{t}_u \times \mathbf{t}_v = \langle 1, 0, -1 \rangle \times \langle 0, 1, -3 \rangle = \langle 1, 3, 1 \rangle$, so that $|\mathbf{t}_u \times \mathbf{t}_v| = \sqrt{11}$. Then

$$\iint_S 1 \, dS = \sqrt{11} \iint_R 1 \, dA = \sqrt{11} \int_0^1 \int_0^{3-3v} 1 \, du \, dv = \sqrt{11} \int_0^1 (3 - 3v) \, dv = \frac{3\sqrt{11}}{2}.$$

17.6.21 The plane has parametric description $\mathbf{r}(u, v) = \langle u, v, 10 - u - v \rangle$, for $-2 \leq u \leq 2$, $-2 \leq v \leq 2$. Then $\mathbf{t}_u \times \mathbf{t}_v = \langle 1, 0, -1 \rangle \times \langle 0, 1, -1 \rangle = \langle 1, 1, 1 \rangle$, so that $\iint_S 1 \, dS = \sqrt{3} \iint_R 1 \, dA = 16\sqrt{3}$.

17.6.22 Using the standard parametric description of the sphere with $0 \leq u \leq \frac{\pi}{2}$, $0 \leq v \leq 2\pi$, we have

$$|\mathbf{t}_u \times \mathbf{t}_v| = a^2 \sin u$$

so that

$$\iint_S 1 \, dS = \iint_R 100 \sin u \, dA = \int_0^{\pi/2} \int_0^{2\pi} 100 \sin u \, dv \, du = 200\pi \int_0^{\pi/2} \sin u \, du = 200\pi.$$

17.6.23 Parameterize the cone by $\mathbf{r}(u, v) = \langle \frac{r}{h}v \cos u, \frac{r}{h}v \sin u, v \rangle$, for $0 \leq v \leq h$, $0 \leq u \leq 2\pi$; then $\mathbf{t}_u \times \mathbf{t}_v = \langle -\frac{r}{h}v \sin u, \frac{r}{h}v \cos u, 0 \rangle \times \langle \frac{r}{h} \cos u, \frac{r}{h} \sin u, 1 \rangle$ and $|\mathbf{t}_u \times \mathbf{t}_v| = \frac{r}{h^2}v\sqrt{h^2 + r^2}$. Then

$$\begin{aligned} \iint_S 1 \, dS &= \frac{r\sqrt{h^2 + r^2}}{h^2} \iint_R v \, dA = \frac{r\sqrt{h^2 + r^2}}{h^2} \int_0^{2\pi} \int_0^h v \, dv \, du \\ &= \frac{r\sqrt{h^2 + r^2}}{h^2} \int_0^{2\pi} \frac{1}{2} h^2 \, du = \frac{2\pi r\sqrt{h^2 + r^2}}{2} = \pi r\sqrt{h^2 + r^2}. \end{aligned}$$

17.6.24 Using the standard parameterization of the sphere for $1 \leq 2 \cos u \leq 2$, or $0 \leq u \leq \frac{\pi}{3}$, and $0 \leq v \leq 2\pi$, we obtain

$$\iint_S 1 \, dS = \iint_R 4 \sin u \, dA = 4 \int_0^{2\pi} \int_0^{\pi/3} \sin u \, du \, dv = 4 \int_0^{2\pi} (-\cos u) \Big|_{u=0}^{u=\pi/3} dv = 4 \int_0^{2\pi} \frac{1}{2} \, dv = 4\pi.$$

17.6.25 Using the standard parameterization of the sphere for $0 \leq u \leq \frac{\pi}{2}$, $0 \leq v \leq 2\pi$, we obtain

$$\begin{aligned} \iint_S (x^2 + y^2) dS &= \int_0^{2\pi} \int_0^{\pi/2} 36 \sin^2 u \cdot 36 \sin u du dv = 1296 \int_0^{2\pi} \int_0^{\pi/2} \sin^3 u du dv \\ &= 1296 \int_0^{2\pi} \frac{2}{3} dv = 1728\pi. \end{aligned}$$

17.6.26 Use the standard parameterization of the cylinder, for $0 \leq u \leq 2\pi$, $0 \leq v \leq 3$; then

$$\iint_S y dS = \int_0^{2\pi} \int_0^3 3 \sin u \cdot 3 dv du = 27 \int_0^{2\pi} \sin u du = 0.$$

17.6.27 Use the standard parameterization, for $0 \leq u \leq 2\pi$, $0 \leq v \leq 3$; then

$$\iint_S x dS = \int_0^{2\pi} \int_0^3 \cos u \cdot 1 dv du = 3 \int_0^{2\pi} \cos u du = 0.$$

17.6.28 Using the standard parameterization (with $u = \varphi$ and $v = \theta$) for $0 \leq u \leq \frac{\pi}{2}$ and $0 \leq v \leq \frac{\pi}{2}$, we have

$$\iint_R \cos u dA = \int_0^{\pi/2} \int_0^{\pi/2} \cos u \cdot \sin u du dv = \frac{\pi}{4}.$$

17.6.29 $z_x = 2$ and $z_y = 2$, so $\sqrt{z_x^2 + z_y^2 + 1} = \sqrt{9} = 3$.

$$\int_0^2 \int_0^{2x} 3 dy dx = \int_0^2 6x dx = 3x^2 \Big|_0^2 = 12.$$

17.6.30 $z_x = 1$ and $z_y = 3$, so $\sqrt{1 + 9 + 1} = \sqrt{11}$.

$$\int_0^{2\pi} \int_1^2 \sqrt{11} r dr d\theta = 2\pi \sqrt{11} \left(\frac{r^2}{2} \right) \Big|_1^2 = 3\pi \sqrt{11}.$$

17.6.31 $2z dz = 8x dx$, so $z_x = \frac{4x}{z}$; similarly, $z_y = \frac{4y}{z}$. Thus $\sqrt{z_x^2 + z_y^2 + 1} = \sqrt{\frac{16x^2 + 16y^2 + z^2}{z^2}} = \sqrt{\frac{20(x^2 + y^2)}{4(x^2 + y^2)}} = \sqrt{5}$. Further, this cone sits over $x^2 + y^2 = 4$. Then $\iint_S 1 dS = \iint_R \sqrt{5} dA = 4\pi\sqrt{5}$.

17.6.32 $z_x = x$ and $z_y = 0$, so that $\iint_S 1 dS = \int_0^4 \int_{-1}^1 \sqrt{x^2 + 1} dx dy = 4\sqrt{2} + 4 \ln(1 + \sqrt{2})$.

17.6.33 $dz = 4x dx$ so that $z_x = 4x$ and similarly $z_y = 4y$. The paraboloid sits over $x^2 + y^2 = 4$. Thus $\iint_S 1 dS = \iint_R \sqrt{16x^2 + 16y^2 + 1} dA = \int_0^{2\pi} \int_0^2 r \sqrt{16r^2 + 1} dr d\theta = \frac{(65\sqrt{65} - 1)\pi}{24}$.

17.6.34 We have $z_x = 2x$, $z_y = -2y$, so

$$\iint_S 1 dS = \iint_R \sqrt{4x^2 + 4y^2 + 1} dA = \int_0^{\pi/2} \int_0^{\sqrt{2}} r \sqrt{4r^2 + 1} dr d\theta = \frac{13\pi}{12}.$$

17.6.35 $z_x = z_y = -1$, so $\iint_S xy \, dS = \sqrt{3} \iint_R xy \, dA = \sqrt{3} \int_0^2 \int_0^{2-x} xy \, dy \, dx = \frac{2\sqrt{3}}{3}$.

17.6.36 $z_x = 2x$, $z_y = 2y$, and the paraboloid sits over $x^2 + y^2 = 4$, so $\iint_S (x^2 + y^2) \, dS =$
 $\iint_R (x^2 + y^2) \sqrt{4x^2 + 4y^2 + 1} \, dA = \int_0^{2\pi} \int_0^1 r^3 \sqrt{4r^2 + 1} \, dr \, d\theta = 2\pi \int_0^1 r^3 \sqrt{4r^2 + 1} \, dr$. Let $u = 4r^2 + 1$ so
 that $du = 8r \, dr$. Substituting gives

$$\begin{aligned} 2\pi \cdot \frac{1}{8} \int_1^5 \frac{u-1}{4} \sqrt{u} \, du &= \frac{\pi}{16} \int_1^5 (u^{3/2} - u^{1/2}) \, du \\ &= \frac{\pi}{16} \left(\frac{2u^{5/2}}{5} - \frac{2u^{3/2}}{3} \right) \Big|_1^5 \\ &= \frac{\pi}{16} \left(10\sqrt{5} - \frac{10\sqrt{5}}{3} - \left(\frac{2}{5} - \frac{2}{3} \right) \right) \\ &= \frac{\pi}{16} \left(\frac{100\sqrt{5}}{15} + \frac{4}{15} \right) = \frac{\pi(1 + 25\sqrt{5})}{60}. \end{aligned}$$

17.6.37 $x^2 + y^2 + z^2 = 25$, so $z_x = -\frac{x}{z}$ and $z_y = -\frac{y}{z}$. Then $\iint_S (25 - x^2 - y^2) \, dS =$
 $\iint_R (25 - x^2 - y^2) \sqrt{\frac{x^2 + y^2 + z^2}{z^2}} \, dA = 5 \iint_R \sqrt{25 - x^2 - y^2} \, dA = 5 \int_0^{2\pi} \int_0^5 r \sqrt{25 - r^2} \, dr \, d\theta = \frac{1250\pi}{3}$.

17.6.38 $z_x = -1$, $z_y = -2$, and the limits of integration are $0 \leq y \leq 4$, $0 \leq x \leq 8 - 2y$. Then $\iint_S e^z \, dS =$
 $\iint_R e^{8-x-2y} \sqrt{6} \, dA = \sqrt{6} \int_0^4 \int_0^{8-2y} e^{8-x-2y} \, dx \, dy = \frac{\sqrt{6}(e^8 - 9)}{2}$.

17.6.39 $\iint_S 1 \, dS = \int_0^1 \int_0^1 \sqrt{1 + 4 + 4} \, dx \, dy = 3$.
 $\iint_S e^{2x+y+z-3} \, dS = 3 \int_0^1 \int_0^1 e^{2x+y+4-2x-2y-3} \, dx \, dy = 3 \int_0^1 \int_0^1 e^{1-y} \, dx \, dy = 3(e - 1)$. So the average
 temperature is $\frac{3(e-1)}{3} = e - 1$.

17.6.40 $z_x = -2x$, $z_y = -2y$. The paraboloid sits over $x^2 + y^2 = 4$. Thus the area of the paraboloid is

$$\iint_S 1 \, dS = \iint_R \sqrt{4x^2 + 4y^2 + 1} \, dA = \int_0^{2\pi} \int_0^2 r \sqrt{4r^2 + 1} \, dr \, d\theta \approx 36.1769,$$

and the integral of the square of the distance from the origin is

$$\begin{aligned} \iint_S (x^2 + y^2 + z^2) \, dS &= \iint_R (x^2 + y^2 + (4 - x^2 - y^2)^2) \sqrt{4x^2 + 4y^2 + 1} \, dA \\ &= \int_0^{2\pi} \int_0^2 r (r^2 + (4 - r^2)^2) \sqrt{4r^2 + 1} \, dr \, d\theta \approx 227.18. \end{aligned}$$

Thus the average squared distance is approximately 6.2797.

17.6.41 $z_x = -\frac{x}{z}$ and $z_y = -\frac{y}{z}$, so that $\sqrt{z_x^2 + z_y^2 + 1} = \sqrt{\frac{x^2 + y^2 + z^2}{z^2}} = (1 - x^2 - y^2)^{-1/2}$. The area of the sphere is $\frac{1}{8} \cdot 4\pi = \frac{\pi}{2}$, and the integral of the function is

$$\iint_S xyz \, dS = \iint_R xy (1 - x^2 - y^2)^{1/2} (1 - x^2 - y^2)^{-1/2} \, dA = \int_0^{\pi/2} \int_0^1 r^3 \sin(\theta) \cos(\theta) \, dr \, d\theta = \frac{1}{8},$$

so that the average value is $\frac{1}{4\pi}$.

17.6.42 $z_x = \frac{x}{z}$ and $z_y = \frac{y}{z}$ so that $\sqrt{z_x^2 + z_y^2 + 1} = \sqrt{2}$. The cone sits over $x^2 + y^2 = 4$. The area of the cone is $\iint_S 1 \, dS = \sqrt{2} \iint_R dA = 4\pi\sqrt{2}$, and the integral of the temperature function is

$$\iint_S (100 - 25z) \, dS = \sqrt{2} \iint_R (100 - 25\sqrt{x^2 + y^2}) \, dA = \sqrt{2} \int_0^{2\pi} \int_0^2 r (100 - 25r) \, dr \, d\theta = \frac{800\pi\sqrt{2}}{3} \text{ so}$$

that the average temperature is $\frac{200}{3}$.

17.6.43 $z_x = z_y = -1$, so the normal vector is $\langle 1, 1, 1 \rangle$, which points in the positive z -direction. Then

$$\iint_S \mathbf{F} \cdot \mathbf{n} \, dS = \iint_R (0 \cdot 1 + 0 \cdot 1 - 1 \cdot 1) \, dA = \int_0^4 \int_0^{4-x} (-1) \, dy \, dx = -8.$$

17.6.44 $z_x = -2$, $z_y = -5$, so the normal vector is $\langle 2, 5, 1 \rangle$, which points in the positive z -direction.

$$\begin{aligned} \iint_S \mathbf{F} \cdot \mathbf{n} \, dS &= \iint_R (x \cdot 2 + y \cdot 5 + z \cdot 1) \, dA = \int_0^2 \int_0^{(10-5y)/2} (2x + 5y + (10 - 2x - 5y)) \, dx \, dy \\ &= \int_0^2 \int_0^{(10-5y)/2} 10 \, dx \, dy = 50. \end{aligned}$$

17.6.45 We have $z = \sqrt{x^2 + y^2}$; then $\mathbf{n} = \left\langle -\frac{x}{z}, -\frac{y}{z}, 1 \right\rangle$, which points upwards.

$$\iint_S \mathbf{F} \cdot \mathbf{n} \, dS = \iint_R \left(\left(-\frac{x}{z} \right) \cdot x + \left(-\frac{y}{z} \right) \cdot y + 1 \cdot z \right) \, dA = \iint_R \left(z - \frac{x^2 + y^2}{z} \right) \, dA = \iint_R (z - z) \, dA = 0.$$

17.6.46 $z_x = 0$, $z_y = -\sin(y)$, so an upward-pointing normal is $\langle 0, \sin y, 1 \rangle$. We have $\iint_S \mathbf{F} \cdot \mathbf{n} \, dS =$

$$\iint_R (2 \sin y \cos y + xy) \, dA = \int_0^4 \int_{-\pi}^{\pi} (2 \sin y \cos y + xy) \, dy \, dx = 0.$$

17.6.47 An outward-pointing normal is $\frac{\mathbf{r}}{|\mathbf{r}|}$. The sphere has radius a , so the vector field is in fact $\frac{\mathbf{r}}{|\mathbf{r}|^3}$.

$$\iint_S \mathbf{F} \cdot \mathbf{n} \, dS = \iint_S \frac{\mathbf{r}}{|\mathbf{r}|^3} \cdot \frac{\mathbf{r}}{|\mathbf{r}|} \, dS = \iint_S \frac{1}{|\mathbf{r}|^2} \, dS = \iint_S \frac{1}{a^2} \, dS = \frac{1}{a^2} \iint_S 1 \, dS = \frac{1}{a^2} 4\pi a^2 = 4\pi.$$

17.6.48 The parametric form is $\langle u, u^2, v \rangle$ for $0 \leq u \leq 1$, $0 \leq v \leq 4$. We have $\mathbf{t}_u = \langle 1, 2u, 0 \rangle$ and $\mathbf{t}_v = \langle 0, 0, 1 \rangle$, so that $\mathbf{t}_u \times \mathbf{t}_v = \langle 2u, -1, 0 \rangle$; since we want normal vectors to point in the positive y direction, we choose $\langle -2u, 1, 0 \rangle$ for the normal vector. Then $\iint_S \mathbf{F} \cdot \mathbf{n} \, dS = \iint_R \langle -u^2, u, 1 \rangle \cdot \langle -2u, 1, 0 \rangle \, dA =$

$$\iint_R (u + 2u^3) \, dA = \int_0^4 \int_0^1 (u + 2u^3) \, du \, dv = 4.$$

17.6.49

a. True. The formula in Theorem 17.12 gives $\iint_S f(x, y, z) \, dS = \iint_R f(x, y, 10) \sqrt{0 + 0 + 1} \, dA$.

b. False. The formula in Theorem 17.12 gives

$$\iint_S f(x, y, z) \, dS = \iint_R f(x, y, x) \sqrt{1 + 0 + 1} \, dA = \sqrt{2} \iint_R f(x, y, x) \, dA.$$

c. True. Substituting $2u$ for u and $\sqrt{v}$ for v in the first parameterization gives $\langle \sqrt{v} \cos 2u, \sqrt{v} \sin 2u, v \rangle$, $0 \leq 2u \leq \pi$, $0 \leq \sqrt{v} \leq 2$. Simplifying the bounds conditions gives the second parameterization.

d. True. The standard parameterization is $\langle a \sin u \cos v, a \sin u \sin v, a \cos u \rangle$ for $0 \leq u \leq \pi$, $0 \leq v \leq 2\pi$. Then $\mathbf{t}_u \times \mathbf{t}_v = \langle a^2 \sin^2 u \cos v, a^2 \sin^2 u \sin v, a^2 \cos u \sin u \rangle$, and it is easily seen that these are outward-pointing vectors, by considering various ranges for u and v .

17.6.50 $\nabla \ln |\mathbf{r}| = \nabla \ln \sqrt{x^2 + y^2 + z^2} = \frac{1}{|\mathbf{r}|^2} \langle x, y, z \rangle = \frac{1}{a^2} \langle x, y, z \rangle$ on the sphere of radius a ; using the explicit description for the sphere, we have $\mathbf{n} = \langle \frac{x}{z}, \frac{y}{z}, 1 \rangle$, so that

$$\begin{aligned} \iint_S \nabla \ln |\mathbf{r}| \cdot \mathbf{n} \, dS &= \iint_R \frac{1}{a^2} \langle x, y, z \rangle \cdot \left\langle \frac{x}{z}, \frac{y}{z}, 1 \right\rangle \, dA = \frac{1}{a^2} \iint_R \left(\frac{x^2 + y^2}{z} + z \right) \, dA = \\ &= \frac{1}{a^2} \iint_R \left(\frac{x^2 + y^2 + z^2}{z} \right) \, dA = \frac{1}{a^2} \iint_R \frac{a^2}{z} \, dA = \iint_R \frac{1}{z} \, dA = \int_0^{2\pi} \int_0^a \frac{1}{\sqrt{a^2 - r^2}} r \, dr \, d\theta = 2\pi a. \end{aligned}$$

17.6.51 Parameterize the surface by $\langle 2 \cos u, 2 \sin u, v \rangle$ for $0 \leq u \leq 2\pi$, $0 \leq v \leq 8$. Then the normal vector has magnitude 2, and

$$\iint_S |\mathbf{r}| \, dS = \iint_R \sqrt{x^2 + y^2 + z^2} \, dA = 2 \int_0^{2\pi} \int_0^8 \sqrt{4 + v^2} \, dv \, du = 8\pi \left(4\sqrt{17} + \ln(4 + \sqrt{17}) \right).$$

17.6.52 $z_x = 0$, $z_y = -1$, so

$$\begin{aligned} \iint_S xyz \, dS &= \iint_R xyz \sqrt{2} \, dA = \sqrt{2} \int_0^{2\pi} \int_0^2 r \cdot r \cos \theta \cdot r \sin \theta \cdot (6 - r \sin \theta) \, dr \, d\theta = \\ &= \sqrt{2} \int_0^{2\pi} \int_0^2 r^3 \cos \theta \sin \theta (6 - r \sin \theta) \, dr \, d\theta = 0. \end{aligned}$$

17.6.53 The normal vector is $\langle x, 0, z \rangle$, so

$$\iint_S \frac{1}{\sqrt{x^2 + z^2}} \langle x, 0, z \rangle \cdot \langle x, 0, z \rangle \, dS = \iint_R \sqrt{x^2 + z^2} \, dA = \int_{-2}^2 \int_0^{2\pi} a \, dA = 8\pi a.$$

17.6.54 The two curves intersect where $x^2 + y^2 = 16 - x^2 - y^2$, so on the plane $z = 2\sqrt{2}$. The projection on the xy -plane of the circle of intersection is $x^2 + y^2 = 8$. The total surface area of the sphere, of radius 4, is 64π . Finally, the outward normals to the sphere are $\langle \frac{x}{z}, \frac{y}{z}, 1 \rangle$.

a. From part (b) below and the fact that the total surface area is 64π , we get an answer of $64\pi - 32\pi + 16\pi\sqrt{2} = 16\pi(2 + \sqrt{2})$.

b.

$$\begin{aligned}\iint_S 1 \, dS &= \iint_R \sqrt{\frac{x^2}{z^2} + \frac{y^2}{z^2} + 1} \, dA = \iint_R \frac{4}{z} \, dA = \iint_R \frac{4}{\sqrt{16 - x^2 - y^2}} \, dA \\ &= \int_0^{2\pi} \int_0^{\sqrt{8}} \frac{4r}{\sqrt{16 - r^2}} \, dr \, d\theta = 16\pi (2 - \sqrt{2}).\end{aligned}$$

c. For the cone $z^2 = x^2 + y^2$, we have $\sqrt{z_x^2 + z_y^2 + 1} = \sqrt{2}$, so

$$\iint_S 1 \, dS = \iint_R \sqrt{2} \, dA = \sqrt{2} \cdot \pi \cdot (\sqrt{8})^2 = 8\pi\sqrt{2}.$$

17.6.55 The surface of the cylinder inside the sphere is defined parametrically by $\langle 1 + \cos u, \sin u, v \rangle$ where $0 \leq u \leq 2\pi$ and also (because the z -coordinate must stay inside the sphere of radius 2),

$$0 \leq v \leq \sqrt{4 - (1 + \cos u)^2 - \sin^2 u},$$

or

$$0 \leq v \leq \sqrt{2 - 2\cos u}.$$

The normal is $\langle \cos u, \sin u, 0 \rangle$, which has magnitude 1, so we have

$$\iint_S 1 \, dS = \iint_R 1 \, dA = \int_0^{2\pi} \int_0^{\sqrt{2-2\cos u}} 1 \, dv \, du = \int_0^{2\pi} \sqrt{2-2\cos u} \, du = 8.$$

17.6.56 We have $z = -\frac{c}{a}x + \left(-\frac{c}{b}\right)y + c$, so that $z_x = -\frac{c}{a}$, $z_y = -\frac{c}{b}$, and an upward-pointing normal is $\left\langle \frac{c}{a}, \frac{c}{b}, 1 \right\rangle$. Then $\iint_S \mathbf{F} \cdot \mathbf{n} \, dS = \iint_R \left(\frac{c}{a}x + \frac{c}{b}y + z \right) \, dA = \iint_R \left(\frac{c}{a}x + \frac{c}{b}y + \left(-\frac{c}{a}x + \left(-\frac{c}{b}\right)y + c \right) \right) \, dA = \iint_R c \, dA$, which is c times the area of A . Recall that if the vector field is vertical, then the flux is equal to the area of the base. As c increases, the slope of the plane gets closer to vertical, so that the x and y components of the vector field $\langle x, y, z \rangle$ contribute more to the flux; also, the values of z get larger. Thus the flux increases as c does.

17.6.57

a. Using the standard parameterization, $\iint_S \mathbf{F} \cdot \mathbf{n} \, dS = \iint_R \left(\frac{-x^2}{z} + \frac{-y^2}{z} + z \right) \, dA = \iint_R 0 \, dA = 0$,

because $x^2 + y^2 = z^2$, so that the flux is zero. This is due to the fact that the field $\mathbf{F}$ is aligned with the cone at all points on the cone.

b. $2z \, dz = \left(\frac{2x}{a^2} \right) dx$ so that $z_x = \frac{x}{a^2 z}$. Then

$$\iint_S \mathbf{F} \cdot \mathbf{n} \, dS = \iint_R \left(\frac{-x^2}{a^2 z} + \frac{-y^2}{a^2 z} + z \right) \, dA = \iint_R \left(\frac{-a^2 z^2}{a^2 z} + z \right) \, dA = 0,$$

so the flux is again zero. This is because the flow is a radial flow, so is always tangent to this surface.

17.6.58 Parameterize the cone by $\mathbf{r}(u, v) = \left\langle \frac{a}{h}v \cos u, \frac{a}{h}v \sin u, v \right\rangle$ for $0 \leq v \leq h$, $0 \leq u \leq 2\pi$; then $\mathbf{t}_u \times \mathbf{t}_v = \left\langle -\frac{a}{h}v \sin u, \frac{a}{h}v \cos u, 0 \right\rangle \times \left\langle \frac{a}{h} \cos u, \frac{a}{h} \sin u, 1 \right\rangle$ and $|\mathbf{t}_u \times \mathbf{t}_v| = \frac{a}{h^2}v\sqrt{h^2 + a^2}$. Then

$$\begin{aligned} \iint_S 1 \, dS &= \frac{a\sqrt{h^2 + a^2}}{h^2} \iint_R v \, dA = \frac{a\sqrt{h^2 + a^2}}{h^2} \int_0^{2\pi} \int_0^h v \, dv \, du \\ &= \frac{a\sqrt{h^2 + a^2}}{h^2} \int_0^{2\pi} \frac{1}{2} h^2 \, du = \frac{2\pi a\sqrt{h^2 + a^2}}{2} = \pi a\sqrt{h^2 + a^2}. \end{aligned}$$

17.6.59 Because the cap has height h , the circle at the boundary of the cap has radius $\sqrt{a^2 - (a - h)^2} = \sqrt{2ah - h^2}$, so that the equation of the base of the region is $x^2 + y^2 = 2ah - h^2$. The outward normals to the sphere are $\left\langle \frac{x}{z}, \frac{y}{z}, 1 \right\rangle$. Thus $\iint_S 1 \, dS = \iint_R \sqrt{\frac{x^2}{z^2} + \frac{y^2}{z^2} + 1} \, dA = \iint_R \sqrt{\frac{a^2 - z^2}{z^2} + 1} \, dA = \iint_R \frac{a}{z} \, dA = \iint_R \frac{a}{\sqrt{a^2 - x^2 - y^2}} \, dA = \int_0^{2\pi} \int_0^{\sqrt{2ah - h^2}} \frac{ar}{\sqrt{a^2 - r^2}} \, dr \, d\theta = 2\pi ah$. (See also problem 54(b)).

17.6.60 Using a parametric description, we have

$$\begin{aligned} \iint_S \mathbf{F} \cdot \mathbf{n} \, dS &= \iint_R \frac{1}{(x^2 + y^2 + z^2)^{p/2}} \langle x, y, z \rangle \cdot (\mathbf{t}_u \times \mathbf{t}_v) \, dA \\ &= \frac{1}{a^p} \iint_R \langle a \sin u \cos v, a \sin u \sin v, a \cos u \rangle \cdot \langle a^2 \sin^2 u \cos v, a^2 \sin^2 u \sin v, a^2 \cos u \sin u \rangle \, dA \\ &= \frac{1}{a^p} \iint_R (a^3 \sin^3 u \cos^2 v + a^3 \sin^3 u \sin^2 v + a^3 \cos^2 u \sin u) \, dA \\ &= a^{3-p} \int_0^\pi \int_0^{2\pi} (\sin^3 u + \cos^2 u \sin u) \, dv \, du \\ &= a^{3-p} \int_0^\pi \int_0^{2\pi} \sin u \, dv \, du = \frac{4\pi}{a^{p-3}}. \end{aligned}$$

Using an explicit description, compute the flux on the upper half hemisphere and double it. There, for $z \geq 0$, we have $z_x = -\frac{x}{z}$, $z_y = -\frac{y}{z}$, so that

$$\begin{aligned} \iint_S \mathbf{F} \cdot \mathbf{n} \, dS &= \iint_R \frac{1}{(x^2 + y^2 + z^2)^{p/2}} \left(\frac{x^2}{z} + \frac{y^2}{z} + z \right) \, dA \\ &= \frac{1}{a^p} \iint_R \frac{a^2}{z} \, dA = a^{2-p} \int_0^{2\pi} \int_0^a \frac{r}{\sqrt{a^2 - r^2}} \, dr \, d\theta = \frac{2\pi}{a^{p-3}}. \end{aligned}$$

After doubling, we get the same answer.

17.6.61 $\mathbf{F} = -\nabla T = -\langle T_x, T_y, T_z \rangle = \langle 100e^{-x-y}, 100e^{-x-y}, 0 \rangle$. Thus the flow is parallel to the two sides where $z = \pm 1$ so that the flux is zero there. We thus need only compute the flux on the remaining four sides. Parameterize the sides as

$$\begin{array}{ll} S_1 : \langle -1, y, z \rangle & \mathbf{t}_y \times \mathbf{t}_z = \langle 1, 0, 0 \rangle \\ S_2 : \langle 1, y, z \rangle & \mathbf{t}_y \times \mathbf{t}_z = \langle 1, 0, 0 \rangle \\ S_3 : \langle x, -1, z \rangle & \mathbf{t}_x \times \mathbf{t}_z = \langle 0, -1, 0 \rangle \\ S_4 : \langle x, 1, z \rangle & \mathbf{t}_x \times \mathbf{t}_z = \langle 0, -1, 0 \rangle \end{array}$$

for $-1 \leq x, y, z \leq 1$.

We are looking for the outward flux, so we must choose outward normals, which are (respectively) $\langle -1, 0, 0 \rangle$, $\langle 1, 0, 0 \rangle$, $\langle 0, -1, 0 \rangle$, and $\langle 0, 1, 0 \rangle$. Then

$$\begin{aligned}\iint_{S_1} \mathbf{F} \cdot \mathbf{n} dS_1 &= \iint_R -100e^{-x-y} dA = -100 \int_{-1}^1 \int_{-1}^1 e^{1-y} dz dy = -200e^2 + 200 \\ \iint_{S_2} \mathbf{F} \cdot \mathbf{n} dS_2 &= \iint_R -100e^{-x-y} dA = 100 \int_{-1}^1 \int_{-1}^1 e^{-1-y} dz dy = -200e^{-2} + 200 \\ \iint_{S_3} \mathbf{F} \cdot \mathbf{n} dS_3 &= \iint_R -100e^{-x-y} dA = -100 \int_{-1}^1 \int_{-1}^1 e^{-x+1} dz dx = -200e^2 + 200 \\ \iint_{S_4} \mathbf{F} \cdot \mathbf{n} dS_4 &= \iint_R -100e^{-x-y} dA = 100 \int_{-1}^1 \int_{-1}^1 e^{-x-1} dz dx = -200e^{-2} + 200\end{aligned}$$

so that the total flux is $-400(e^2 + e^{-2} - 2) = -400\left(e - \frac{1}{e}\right)^2$.

17.6.62 $\mathbf{F} = -\nabla T = -\langle T_x, T_y, T_z \rangle = \langle 200xe^{-x^2-y^2-z^2}, 200ye^{-x^2-y^2-z^2}, 200ze^{-x^2-y^2-z^2} \rangle$. Thus integrating over the top half of the sphere gives

$$\begin{aligned}\iint_S \mathbf{F} \cdot \mathbf{n} dS &= \iint_R \langle 200xe^{-x^2-y^2-z^2}, 200ye^{-x^2-y^2-z^2}, 200ze^{-x^2-y^2-z^2} \rangle \cdot \left\langle \frac{x}{z}, \frac{y}{z}, 1 \right\rangle dA \\ &= 200e^{-a^2} \iint_R \left(\frac{x^2}{z} + \frac{y^2}{z} + z \right) dA = 200a^2e^{-a^2} \iint_R \frac{1}{z} dA \\ &= 200a^2e^{-a^2} \int_0^{2\pi} \int_0^a \frac{r}{\sqrt{a^2-r^2}} dr d\theta = 400\pi a^3 e^{-a^2},\end{aligned}$$

and because the vector field is symmetric, the answer is $800\pi a^3 e^{-a^2}$.

17.6.63 $\mathbf{F} = -\nabla T = \frac{2}{x^2+y^2+z^2} \langle x, y, z \rangle$. Thus integrating on the top half of the sphere gives $\iint_S \mathbf{F} \cdot \mathbf{n} dS = \iint_R \left(\frac{x^2}{z} + \frac{y^2}{z} + z \right) dA = 2 \iint_R \frac{1}{z} dA = 2 \int_0^{2\pi} \int_0^a \frac{r}{\sqrt{a^2-r^2}} dr d\theta = 4\pi a$, and because the vector field is symmetric, the answer is $2 \cdot 4\pi a = 8\pi a$.

17.6.64

a.
$$\iint_S \mathbf{F} \cdot \mathbf{n} dS = \iint_R \langle x, y, 0 \rangle \cdot \langle x, y, 0 \rangle dA = \iint_R (x^2 + y^2) dA = a^2 \int_0^{2\pi} \int_0^a r dr d\theta = \pi a^4.$$

b. On the cylinder, this field is $(x^2 + y^2)^{-p/2} = a^{-p}$ times as large as the field in part (a), so the flux is $4\pi L a^{2-p}$.

c. As $a \rightarrow \infty$, this converges for $2 - p \leq 0$, or $p \geq 2$.

d. As $L \rightarrow \infty$, the flux never converges.

17.6.65

a. From problem 60, the outward flux across a sphere of radius b is $\frac{4\pi}{b^{p-3}}$, so the total flux across the concentric spheres when $p = 0$ is $4\pi b^3 - 4\pi a^3 = 4\pi(b^3 - a^3)$.

b. For $p = 3$, the flux across the sphere of radius b is 4π , so the net flux is zero across S .

17.6.66 By symmetry, $\bar{x} = \bar{y} = 0$. Since the shell has constant density, we assume the density is 1; then its mass is $2\pi a^2$, and $M_{xy} = \iint_S z \, dS = \iint_R z \sqrt{\left(\frac{-x}{z}\right)^2 + \left(\frac{-y}{z}\right)^2 + 1} \, dA = \iint_R z \cdot \frac{a}{z} \, dA = a \int_0^{2\pi} \int_0^a r \, dr \, d\theta = \pi a^3$ so that $\bar{z} = \frac{a}{2}$.

17.6.67 The cone is rotationally symmetric around the z axis, so $\bar{x} = \bar{y} = 0$. Parameterize the cone by $\left\langle \frac{r}{h}v \cos u, \frac{r}{h}v \sin u, v \right\rangle$. Then from problem 23, the surface area of the cone is $\pi r \sqrt{h^2 + r^2}$, so its mass is $\rho \pi r \sqrt{h^2 + r^2}$. Using the parameterization from that problem, $|\mathbf{t}_u \times \mathbf{t}_v| = \frac{r}{h^2} v \sqrt{h^2 + r^2}$, so that $M_{xy} = \rho \iint_S z \, dS = \rho \frac{r \sqrt{h^2 + r^2}}{h^2} \iint_R v z \, dA = \rho \frac{r \sqrt{h^2 + r^2}}{h^2} \int_0^{2\pi} \int_0^h v^2 \, dv \, du = \rho \frac{r \sqrt{h^2 + r^2}}{h^2} \int_0^{2\pi} \frac{1}{3} h^3 \, du = \rho \frac{2\pi r h \sqrt{h^2 + r^2}}{3}$ so that $\bar{z} = \frac{M_{xy}}{m} = \rho \frac{2\pi r h \sqrt{h^2 + r^2}}{3} \cdot \frac{1}{\rho \pi r \sqrt{h^2 + r^2}} = \frac{2h}{3}$.

17.6.68 Assume the shell has density 1. Then the mass of the shell, which is half the area of the entire cylinder, is $\pi a h$. Further, by symmetry, $\bar{x} = \bar{y} = 0$. Use the parameterization $\langle a \cos u, v, a \sin u \rangle$; then $|\mathbf{t}_u \times \mathbf{t}_v| = a$ and $M_{xy} = \iint_S z \, dS = a \iint_R z \, dA = a \int_{-h/2}^{h/2} \int_0^\pi a \sin u \, du \, dv = 2a^2 h$ and $\bar{z} = \frac{2}{\pi} a$.

17.6.69 Using the standard parameterization, the mass of the shell is

$$m = \iint_S (1+z) \, dS = a \iint_R (1+z) \, dA = a \int_0^{2\pi} \int_0^2 (1+v) \, dv \, du = 8\pi a.$$

The density does not depend on either x or y , and the cylinder is symmetric about the z axis, so $\bar{x} = \bar{y} = 0$. Then

$$M_{xy} = \iint_S z(1+z) \, dS = a \iint_R z(1+z) \, dA = a \int_0^{2\pi} \int_0^2 v(1+v) \, dv \, du = \frac{28\pi a}{3}.$$

Then $\bar{z} = \frac{7}{6}$.

17.6.70 $\mathbf{t}_u = \langle a \cos u \cos v, a \cos u \sin v, -a \sin u \rangle$ and $\mathbf{t}_v = \langle -a \sin u \sin v, a \sin u \cos v, 0 \rangle$, and then $\mathbf{t}_u \times \mathbf{t}_v = \langle a^2 \sin^2 u \cos v, a^2 \sin^2 u \sin v, a^2 \cos u \sin u \rangle$ so that $|\mathbf{t}_u \times \mathbf{t}_v| = a^2 |\langle \sin^2 u \cos v, \sin^2 u \sin v, \cos u \sin u \rangle| = a^2 \sqrt{\sin^4 u \cos^2 v + \sin^4 u \sin^2 v + \cos^2 u \sin^2 u} = a^2 \sqrt{\sin^4 u + \sin^2 u \cos^2 u} = a^2 \sin u$.

17.6.71 The explicit formula $z = g(x, y)$ becomes, on regarding x and y as parameters, the parametric form $\langle x, y, g(x, y) \rangle$, and now $\mathbf{t}_x = \langle 1, 0, z_x \rangle$ and $\mathbf{t}_y = \langle 0, 1, z_y \rangle$. Then $\mathbf{t}_x \times \mathbf{t}_y = \langle -z_x, -z_y, 1 \rangle$, so that $|\mathbf{t}_x \times \mathbf{t}_y| = \sqrt{z_x^2 + z_y^2 + 1}$. Now the formula $\iint_S f(x, y, z) \, dS = \iint_R f(x, y, g(x, y)) \sqrt{z_x^2 + z_y^2 + 1} \, dA$ follows from the definition of the surface integral for parameterized surfaces.

17.6.72

a. Each point on the graph of f on $[a, b]$, say $f(x)$ becomes, in the surface of revolution, a circle of radius $f(x)$ with center on the x -axis. Letting $u = x$, that circle is then parameterized by $\langle f(u) \cos v, f(u) \sin v \rangle$ for $0 \leq v \leq 2\pi$, so the entire surface is parameterized by $\langle u, f(u) \cos v, f(u) \sin v \rangle$, $a \leq u \leq b$, $0 \leq v \leq 2\pi$.

- b. $\mathbf{t}_u = \langle 1, f'(u) \cos v, f'(u) \sin v \rangle$ and $\mathbf{t}_v = \langle 0, -f(u) \sin v, f(u) \cos v \rangle$, and then

$$\mathbf{t}_u \times \mathbf{t}_v = \langle f'(u) f(u), -f(u) \cos v, -f(u) \sin v \rangle$$

and

$$|\mathbf{t}_u \times \mathbf{t}_v| = \sqrt{f'(u)^2 f(u)^2 + f(u)^2 (\sin^2 v + \cos^2 v)} = f(u) \sqrt{f'(u)^2 + 1}.$$

We have

$$\begin{aligned} \iint_S f(x, y, z) dS &= \iint_R f(u) \sqrt{f'(u)^2 + 1} dA = \int_a^b \int_0^{2\pi} f(u) \sqrt{f'(u)^2 + 1} dv du \\ &= 2\pi \int_a^b f(u) \sqrt{f'(u)^2 + 1} du. \end{aligned}$$

- c. The area of the surface is $2\pi \int_1^2 x^3 \sqrt{9x^4 + 1} dx = \frac{\pi}{27} (145^{3/2} - 10^{3/2})$.

- d. The area of the surface is

$$\begin{aligned} 2\pi \int_3^4 (25 - x^2)^{1/2} \sqrt{(-x(25 - x^2)^{-1/2})^2 + 1} dx &= 2\pi \int_3^4 (25 - x^2)^{1/2} \sqrt{\frac{x^2}{25 - x^2} + 1} dx \\ &= 2\pi \int_3^4 (25 - x^2)^{1/2} \sqrt{\frac{25}{25 - x^2}} dx = 2\pi \int_3^4 5 dx = 10\pi. \end{aligned}$$

17.6.73 We have $z = s(x, y)$, so a normal vector is $\langle -z_x, -z_y, 1 \rangle$. Since we are interested in the downward flux, we choose a downward-pointing normal, which is $\langle s_x(x, y), s_y(x, y), -1 \rangle$. Then $\iint_S \mathbf{F} \cdot \mathbf{n} dS = \iint_R \langle 0, 0, -1 \rangle \cdot \langle s_x(x, y), s_y(x, y), -1 \rangle dA = \iint_R 1 dA$, which is the area of R . Since the vector field is constant and pointed downwards vertically, everything that goes through the surface is matched by something going through R .

17.6.74

- a. Imagine the torus as built by starting with a circle of radius R in the xy plane, centered at the origin. From each point on this circle, parameterize as $\langle R \cos v, R \sin v, 0 \rangle$, we can reach a circle of points on the surface by making it the center of a circle of radius r , parameterized by u . This second circle is drawn in a vertical plane that includes the z -axis. Each point of this second circle is thus in the plane determined by $\langle R \cos v, R \sin v, 0 \rangle$ and the z -axis; its z -coordinate will be $r \sin u$ and its x and y -coordinates will then be (from its center) $r \cos v \cos u$ and $r \sin v \cos u$. Thus the set of points on the torus can be parameterized by the sum of these vectors, which is $\langle R \cos v, R \sin v, 0 \rangle + \langle r \cos v \cos u, r \sin v \cos u, r \sin u \rangle = \langle (R + r \cos u) \cos v, (R + r \cos u) \sin v, r \sin u \rangle$

- b. From the parameterization above, we have

$$\begin{aligned} \mathbf{t}_u &= \langle -r \sin u \cos v, -r \sin u \sin v, r \cos u \rangle \\ \mathbf{t}_v &= \langle -(R + r \cos u) \sin v, (R + r \cos u) \cos v, 0 \rangle \end{aligned}$$

so that

$$\mathbf{t}_u \times \mathbf{t}_v = (R + r \cos u) \langle -r \cos u \cos v, -r \cos u \sin v, -r \sin u \rangle$$

and $|\mathbf{t}_u \times \mathbf{t}_v| = (R + r \cos u) \sqrt{r^2 \cos^2 u \cos^2 v + r^2 \cos^2 u \sin^2 v + r^2 \sin^2 u} = r(R + r \cos u)$, so that the area of the torus is $\iint_S 1 dS = \int_0^{2\pi} \int_0^{2\pi} r(R + r \cos u) du dv = 4\pi^2 Rr$.

17.6.75 The goal is to start with the surface area formula $A = \iint_R \sqrt{z_x^2 + z_y^2 + 1} dA$ of Theorem 17.12

(where we have set $f(x, y, g(x, y)) = 1$) and derive the surface area formula $A = 2\pi \int_a^b f(x) \sqrt{1 + f'(x)^2} dx$ of section 6.6. Because S is generated by revolving the graph of f about the x -axis, we can use symmetry and take $R = \{(x, y) : a \leq x \leq b, 0 \leq y \leq f(x)\}$. The resulting surface area is then one-quarter of the desired surface area. The key observation is that the surface generated is given by $z^2 = f(x)^2 - y^2$ over the region R . It follows that $2zz_x = 2ff'$, or $z_x = \frac{ff'}{z}$ and $2zz_y = -2y$, or $z_y = \frac{-y}{z}$. Substituting,

we have that the surface area is $A = \iint_R \sqrt{z_x^2 + z_y^2 + 1} dA = 4 \int_a^b \int_0^{f(x)} \sqrt{\left(\frac{ff'}{z}\right)^2 + \left(\frac{-y}{z}\right)^2 + 1} dy dx = 4 \int_a^b \int_0^{f(x)} \sqrt{\frac{f^2 f'^2 + y^2 + z^2}{z^2}} dy dx = 4 \int_a^b \int_0^{f(x)} \sqrt{\frac{f^2 f'^2 + f^2}{f^2 - y^2}} dy dx$, because $z^2 = f^2 - y^2$.

Continuing, we have

$$\begin{aligned} A &= 4 \int_a^b \int_0^{f(x)} \frac{f(x) \sqrt{1 + f'(x)^2}}{\sqrt{f(x)^2 - y^2}} dy dx = 4 \int_a^b f(x) \sqrt{1 + f'(x)^2} \int_0^{f(x)} \frac{1}{\sqrt{f(x)^2 - y^2}} dy dx \\ &= 4 \int_a^b f(x) \sqrt{1 + f'(x)^2} \left(\sin^{-1} \frac{y}{f(x)} \Big|_0^{f(x)} \right) dx = 2\pi \int_a^b f(x) \sqrt{1 + f'(x)^2} dx. \end{aligned}$$

17.7 Stokes' Theorem

17.7.1 It measures the circulation of the vector field $\mathbf{F}$ along the closed curve C .

17.7.2 It measures the accumulated rotation of the vector field $\mathbf{F}$ over the surface S .

17.7.3 It says that the circulation of a vector field along a closed curve is equal to the net circulation of the field over a surface whose boundary is that curve, so that either can be calculated from the other.

17.7.4 This is the fundamental theorem of line integrals - the integral of any conservative vector field around a closed curve is zero.

17.7.5 The line integral is $\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R \langle \sin t, -\cos t, 10 \rangle \cdot \langle -\sin t, \cos t, 0 \rangle dA = \int_0^{2\pi} (-1) dt = -2\pi$. For the surface integral, use the standard parameterization of the sphere; then

$$\mathbf{n} = \langle \sin^2 u \cos v, \sin^2 u \sin v, \cos u \sin u \rangle$$

and $\nabla \times \mathbf{F} = \langle 0, 0, -2 \rangle$ so that $\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} dS = \int_0^{2\pi} \int_0^{\pi/2} (-2 \cos u \sin u) du dv = -2\pi$.

17.7.6 The line integral is

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R \langle 0, -2 \cos t, 2 \sin t \rangle \cdot \langle -2 \sin t, 2 \cos t, 0 \rangle dA = \int_0^{2\pi} (-4 \cos^2 t) dt = -4\pi.$$

$\nabla \times \mathbf{F} = \langle 1, 0, -1 \rangle$. The outward normal to the sphere is $\left\langle \frac{x}{z}, \frac{y}{z}, 1 \right\rangle$, so the surface integral is

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} dS = \iint_R \langle 1, 0, -1 \rangle \cdot \left\langle \frac{x}{z}, \frac{y}{z}, 1 \right\rangle dA = \int_0^{2\pi} \int_0^2 \left(\frac{r^2 \cos \theta}{\sqrt{4 - r^2}} - r \right) dr d\theta = -4\pi.$$

17.7.7 The line integral is

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \langle 2\sqrt{2} \cos t, 2\sqrt{2} \sin t, 0 \rangle \cdot \langle -2\sqrt{2} \sin t, 2\sqrt{2} \cos t, 0 \rangle dt = \int_0^{2\pi} 0 dt = 0.$$

For the surface integral, we have $\nabla \times \mathbf{F} = \mathbf{0}$ so that the surface integral is also zero.

17.7.8 The boundary of the region is the intersection of the sphere with the plane $z = 12$, which has the equation $x^2 + y^2 = 25$ and $z = 12$. Thus the line integral is $\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \langle 24, -4 \cdot 5 \cos t, 3 \cdot 5 \sin t \rangle \cdot \langle -5 \sin t, 5 \cos t, 0 \rangle dt = \int_0^{2\pi} (-120 \sin t - 100 \cos^2 t) dt = -100\pi$.

The surface sits over $x^2 + y^2 = 25$ and the normal to the sphere is $\langle \frac{x}{z}, \frac{y}{z}, 1 \rangle$. $\nabla \times \mathbf{F} = \langle 3, 2, -4 \rangle$, so the surface integral is $\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} dS = \int_0^{2\pi} \int_0^5 r \left(\frac{3r \cos \theta}{\sqrt{169 - r^2}} + \frac{2r \sin \theta}{\sqrt{169 - r^2}} - 4 \right) dr d\theta = -100\pi$.

17.7.9 The boundary of the region is the intersection of the sphere with the plane $z = \sqrt{7}$, which has the equation $x^2 + y^2 = 9$ and $z = \sqrt{7}$. Then the line integral is

$$\begin{aligned} \oint_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^{2\pi} \langle 3 \sin t - \sqrt{7}, \sqrt{7} - 3 \cos t, 3 \cos t - 3 \sin t \rangle \cdot \langle -3 \sin t, 3 \cos t, 0 \rangle dt \\ &= \int_0^{2\pi} (-9 \sin^2 t + 3\sqrt{7} \sin t + 3\sqrt{7} \cos t - 9 \cos^2 t) dt = \int_0^{2\pi} (-9 + 3\sqrt{7} \sin t + 3\sqrt{7} \cos t) dt = -18\pi. \end{aligned}$$

The surface sits over $x^2 + y^2 = 9$ and the normal to the sphere is $\langle \frac{x}{z}, \frac{y}{z}, 1 \rangle$. $\nabla \times \mathbf{F} = \langle -2, -2, -2 \rangle$, so the surface integral is

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} dS = -2 \int_0^{2\pi} \int_0^3 r \left(\frac{r \cos \theta}{\sqrt{16 - r^2}} + \frac{r \sin \theta}{\sqrt{16 - r^2}} + 1 \right) dr d\theta = -18\pi.$$

17.7.10 The boundary of the region can be parameterized by $\langle 4 \cos t, 4 \sin t, 6 - 4 \sin t \rangle$, so the line integral is

$$\begin{aligned} \oint_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^{2\pi} \langle -4 \sin t, 4 \sin t - 4 \cos t - 6, 4 \sin t - 4 \cos t \rangle \cdot \langle -4 \sin t, 4 \cos t, -4 \cos t \rangle dt \\ &= \int_0^{2\pi} (16 \sin^2 t + 16 \sin t \cos t - 16 \cos^2 t - 24 \cos t - 16 \sin t \cos t + 16 \cos^2 t) dt \\ &= \int_0^{2\pi} (16 \sin^2 t - 24 \cos t) dt = 16\pi. \end{aligned}$$

$\nabla \times \mathbf{F} = \langle 2, 1, 0 \rangle$, and an outward pointing normal to the plane is $\langle 0, 1, 1 \rangle$, so the surface integral is

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} dS = \iint_R 1 dA = \int_0^{2\pi} \int_0^4 r dr d\theta = 16\pi.$$

17.7.11 $\nabla \times \mathbf{F} = \langle 1, -1, -2 \rangle$; for S take the disk $x^2 + y^2 \leq 12$ with upward-oriented normal vector $\langle 0, 0, 1 \rangle$. Then $\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} dS = \iint_R (-2) dA = -24\pi$.

17.7.12 $\nabla \times \mathbf{F} = \langle -1 - x, 0, z - 1 \rangle$; for S take the region $x^2 + \frac{y^2}{4} \leq 1$ in the plane $z = 1$, with normal vector $\langle 0, 0, 1 \rangle$; then $\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} dS = \iint_R (z - 1) dA = 0$ because $z = 1$ on the region of integration.

17.7.13 $\nabla \times \mathbf{F} = \langle 0, -4z, 0 \rangle$. For S take the plane in the first octant, which sits over $0 \leq x \leq 4$, $0 \leq y \leq 4 - x$. The upward-pointing normal to this plane is $\langle 1, 1, 1 \rangle$. Then $\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, dS = \iint_R (-4z) \, dA =$

$$\int_0^4 \int_0^{4-x} (-4)(4-x-y) \, dy \, dx = -\frac{128}{3}.$$

17.7.14 $\nabla \times \mathbf{F} = \langle 2y - 2z, 0, 2y - 2x \rangle$. Take S to be the square bounded by C , with normal $\langle 0, 0, 1 \rangle$. Then

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, dS = \iint_R (2y - 2x) \, dA = \int_{-1}^1 \int_{-1}^1 (2y - 2x) \, dy \, dx = 0.$$

17.7.15 $\nabla \times \mathbf{F} = \langle 2z, -1, -2y \rangle$. Take S to be the disk $\langle 3r \cos t, 4r \cos t, 5r \sin t \rangle$ for $0 \leq r \leq 1$, $0 \leq t \leq 2\pi$.

$\mathbf{t}_r \times \mathbf{t}_t = \langle 20r, -15r, 0 \rangle$ is a normal vector. Then $\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, dS = \iint_R \langle 2z, -1, -2y \rangle \cdot \langle 20r, -15r, 0 \rangle \, dA =$

$$\int_0^{2\pi} \int_0^1 ((10r \sin t) \cdot 20r + 15r) \, dr \, dt = 15\pi.$$

17.7.16 $\nabla \times \mathbf{F} = \mathbf{0}$, so the surface integral is zero.

17.7.17 The boundary of the surface is the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$, found by setting $z = 0$. Parameterize the

path by $\mathbf{r}(t) = \langle 2 \cos t, 3 \sin t, 0 \rangle$; then $\oint \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \mathbf{F} \cdot \mathbf{r}'(t) \, dt = \int_0^{2\pi} (-4 \cos t \sin t + 9 \cos t \sin t) \, dt =$

$$\int_0^{2\pi} 5 \cos t \sin t \, dt = 0.$$

17.7.18 The boundary of the surface is on the plane $x = 0$, and it is the circle $y^2 + z^2 = 9$. Parameterize

the circle by $\mathbf{r}(t) = \langle 0, 3 \cos t, 3 \sin t \rangle$; then $\mathbf{r}'(t) = \langle 0, -3 \sin t, 3 \cos t \rangle$. $\mathbf{F} = \frac{\mathbf{r}}{|\mathbf{r}|} = \frac{1}{\sqrt{x^2 + y^2 + z^2}} \langle x, y, z \rangle$,

and $\oint \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \mathbf{F} \cdot \mathbf{r}'(t) \, dt = \frac{1}{3} \int_0^{2\pi} (-9 \sin t \cos t + 9 \sin t \cos t) \, dt = 0$.

17.7.19 The boundary of the surface is the intersection of the plane $x = 3$ with the sphere $x^2 + y^2 + z^2 = 25$, so is the circle $y^2 + z^2 = 16$ at $x = 3$. Parametrize the circle with $x = 3$, $y = 4 \cos t$ and $z = 4 \sin t$. We have

$\mathbf{r}'(t) = \langle 0, -4 \sin t, 4 \cos t \rangle$, so $\oint \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \langle 8 \cos t, -4 \sin t, 3 - 4 \cos t - 4 \sin t \rangle \cdot \langle 0, -4 \sin t, 4 \cos t \rangle \, dt =$

$$\int_0^{2\pi} (16 \sin^2 t + 12 \cos t - 16 \cos^2 t - 16 \sin t \cos t) \, dt = 0.$$

17.7.20 The boundary of the surface is given in the problem: $\mathbf{r}(t) = \langle \cos t, 2 \sin t, \sqrt{3} \cos t \rangle$; so $\mathbf{r}'(t) = \langle -\sin t, 2 \cos t, -\sqrt{3} \sin t \rangle$ and the integral is

$$\begin{aligned} \oint \mathbf{F} \cdot d\mathbf{r} &= \int_0^{2\pi} \langle \cos t + 2 \sin t, 2 \sin t + \sqrt{3} \cos t, (1 + \sqrt{3}) \cos t \rangle \cdot \langle -\sin t, 2 \cos t, -\sqrt{3} \sin t \rangle \, dt \\ &= \int_0^{2\pi} (-\sin t \cos t - 2 \sin^2 t + 4 \sin t \cos t + 2\sqrt{3} \cos^2 t - (3 + \sqrt{3}) \cos t \sin t) \, dt \\ &= \int_0^{2\pi} (-\sqrt{3} \sin t \cos t - 2 \sin^2 t + 2\sqrt{3} \cos^2 t) \, dt = 2\pi (\sqrt{3} - 1). \end{aligned}$$

17.7.21 C is given by $\mathbf{r}(t) = \langle \cos t, \sin t, \cos^2 t \rangle$. Then

$$\begin{aligned} \mathbf{F} \cdot \mathbf{r}'(t) &= \langle \sin t, \cos^2 t - \cos t, -\sin t \rangle \cdot \langle -\sin t, \cos t, -2 \sin t \cos t \rangle \\ &= -\sin^2 t + \cos^3 t - \cos^2 t + 2 \sin^2 t \cos t = -1 + \cos^3 t + 2 \sin^2 t \cos t. \end{aligned}$$

We have $\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} dS = \oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} (-1 + \cos^3 t + 2 \sin^2 t \cos t) dt = -2\pi + \int_0^{2\pi} \cos t (2 \sin^2 t + (1 - \sin^2 t)) dt = -2\pi + \int_0^{2\pi} \cos t (\sin^2 t + 1) dt$. Let $u = \sin t$ so that $du = \cos t dt$. Then we have

$$-2\pi + \int_0^0 (u^2 + 1) du = -2\pi.$$

17.7.22 C is given by $\mathbf{r}(t) = \langle \frac{1}{2} \cos t, \frac{1}{2} \sin t, \frac{1}{2} \cos^2 t + \frac{1}{4} \sin^2 t \rangle = \langle \frac{1}{2} \cos t, \frac{1}{2} \sin t, \frac{1}{4} \cos^2 t + \frac{1}{4} \rangle$. Then $\mathbf{F} \cdot \mathbf{r}'(t) = \langle 2 \cos t, -2 \cos^2 t - 2, 2 \sin t \rangle \cdot \langle -\frac{1}{2} \sin t, \frac{1}{2} \cos t, -\frac{1}{2} \cos t \sin t \rangle = -\sin t \cos t - 2 \cos t$. We have

$$\begin{aligned} \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} dS &= \oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} (-\sin t \cos t - 2 \cos t) dt \\ &= -\int_0^{2\pi} \cos t (\sin t + 2) dt. \end{aligned}$$

A substitution of $u = \sin t + 2$ reveals that the value of this integral is 0.

17.7.23 C is given by $\mathbf{r}(t) = \langle 2 + 2 \cos t, 2 \sin t, 2\sqrt{2 + 2 \cos t} \rangle$. Then

$$\begin{aligned} \mathbf{F} \cdot \mathbf{r}'(t) &= \langle 2 \sin t, 1, 2\sqrt{2 + 2 \cos t} \rangle \cdot \left\langle -2 \sin t, 2 \cos t, \frac{-2 \sin t}{\sqrt{2 + 2 \cos t}} \right\rangle \\ &= -4 \sin^2 t + 2 \cos t - 4 \sin t. \end{aligned}$$

We have

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} dS = \oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} (-4 \sin^2 t + 2 \cos t - 4 \sin t) dt = \int_0^{2\pi} (-4 \sin^2 t) dt = -4\pi.$$

17.7.24 C is given by $\mathbf{r}(t) = \langle 2 \cos t, \sin t, 4 - 3 \sin^2 t \rangle$. Then

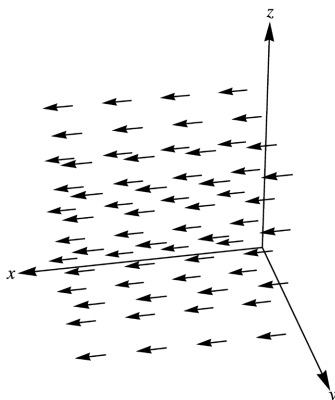
$$\begin{aligned} \mathbf{F} \cdot \mathbf{r}'(t) &= \left\langle e^{2 \cos t}, \frac{1}{4 - 3 \sin^2 t}, \sin t \right\rangle \cdot \langle -2 \sin t, \cos t, -6 \sin t \cos t \rangle \\ &= -2 \sin t e^{2 \cos t} + \frac{\cos t}{4 - 3 \sin^2 t} - 6 \sin^2 t \cos t. \end{aligned}$$

We have

$$\begin{aligned} \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} dS &= \oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \left(-2(\sin t) e^{2 \cos t} + \frac{\cos t}{4 - 3 \sin^2 t} - 6 \sin^2 t \cos t \right) dt \\ &= \int_0^{2\pi} -2 \sin t e^{2 \cos t} dt + \int_0^{2\pi} \frac{\cos t}{4 - 3 \sin^2 t} dt - 6 \int_0^{2\pi} \sin^2 t \cos t dt = 0 + 0 - 0 = 0, \end{aligned}$$

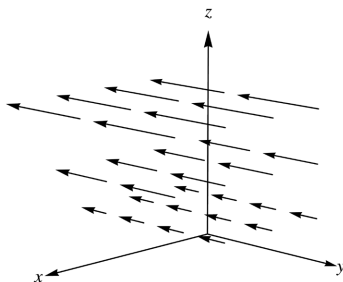
where the integrals are all evaluated either via the substitution $u = \sin t$ or $u = \cos t$.

17.7.25 $\nabla \times \mathbf{v} = \langle 1, 0, 0 \rangle$. The curl looks like:



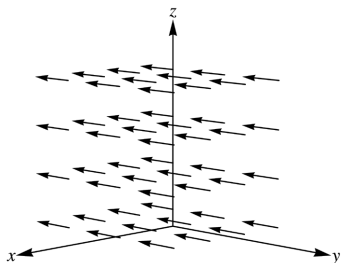
This means that the maximum rotation of the field is in the direction $\langle 1, 0, 0 \rangle$. The rotation is counterclockwise looking in the negative x direction.

17.7.26 $\nabla \times \mathbf{v} = \langle 0, -2z, 0 \rangle$. The curl looks like:



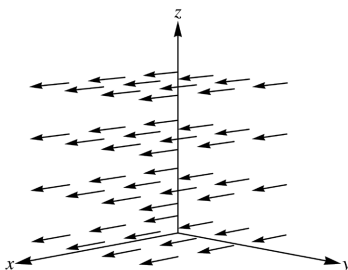
This means that the maximum rotation of the field is in the direction of the y -axis, and the amount of rotation, clockwise or counterclockwise, increases the further from the z -axis one gets. It rotates counterclockwise (viewed from the positive z -axis) for $z < 0$, and clockwise for $z > 0$.

17.7.27 $\nabla \times \mathbf{v} = \langle 0, -2, 0 \rangle$. The curl looks like:



This means that the maximum rotation of the field is in the direction of the y -axis. It is constant at all points, and is clockwise viewed from the positive y -axis.

17.7.28 $\nabla \times \mathbf{v} = \langle 2, 0, 0 \rangle$. The curl looks like:



The maximum rotation of the vector field is in the direction of the x -axis; it is constant at all points, and is counterclockwise viewed from the positive x -axis.

17.7.29

- False. This is a rotation field with axis of rotation $\langle 1, 1, 2 \rangle$, but $\langle 0, 1, -1 \rangle \cdot \langle 1, 1, 2 \rangle \neq 0$, so the paddle wheel axis is not perpendicular to the axis of rotation.
- False. It relates the curl of $\mathbf{F}$, not its flux.
- True. This is because it is conservative: it is the gradient of $ax + F(x) + by + G(y) + cz + H(z)$, where F, G, H are the antiderivatives of f, g, h , respectively.

d. True. See Theorem 14.14.

17.7.30 This is a conservative vector field, with $\varphi = x^2 - y^2 + z^2$, so the integral $\oint \mathbf{F} \cdot d\mathbf{r} = 0$ for any closed curve C .

17.7.31 This is a conservative vector field, so the integral around any closed curve is zero.

17.7.32 This is a conservative vector field with $\varphi = x^3y + y^2z^2$, so the integral around any closed curve is zero.

17.7.33 This is a conservative vector field with $\varphi = xy^2z^3$, so the integral around any closed curve is zero.

17.7.34 The surface S is $\langle r \cos \varphi \cos t, r \sin t, r \sin \varphi \cos t \rangle$; computing the normal vector gives $\mathbf{t}_r \times \mathbf{t}_t = \langle -r \sin \varphi, 0, r \cos \varphi \rangle$. Thus the surface area is $\iint_S 1 dS = \int_0^1 \int_0^{2\pi} |\mathbf{t}_r \times \mathbf{t}_t| dt dr = \int_0^1 \int_0^{2\pi} r dt dr = \pi$. This makes sense because the surface is simply the unit circle inclined at the angle φ to the xy -plane.

17.7.35 $\mathbf{r}'(t) = \langle -\cos \varphi \sin t, \cos t, -\sin \varphi \sin t \rangle$, so that $|\mathbf{r}'(t)| = 1$. Then the length of C is $\int_C 1 ds = \int_0^{2\pi} 1 dt = 2\pi$, again as expected because C is just an inclined unit circle.

17.7.36 By Stokes' theorem, $\oint \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} dS$, $\nabla \times \mathbf{F} = \langle 0, 0, 2 \rangle$; using the normal $\mathbf{t}_r \times \mathbf{t}_t = \langle -r \sin \varphi, 0, r \cos \varphi \rangle$ we have $\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} dS = \int_0^1 \int_0^{2\pi} 2r \cos \varphi dt dr = 2\pi \cos \varphi$. This is maximal for $\varphi = 0$, when it is 2π .

17.7.37 We have

$$\begin{aligned} \oint_C \mathbf{F} \cdot d\mathbf{r} &= \int_C \langle -y, -z, x \rangle \cdot \langle -\cos \varphi \sin t, \cos t, -\sin \varphi \sin t \rangle dt \\ &= \int_0^{2\pi} (\cos \varphi \sin^2 t - \sin \varphi \cos^2 t + \cos \varphi \sin \varphi \cos t \sin t) dt = \pi (\cos \varphi - \sin \varphi). \end{aligned}$$

This is maximal for $\varphi = 0$, when it is π .

17.7.38 $\nabla \times (\mathbf{a} \times \mathbf{r}) = \langle 2a_1, 2a_2, 2a_3 \rangle$. Thus

$$\begin{aligned} \oint_S \mathbf{F} \cdot d\mathbf{r} &= \iint_S \langle 2a_1, 2a_2, 2a_3 \rangle \cdot \langle -r \sin \varphi, 0, r \cos \varphi \rangle dS \\ &= 2 \int_0^{2\pi} \int_0^1 (-a_1 r \sin \varphi + a_3 r \cos \varphi) dr dt = 2\pi (a_3 \cos \varphi - a_1 \sin \varphi). \end{aligned}$$

This is a maximum when its derivative vanishes, i.e. when $a_3 \cos \varphi - a_1 \sin \varphi = 0$. Now, $\langle a_1, a_2, a_3 \rangle$ points in the direction of the normal if their cross-product is zero, i.e. if $\langle a_1, a_2, a_3 \rangle \times \langle -r \sin \varphi, 0, r \cos \varphi \rangle = \langle a_2 r \cos \varphi, -a_3 r \sin \varphi - a_1 r \cos \varphi, a_2 r \sin \varphi \rangle = 0$. This happens when $a_2 = 0$ and $a_3 \sin \varphi + a_1 \cos \varphi = 0$.

17.7.39 $\nabla \times \mathbf{F} = \langle 3, 0, 0 \rangle$. To evaluate the circulation around C , we instead (using Stokes' theorem) evaluate $\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} dS$ for the surface of the disk of which C is a boundary. Note that $\mathbf{n} = \langle 1, 1, 1 \rangle$, so that $\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} dS = \iint_R 3 dA = 3 \cdot \text{area of } A = 48\pi$. From this calculation, it is clear that the result depended on the radius of the circle, because that affects the area of A , but not on the center of the circle.

17.7.40

- a. $\nabla \times \mathbf{F} = \langle 1, 1, 0 \rangle$, so the integrand of the surface integral is just $\frac{x+y}{z}$; by symmetry, the integral vanishes on each level curve, so it vanishes altogether.
- b. On the boundary of S , we have $z = 0$, so that $\mathbf{F} = \langle 0, 0, 2y + x \rangle$, and thus $\mathbf{r}'(t) = \langle -\sin t, \cos t, 0 \rangle$ so that the dot product and thus the line integral is zero.

17.7.41

- a. The normal vectors point toward the z -axis on the curved surface of S and in the direction of $\langle 0, 1, 0 \rangle$ on the flat surface of S .
- b. To evaluate the integral, we must add up the integrals on each of the surfaces. $\nabla \times \mathbf{F} = \langle 1, 1, 1 \rangle$. Let S_1 be the surface in the xz -plane, parameterized by $\langle x, 0, z \rangle$ for $-2 \leq x \leq 2, x^2 \leq z \leq 4$; then the normal to S_1 is $\mathbf{t}_x \times \mathbf{t}_z = \langle 0, 1, 0 \rangle$, so that the integral over S_1 is

$$\iint_{S_1} \langle 1, 1, 1 \rangle \cdot \langle 0, 1, 0 \rangle dS = \int_{-2}^2 \int_{x^2}^4 (-1) dy dx = - \int_{-2}^2 (4 - x^2) dx = -\frac{32}{3}.$$

S_2 is the half of the paraboloid for $y \geq 0$, parameterized as $\langle r \cos u, r \sin u, r^2 \rangle$, $0 \leq r \leq 2, -\pi \leq u \leq 0$. The normal to S_2 is $\mathbf{t}_r \times \mathbf{t}_u = \langle -2r^2 \cos u, -2r^2 \sin u, r \rangle$. The integral over S_2 is

$$\iint_{S_2} \langle 1, 1, 1 \rangle \cdot \langle -2r^2 \cos u, -2r^2 \sin u, r \rangle dS = \int_0^2 \int_{-\pi}^0 (-2r^2 (\cos u + \sin u) + r) du dr = \frac{32}{3} + 2\pi.$$

Thus the total is 2π .

- c. The line integral is the sum of two line integrals:

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_{C_1} \mathbf{F} \cdot d\mathbf{r}_1 + \oint_{C_2} \mathbf{F} \cdot d\mathbf{r}_2$$

where $C_1 = \langle t, 0, 4 \rangle$ for $-2 \leq t \leq 2$ and $C_2 = \langle 2 \cos t, 2 \sin t, 4 \rangle$ for $-\pi \leq t \leq 0$. Then

$$\begin{aligned} \oint_{C_1} \mathbf{F} \cdot d\mathbf{r}_1 &= \int_{-2}^2 \langle 2z + y, 2x + z, 2y + x \rangle \cdot \langle 1, 0, 0 \rangle dt = \int_{-2}^2 8 dt = -32 \\ \oint_{C_2} \mathbf{F} \cdot d\mathbf{r}_2 &= \int_{-\pi}^0 \langle 2z + y, 2x + z, 2y + x \rangle \cdot \langle -2 \sin t, 2 \cos t, 0 \rangle dt \\ &= \int_{-\pi}^0 (-16 \sin t - 4 \sin^2 t + 8 \cos t + 8 \cos^2 t) dt = 32 + 2\pi, \end{aligned}$$

so the total line integral is 2π .

17.7.42 We have, from Stokes' theorem and Ampere's Law, $\iint_S (\nabla \times \mathbf{B}) \cdot \mathbf{n} dS = \oint_C \mathbf{B} \cdot d\mathbf{r} = \mu I =$

$\mu \iint_S \mathbf{J} \cdot \mathbf{n} dS$ Thus we have

$$\iint_S ((\nabla \times \mathbf{B}) - \mu \mathbf{J}) \cdot \mathbf{n} dS = 0$$

for all surfaces S bounded by any given closed curve C . It is clear that given the freedom to choose C and S , that it follows that the integrand is identically zero, i.e. that for any surface S , $((\nabla \times \mathbf{B}) - \mu \mathbf{J}) \cdot \mathbf{n} = 0$. From this it is easy to see that we must have $\nabla \times \mathbf{B} = \mu \mathbf{J}$, since we are free to make the normal vector point in any direction at any given point by choosing S appropriately.

17.7.43 The boundary of the region is the circle $C: x^2 + y^2 = 1$ for $z = 0$. With the usual parameterization, we have

$$\begin{aligned}\oint_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^{2\pi} \langle \cos t - \sin t, \sin t, -\cos t \rangle \cdot \langle -\sin t, \cos t, 0 \rangle dt \\ &= \int_0^{2\pi} (-\sin t \cos t + \sin^2 t + \sin t \cos t) dt = \int_0^{2\pi} \sin^2 t dt = \pi.\end{aligned}$$

So the integral is independent of a .

17.7.44

a. $\nabla \times \mathbf{F} = \left\langle \frac{\partial}{\partial y}(ay) - \frac{\partial}{\partial z}(cx), \frac{\partial}{\partial z}(bz) - \frac{\partial}{\partial x}(ay), \frac{\partial}{\partial x}(cx) - \frac{\partial}{\partial y}(bz) \right\rangle = \langle a, b, c \rangle.$

b. The area of R is $\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} dS = \iint_S \mathbf{n} \cdot \mathbf{n} dS = \iint_R |\mathbf{n}|^2 dA = \text{area of } R$ because $|\mathbf{n}| = 1$, so that

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} dS = \text{area of } R.$$

c. $\mathbf{r}'(t) = \langle 5 \cos t, -13 \sin t, 12 \cos t \rangle$, so that $\mathbf{r}(t) \times \mathbf{r}'(t) = \langle 156, 0, -65 \rangle$ and thus, because $\mathbf{r}(t) \times \mathbf{r}'(t)$ is constant, it points in a constant direction, so that $\mathbf{r}$ must lie in a plane.

d. By parts (b) and (c), we have a normal vector $\langle 156, 0, -65 \rangle$; its magnitude is 169, so we take $\mathbf{F} = \left\langle 0, -\frac{5x}{13}, \frac{12y}{13} \right\rangle$; then the area of R is

$$\begin{aligned}\oint_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^{2\pi} \left\langle 0, -\frac{25}{13} \sin t, 12 \cos t \right\rangle \cdot \langle 5 \cos t, -13 \sin t, 12 \cos t \rangle dt \\ &= \int_0^{2\pi} (25 \sin^2 t + 144 \cos^2 t) dt = 169\pi.\end{aligned}$$

17.7.45

a. The boundary of this surface is the circle $x^2 + y^2 = 1$ at $z = 0$, so we choose instead the surface of the disk bounded by that circle. $\nabla \times \mathbf{F} = \langle 2x, 0, -2z \rangle$, which is $\langle 2x, 0, 0 \rangle$ at $z = 0$, and the normal to the disk is $\langle 0, 0, 1 \rangle$. Thus, the integral is equal to $\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} dS = \iint_S 0 dS = 0$.

b. With the usual parameterization of the boundary circle (and remembering that $z = 0$), we have

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \langle 0, 0, 1 \rangle \cdot \langle -\sin t, \cos t, 0 \rangle dt = 0.$$

17.7.46

a. Let C be the circle $x^2 + y^2 = a^2$. Parameterize the circle in the usual way; then $\mathbf{F} = \frac{1}{a^p} \langle x, y, 0 \rangle$ and $\mathbf{r}(t) = \langle a \cos t, a \sin t, 0 \rangle$. Then $\oint_C \mathbf{F} \cdot d\mathbf{r} = \frac{1}{a^p} \int_0^{2\pi} (-a^2 \sin t \cos t + a^2 \sin t \cos t) dt = 0$.

b. Stokes' Theorem will apply when the vector field is defined throughout the disk of radius a , which happens only when $p \leq 0$. In that case, $\nabla \times \mathbf{F} = a^{-p} \langle 0, 0, 0 \rangle$, so that the surface integral is zero.

17.7.47

a. $\nabla \times \mathbf{F} = \left\langle 0 - 0, 0 - 0, \frac{y^2 - x^2}{(x^2 + y^2)^2} - \frac{y^2 - x^2}{(x^2 + y^2)^2} \right\rangle = \mathbf{0}.$

- b. Let C be the unit circle with the usual parameterization; then $\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \langle -\sin t, \cos t, 0 \rangle \cdot \langle -\sin t, \cos t, 0 \rangle dt = 2\pi$.
- c. The theorem does not apply because the vector field is not defined at the origin, which is inside the curve C . For example, the limit of the y -coordinate is different depending on the direction.

17.7.48

- a. The circumference of the disk is $2\pi R$, so the average circulation is $\frac{1}{2\pi R} \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} dS$
- b. As R becomes small, because $\mathbf{F}$ and thus $\nabla \times \mathbf{F}$ are continuous, $\nabla \times \mathbf{F}$ can be made arbitrarily close to $(\nabla \times \mathbf{F})_P$ everywhere on S by taking R small enough. Approximately, then, $(\nabla \times \mathbf{F}) \cdot \mathbf{n} \approx (\nabla \times \mathbf{F})_P \cdot \mathbf{n}$, so that

$$\frac{1}{2\pi R} \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} dS \approx \frac{1}{2\pi R} \iint_S (\nabla \times \mathbf{F})_P \cdot \mathbf{n} dS = (\nabla \times \mathbf{F})_P \cdot \mathbf{n} \frac{1}{2\pi R} \iint_S 1 dS = (\nabla \times \mathbf{F})_P \cdot \mathbf{n}.$$

As R becomes smaller, the goodness of the approximation of $\nabla \times \mathbf{F}$ becomes better, so the value of the integral does as well.

17.7.49 By the chain rule, $\frac{df}{dy} = \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} \frac{\partial z}{\partial y}$ and similarly for g, h , so

$$M_y = \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} \frac{\partial z}{\partial y} + h z_{xy} + z_x \left(\frac{\partial h}{\partial y} + \frac{\partial h}{\partial z} \frac{\partial z}{\partial y} \right) = f_y + f_z z_y + h z_{xy} + z_x (h_y + h_z z_y)$$

$$N_x = \frac{\partial g}{\partial x} + \frac{\partial g}{\partial z} \frac{\partial z}{\partial x} + h z_{yx} + z_y \left(\frac{\partial h}{\partial x} + \frac{\partial h}{\partial z} \frac{\partial z}{\partial x} \right) = g_x + g_z z_x + h z_{yx} + z_y (h_x + h_z z_x).$$

17.7.50 One argument is quite simple: a closed surface has a closed (empty!) boundary, so the integral of $\mathbf{F}$ over that boundary is zero. Alternatively, choose any closed curve C dividing the surface into two pieces. On one half, the outward-pointing normals give a counterclockwise orientation to the boundary (viewed from above); on the other half, they give a clockwise orientation. Thus the integral $\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} dS =$

$$\oint_C \mathbf{F} \cdot d\mathbf{r} + \oint_{-C} \mathbf{F} \cdot d\mathbf{r}, \text{ which is zero.}$$

17.7.51 Let $\mathbf{F} = \langle 0, g(y, z), h(y, z) \rangle$ be a vector field in the yz -plane; for a region R in that plane, with boundary C , the normal is $\langle 1, 0, 0 \rangle$. Now, $\nabla \times \mathbf{F} = \left\langle \frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}, 0, 0 \right\rangle$, so by Stokes' theorem, $\oint_C \mathbf{F} \cdot d\mathbf{r} =$

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} dS = \iint_R \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z} \right) dA.$$

17.8 Divergence Theorem

17.8.1 The surface integral measures the flow across the boundary.

17.8.2 The volume integral measures the net expansion or contraction of the vector field in the region.

17.8.3 The Divergence Theorem says that the flow across the boundary equals the net expansion or contraction of the field within the solid, so that either can be computed from the other.

17.8.4 Since $\nabla \cdot \langle 2z + y, -x, -2x \rangle = 0$, the net flux is zero.

17.8.5 Since $\nabla \cdot \langle x, y, z \rangle = 3$, the Divergence theorem says that the net outward flux is equal to

$$\iiint_D 3 \, dV = 3 \cdot \text{volume of } S = 32\pi.$$

17.8.6 From Example 4 (or Exercise 71 in section 14.5), the divergence is zero.

17.8.7 The outward fluxes must be equal, since by the Divergence theorem the net flux, which is the difference of the two, is zero.

17.8.8 Outward, since it is equal to the integral of $\text{div } \mathbf{F}$ over the cube.

17.8.9 For the volume integral, $\nabla \cdot \mathbf{F} = 9$, so that $\iiint_D \nabla \cdot \mathbf{F} \, dV = \iiint_D 9 \, dV = 9 \cdot \frac{4}{3} 2^3 \pi = 96\pi$.

For the surface integral, with the usual parameterization of the sphere,

$$\begin{aligned} \mathbf{F} \cdot \mathbf{n} &= \langle 2a \sin u \cos v, 3a \sin u \sin v, 4a \cos u \rangle \cdot \langle a^2 \sin^2 u \cos v, a^2 \sin^2 u \sin v, a^2 \cos u \sin u \rangle \\ &= 2a^3 \sin^3 u \cos^2 v + 3a^3 \sin^3 u \sin^2 v + 4a^3 \cos^2 u \sin u \end{aligned}$$

and here $a = 2$, so that

$$\begin{aligned} \iint_S \mathbf{F} \cdot \mathbf{n} \, dS &= 8 \iint_R (2 \sin^3 u \cos^2 v + 3 \sin^3 u \sin^2 v + 4 \cos^2 u \sin u) \, dA \\ &= 8 \int_0^{2\pi} \int_0^\pi (2 \sin^3 u \cos^2 v + 3 \sin^3 u \sin^2 v + 4 \cos^2 u \sin u) \, du \, dv = 96\pi. \end{aligned}$$

17.8.10 For the volume integral, $\nabla \cdot \mathbf{F} = -3$, so that $\iiint_D (-3) \, dV = -3 \cdot \text{volume of } D = -24$

For the surface integral, we have six surfaces:

$S_1 : x = -1$	$\mathbf{n} = \langle -1, 0, 0 \rangle$
$S_2 : x = 1$	$\mathbf{n} = \langle 1, 0, 0 \rangle$
$S_3 : y = -1$	$\mathbf{n} = \langle 0, -1, 0 \rangle$
$S_4 : y = 1$	$\mathbf{n} = \langle 0, 1, 0 \rangle$
$S_5 : z = -1$	$\mathbf{n} = \langle 0, 0, -1 \rangle$
$S_6 : z = 1$	$\mathbf{n} = \langle 0, 0, 1 \rangle$

and on each of those surfaces a simple computation shows that we have $\mathbf{F} \cdot \mathbf{n} = -1$. Thus

$$\sum_{i=1}^6 \iint_{S_i} \mathbf{F} \cdot \mathbf{n} \, dS_i = \sum_{i=1}^6 (-1 \cdot \text{area of } S_i) = -24.$$

17.8.11 For the volume integral, $\nabla \cdot \mathbf{F} = 0$, so the volume integral is zero. For the surface integral, the boundary ellipsoid can be parameterized by $\langle 2 \sin u \cos v, 2\sqrt{2} \sin u \sin v, 2\sqrt{3} \cos u \rangle$, and $\mathbf{n} = \mathbf{t}_u \times \mathbf{t}_v = \langle 4\sqrt{6} \sin^2 u \cos v, 4\sqrt{3} \sin^2 u \sin v, 4\sqrt{2} \sin u \cos u \rangle$ so that

$$\begin{aligned} \mathbf{F} \cdot \mathbf{n} &= \langle 2\sqrt{3} \cos u - 2\sqrt{2} \sin u \sin v, 2 \sin u \cos v, -2 \sin u \cos v \rangle \cdot \\ &\quad \langle 4\sqrt{6} \sin^2 u \cos v, 4\sqrt{3} \sin^2 u \sin v, 4\sqrt{2} \sin u \cos u \rangle = -8 \sin^2 u \cos v \left(-2\sqrt{2} \cos u + \sqrt{3} \sin u \sin v \right) \end{aligned}$$

and then $\iint_S \mathbf{F} \cdot \mathbf{n} \, dS = \int_0^{2\pi} \int_0^\pi \left(-8 \sin^2 u \cos v \left(-2\sqrt{2} \cos u + \sqrt{3} \sin u \sin v \right) \right) \, du \, dv = 0$.

17.8.12 For the volume integral, $\nabla \cdot \mathbf{F} = 2(x + y + z)$, so that

$$\iiint_D \nabla \cdot \mathbf{F} dV = 2 \int_{-1}^1 \int_{-2}^2 \int_{-3}^3 (x + y + z) dz dy dx = 0.$$

For the surface integral, we have six surfaces:

$$\begin{array}{ll} S_1 : x = -1 & \mathbf{n} = \langle -1, 0, 0 \rangle \\ S_2 : x = 1 & \mathbf{n} = \langle 1, 0, 0 \rangle \\ S_3 : y = -2 & \mathbf{n} = \langle 0, -1, 0 \rangle \\ S_4 : y = 2 & \mathbf{n} = \langle 0, 1, 0 \rangle \\ S_5 : z = -3 & \mathbf{n} = \langle 0, 0, -1 \rangle \\ S_6 : z = 3 & \mathbf{n} = \langle 0, 0, 1 \rangle \end{array}$$

A short computation shows that for S_1 : $\mathbf{F} \cdot \mathbf{n} = -1$, for S_2 : $\mathbf{F} \cdot \mathbf{n} = 1$, for S_3 : $\mathbf{F} \cdot \mathbf{n} = -4$, for S_4 : $\mathbf{F} \cdot \mathbf{n} = 4$, for S_5 : $\mathbf{F} \cdot \mathbf{n} = -9$, and for S_6 : $\mathbf{F} \cdot \mathbf{n} = 9$. Thus, the surface integral is zero.

17.8.13 $\nabla \cdot \mathbf{F} = 0$, so by the Divergence Theorem, the net outward flux is zero since the volume integral of $\nabla \cdot \mathbf{F}$ is zero.

17.8.14 $\nabla \cdot \mathbf{F} = 0$, so by the Divergence Theorem, the net outward flux is zero since the volume integral of $\nabla \cdot \mathbf{F}$ is zero.

17.8.15 $\nabla \cdot \mathbf{F} = 0$, so by the Divergence Theorem, the net outward flux is zero since the volume integral of $\nabla \cdot \mathbf{F}$ is zero.

17.8.16 If $\mathbf{a} = \langle a, b, c \rangle$, then $\mathbf{a} \times \mathbf{r}$ is the field $\mathbf{F}$ in Exercise 15, so the net outward flux is zero.

17.8.17 By the Divergence theorem, we can compute the integral of $\nabla \cdot \mathbf{F}$ over the ball of radius $\sqrt{6}$: $\nabla \cdot \mathbf{F} = 2$, so the volume integral is

$$\iiint_D \nabla \cdot \mathbf{F} dV = 2 \cdot \text{volume of sphere of radius } \sqrt{6} = 2 \cdot \frac{4}{3}\pi \cdot 6\sqrt{6} = 16\pi\sqrt{6}.$$

17.8.18 $\nabla \cdot \mathbf{F} = 2x$, so by the Divergence theorem, the outward flux is

$$\iiint_D 2x dV = \int_0^1 \int_0^1 \int_0^1 2x dx dy dz = 1.$$

17.8.19 $\nabla \cdot \mathbf{F} = 4$, so by the Divergence theorem, the outward flux is 4 times the volume of the tetrahedron, which is (by the formula for the volume of a pyramid), $\frac{1}{3}$ times the area of the base times the height, or $\frac{1}{6}$. So the outward flux is $\frac{2}{3}$.

17.8.20 $\nabla \cdot \mathbf{F} = 2(x + y + z)$, so by the Divergence theorem, the outward flux is

$$\iiint_D 2(x + y + z) dV = 2 \int_0^5 \int_0^{2\pi} \int_0^\pi r(5 \sin u \cos v + 5 \sin u \sin v + 5 \cos u) du dv dr = 0.$$

17.8.21 $\nabla \cdot \mathbf{F} = -4$, so by the Divergence theorem, the outward flux is -4 times the volume of the sphere, so is $-\frac{128}{3}\pi$.

17.8.22 $\nabla \cdot \mathbf{F} = 0$, so the outward flux is zero by the Divergence theorem.

17.8.23 $\nabla \cdot \mathbf{F} = 3$, so the outward flux is 3 times the volume of the paraboloid, which is

$$\int_0^2 \int_0^{2\pi} r(4 - r^2) d\theta dr = 8\pi,$$

so the outward flux is 24π .

17.8.24 $\nabla \cdot \mathbf{F} = 3$, so the outward flux is 3 times the volume of the cone. The area of the base of the cone is 16π , so the outward flux is $3 \cdot \frac{1}{3} \cdot 16\pi \cdot 4 = 64\pi$.

17.8.25 $\nabla \cdot \mathbf{F} = -3$, so the outward flux across the boundary of D is the outward flux across the sphere of radius 4 minus that across the sphere of radius 3, which is $-3 \cdot \frac{4}{3}\pi(4^3 - 3^3) = -224\pi$.

17.8.26 $\nabla \cdot \mathbf{F} = 4|\mathbf{r}|$, so the outward flux across a sphere of radius r is

$$\iiint_D 4\sqrt{x^2 + y^2 + z^2} dV = \int_0^{2\pi} \int_0^\pi \int_0^r 4\rho^2 \sin u d\rho du dv = 4\pi r^4.$$

Thus the net outward flux across the boundary of the given region is 60π .

17.8.27 $\nabla \cdot \mathbf{F} = \frac{2}{|\mathbf{r}|}$, so the outward flux across a sphere of radius r is

$$\iiint_D \frac{2}{\sqrt{x^2 + y^2 + z^2}} dV = \int_0^{2\pi} \int_0^\pi \int_0^r \frac{2}{\rho} \rho^2 \sin u d\rho du dv = 4\pi r^2.$$

Thus the net outward flux across the boundary of the given region is 12π .

17.8.28 $\nabla \cdot \mathbf{F} = 0$, so the net outward flux is zero.

17.8.29 $\nabla \cdot \mathbf{F} = 2(x - y + z)$. The net outward flux is thus the difference in the outward flux across the two planes, so is

$$\begin{aligned} & \iiint_D 2(x - y + z) dV \\ &= 2 \left(\int_0^4 \int_0^{4-x} \int_0^{4-x-y} (x - y + z) dz dy dx - \int_0^2 \int_0^{2-x} \int_0^{2-x-y} (x - y + z) dz dy dx \right) \\ &= 20. \end{aligned}$$

17.8.30 $\nabla \cdot \mathbf{F} = 6$, so the net outward flux is 6 times the difference in the volumes of the cylinders, so is $6 \cdot (4\pi - \pi) \cdot 8 = 144\pi$.

17.8.31

a. False. For example, $\mathbf{F} = \langle y, 0, 0 \rangle$ has $\nabla \cdot \mathbf{F} = 0$ at all points of the unit sphere, but the normal to the unit sphere, $\left\langle \frac{x}{z}, \frac{y}{z}, 1 \right\rangle$, is not perpendicular to $\mathbf{F}$ at all points.

b. False. For example, any rotation field has $\nabla \cdot \mathbf{F} = 0$, so that $\iint_S \mathbf{F} \cdot \mathbf{n} dS = 0$ by the Divergence theorem, but $\mathbf{F}$ is not in general constant.

c. True. This is because it is bounded by $\iiint_D 1 dV$.

17.8.32 $\nabla \cdot \mathbf{F} = 3$; we compute the outward flux from the Divergence theorem as (where S_1 is the upper hemisphere of radius a .)

$$\iiint_{S_1} \nabla \cdot \mathbf{F} dV = 3 \iiint_{S_1} 1 dV = 3 \int_0^{2\pi} \int_0^{\pi/2} \int_0^a r^2 \sin u dr du dv = 2\pi a^3.$$

The the outward flux over the whole sphere is thus $4\pi a^3$.

17.8.33 Because $\nabla \cdot \mathbf{F} = 0$, the outward flux is zero from the Divergence theorem.

17.8.34 Because $\nabla \cdot \mathbf{F} = 0$, the outward flux is zero from the Divergence theorem.

17.8.35 $\nabla \cdot \mathbf{F} = 3 \sin y$, so the outward flux is

$$\iiint_S 3 \sin y dV = \int_0^{\pi/2} \int_0^1 \int_0^x 3 \sin y dz dx dy = \frac{3}{2}.$$

17.8.36

a. We have $\mathbf{F} \cdot \mathbf{n} = \frac{\mathbf{r}}{|\mathbf{r}|^p} \cdot \frac{\mathbf{r}}{|\mathbf{r}|} = \frac{|\mathbf{r}|^2}{|\mathbf{r}|^{p+1}} = |\mathbf{r}|^{1-p}$. Thus the surface integral is $\iint_S \mathbf{F} \cdot \mathbf{n} dS = a^{1-p} \cdot$

area of sphere $= a^{1-p} \cdot 4\pi a^2 = 4\pi a^{3-p}$.

b. The conditions of the Divergence Theorem require that $\mathbf{F}$ be defined and have continuous partials everywhere inside the sphere; in particular, this must hold at the origin. Thus we must have $p \leq -2$. Then the volume integral is

$$\iiint_S \frac{3-p}{|\mathbf{r}|^p} dV = (3-p) \int_0^a \int_0^{2\pi} \int_0^\pi r^{-p} \cdot r^2 \sin u du dv dr = 4\pi a^{3-p}.$$

17.8.37

a. Either use Exercise 36(a), or compute $\mathbf{F} \cdot \mathbf{n} = \frac{\mathbf{r}}{|\mathbf{r}|} \cdot \frac{\mathbf{r}}{|\mathbf{r}|} = 1$, so the surface integral is

$$\iint_S \mathbf{F} \cdot \mathbf{n} dS = \text{area of sphere} = 4\pi a^2.$$

b. $\nabla \cdot \mathbf{F} = 2|\mathbf{r}|^{-1}$, so if D is the shell between the spheres of radius ϵ and a , the volume integral is

$$\iiint_S 2|\mathbf{r}|^{-1} dV = 2 \int_\epsilon^a \int_0^{2\pi} \int_0^\pi r \sin u du dv dr = 4\pi (a^2 - \epsilon^2)$$

and $\lim_{\epsilon \rightarrow 0} 4\pi (a^2 - \epsilon^2) = 4\pi a^2$.

17.8.38

a. $\frac{\partial}{\partial x} \varphi = \frac{x}{x^2 + y^2 + z^2}$, so that $\nabla \varphi = \frac{1}{|\mathbf{r}|^2} \langle x, y, z \rangle = \frac{\mathbf{r}}{|\mathbf{r}|^2}$.

b. $\mathbf{n} = \frac{\mathbf{r}}{|\mathbf{r}|^2}$, so that $\mathbf{F} \cdot \mathbf{n} = \frac{1}{|\mathbf{r}|^2}$. Then $\iint_S \mathbf{F} \cdot \mathbf{n} dS = \frac{1}{a^2} \int_0^{2\pi} \int_0^\pi a^3 \sin u du dv = 4\pi a$.

c. By Exercise 36, $\text{div } \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{1}{|\mathbf{r}|^2}$.

- d. If D is the shell between the spheres of radius ϵ and a , the volume integral is

$$\iiint_D |\mathbf{r}|^{-2} dV = 2 \int_{\epsilon}^a \int_0^{2\pi} \int_0^{\pi} \frac{1}{r^2} r \sin u \, du \, dv \, dr = 4\pi (a - \epsilon)$$

$$\text{and } \lim_{\epsilon \rightarrow 0} 4\pi (a - \epsilon) = 4\pi a.$$

17.8.39

- a. By Exercise 36, the flux of $\mathbf{E}$ across a sphere of radius a is $\frac{Q}{4\pi\epsilon_0} 4\pi a^3 = \frac{Q}{\epsilon_0}$.
- b. The net outward flux across S is the difference of the fluxes across the inner and outer spheres; but by part (a), these are equal, so the net flux across S is zero.
- c. The left-hand side is the flux across the boundary of D , while the right-hand side is the sum of the charge densities at each point of D . The statement says that the flux across the boundary, up to multiplication by a constant, is the sum of the charge densities in the region.
- d. By the Divergence theorem, and using part (c),

$$\frac{1}{\epsilon_0} \iiint_D q(x, y, z) \, dV = \iint_S \mathbf{E} \cdot \mathbf{n} \, dS = \iiint_D \nabla \cdot \mathbf{E} \, dV$$

$$\text{and because this holds for all regions } D, \text{ we conclude that } \nabla \cdot \mathbf{E} = \frac{q(x, y, z)}{\epsilon_0}.$$

$$\text{e. } \nabla^2 \varphi = \nabla \cdot \nabla \varphi = \nabla \cdot \mathbf{E} = \frac{q(x, y, z)}{\epsilon_0}.$$

17.8.40

- a. By Exercise 36, the flux of $\mathbf{F}$ across a sphere of radius a is $4\pi G M a^{3-p} = 4\pi G M$.
- b. Since the outward flux across a sphere, from part (a), is independent of the radius of the sphere, the outward flux across the spheres of radii a and b are equal, so their difference, which is the net flux across the spherical shell bounded by them, is zero.
- c. The left hand side is the flux across the boundary of D , while the right-hand side is the sum of the mass density inside D . The statement says that the flux across the boundary is determined by (is a constant multiple of) the sum of the mass density inside D .
- d. By the Divergence theorem, and using part (c),

$$4\pi G \iiint_D \rho(x, y, z) \, dV = \iint_S \mathbf{F} \cdot \mathbf{n} \, dS = \iiint_D \nabla \cdot \mathbf{F} \, dV$$

$$\text{and because this holds over all regions } D, \text{ we have } \nabla \cdot \mathbf{F} = 4\pi G \rho(x, y, z).$$

$$\text{e. } \nabla^2 \varphi = \nabla \cdot \nabla \varphi = \nabla \cdot \mathbf{F} = 4\pi G \rho(x, y, z).$$

17.8.41 $\mathbf{F} = -\nabla T = \langle -1, -2, -1 \rangle$, so that $\nabla \cdot \mathbf{F} = 0$ and the heat flux is zero.

17.8.42 $\mathbf{F} = -\nabla T = \langle -2x, -2y, -2z \rangle$, so that $\nabla \cdot \mathbf{F} = -6$, and the heat flux is -6 times the volume of the region, or -6 .

17.8.43 $\mathbf{F} = -\nabla T = \langle 0, 0, e^{-z} \rangle$; then $\nabla \cdot \mathbf{F} = -e^{-z}$. The heat flux is then

$$\int_0^1 \int_0^1 \int_0^1 -e^{-z} \, dx \, dy \, dz = e^{-1} - 1.$$

17.8.44 From Exercise 42, $\nabla \cdot \mathbf{F} = -6$, so the heat flux is -6 times the volume of the sphere, or -8π .

17.8.45 $\mathbf{F} = -\nabla T = \langle 200xe^{-x^2-y^2-z^2}, 200ye^{-x^2-y^2-z^2}, 200ze^{-x^2-y^2-z^2} \rangle$. Then

$$\nabla \cdot \mathbf{F} = 200e^{-x^2-y^2-z^2} (3 - 2x^2 - 2y^2 - 2z^2)$$

so that

$$\iiint_D \nabla \cdot \mathbf{F} dV = 200 \int_0^{2\pi} \int_0^\pi \int_0^a e^{-r^2} (3 - 2r^2) r^2 \sin u dr du dv = 800\pi a^3 e^{-a^2}.$$

17.8.46

a. By Exercise 36, the net flux across a sphere of radius a centered at the origin is $4\pi a^{3-p}$, which is independent of a only if $p = 3$.

b. In the general case, we have $\nabla \cdot \mathbf{F} = \frac{3-p}{|\mathbf{r}|^p}$, so

$$\begin{aligned} \iiint_D \nabla \cdot \mathbf{F} dV &= \int_a^b \int_{\varphi_1}^{\varphi_2} \int_{\theta_1}^{\theta_2} \frac{3-p}{r^p} r^2 \sin u du dv dr \\ &= (a^{3-p} - b^{3-p}) (\varphi_1 \cos \theta_1 - \varphi_2 \cos \theta_1 - \varphi_1 \cos \theta_2 + \varphi_2 \cos \theta_2) \\ &= (a^{3-p} - b^{3-p}) (\varphi_1 - \varphi_2) (\cos \theta_1 - \cos \theta_2), \end{aligned}$$

and these are in general zero only if $3 - p = 0$.

17.8.47

a. $\varphi_x(x, y, z) = G'(\rho) \rho_x = G'(\rho) \cdot \frac{x}{\sqrt{x^2 + y^2 + z^2}} = G'(\rho) \frac{x}{\rho}$, so that $\nabla \varphi = \mathbf{F} = G'(\rho) \frac{\mathbf{r}}{\rho}$.

b. The normal to the sphere of radius a is $\langle \frac{x}{z}, \frac{y}{z}, 1 \rangle$, so on that sphere (where $\rho = a$)

$$\mathbf{F} \cdot \mathbf{n} = G'(a) \frac{\frac{x^2}{z} + \frac{y^2}{z} + z}{a} = G'(a) \frac{\frac{a^2 - z^2}{z} + z}{a} = G'(a) \frac{a}{z},$$

and then the surface integral over the upper hemisphere is

$$\iint_S \mathbf{F} \cdot \mathbf{n} dS = aG'(a) \int_0^a \int_0^{2\pi} \frac{r}{\sqrt{a^2 - r^2}} d\theta dr = 2\pi a^2 G'(a),$$

so the total surface integral is twice that, or $4\pi a^2 G'(a)$.

c. By the Chain Rule,

$$\frac{\partial}{\partial x} G'(\rho) \frac{x}{\rho} = G''(\rho) \rho_x \frac{x}{\rho} + G'(\rho) \frac{y^2 + z^2}{\rho^3}$$

so that (noting that $\rho_x = \frac{x}{\rho}$)

$$\nabla \cdot \mathbf{F} = G''(\rho) \left(\frac{x^2 + y^2 + z^2}{\rho^2} \right) + G'(\rho) \frac{2(x^2 + y^2 + z^2)}{\rho^3} = G''(\rho) + \frac{2G'(\rho)}{\rho}.$$

d. By the Divergence theorem, the flux is also given by

$$\begin{aligned} \iiint_D \nabla \cdot \mathbf{F} dV &= \int_0^a \int_0^\pi \int_0^{2\pi} \rho^2 \sin u \left(G''(\rho) + \frac{2G'(\rho)}{\rho} \right) dv du d\rho \\ &= 2\pi \int_0^a \int_0^\pi \sin u (\rho^2 G''(\rho) + 2\rho G'(\rho)) du d\rho \\ &= 2\pi \int_0^a (-\cos u) (\rho^2 G''(\rho) + 2\rho G'(\rho)) \Big|_{u=0}^{u=\pi} d\rho \\ &= 4\pi \int_0^a (\rho^2 G''(\rho) + 2\rho G'(\rho)) d\rho. \end{aligned}$$

It remains to evaluate this integral. Using integration by parts, we have

$$\begin{aligned} 4\pi \left[\int_0^a (\rho^2 G''(\rho) + 2\rho G'(\rho)) d\rho \right] &= 4\pi \left[\rho^2 G'(\rho) \Big|_{\rho=0}^{\rho=a} - \int_0^a 2\rho G'(\rho) d\rho \right] \\ &= 4\pi \left[a^2 G'(a) - \int_0^a 2\rho G'(\rho) d\rho \right] \end{aligned}$$

and the remaining integrals cancel, giving $4\pi a^2 G'(a)$ as the final result.

17.8.48

- a. Rearrange the given equation and integrate over D to get

$$\iiint_D u \nabla \cdot \mathbf{F} dS = \iiint_D \nabla \cdot (u \mathbf{F}) dS - \iiint_D \mathbf{F} \cdot \nabla u dS.$$

By the Divergence theorem, the first term on the right side is equal to $\iint_S u \mathbf{F} \cdot \mathbf{n} dS$ where S is the boundary of D . The result follows.

- b. If you set $\mathbf{F} = \langle f(x), 0, 0 \rangle$ and $u(x, y, z) = g(x)$, then

$$\begin{aligned} \nabla \cdot \mathbf{F} &= f'(x) & u \nabla \cdot \mathbf{F} &= f'(x) g(x) \\ u \mathbf{F} &= \langle f(x) g(x), 0, 0 \rangle & \nabla \cdot (u \mathbf{F}) &= (fg)'(x) \\ \nabla u &= \langle g'(x), 0, 0 \rangle & \mathbf{F} \cdot \nabla u &= f(x) g'(x) \end{aligned}$$

so that

$$\iiint_D f'(x) g(x) dV = \iiint_D (fg)'(x) dV - \iiint_D f(x) g'(x) dV = f(x) g(x) - \iiint_D f(x) g'(x) dV,$$

which is the usual rule for integration by parts.

- c. One approach is to set $u = 1$ and $\mathbf{F} = \frac{1}{2} \langle x^2 z^2, x^2 y^2, y^2 z^2 \rangle$. Gauss' formula then gives

$$\iiint_D (x^2 y + y^2 z + z^2 x) dV = \frac{1}{2} \iint_S \langle x^2 z^2, x^2 y^2, y^2 z^2 \rangle dS.$$

The integral on the right is computed by integrating over each face of the cube; on faces where x is constant, the normal is either $\langle 1, 0, 0 \rangle$ or $\langle -1, 0, 0 \rangle$ depending on whether x is 1 or 0; similarly for y and z . Considering x first, when $x = 0$, this surface integral becomes 0 on that face (the integrand is $x^2 z^2$ at $x = 0$), while for $x = 1$ it becomes z^2 . In that case, the integral is $\frac{1}{3}$. This holds for each dimension, so the total integral on the right side is $3 \cdot \frac{1}{3} = 1$, and thus the integral on the left is $\frac{1}{2}$.

17.8.49 Suppose $\mathbf{F} = \langle f, g \rangle$ where $f = f(x, y)$, $g = g(x, y)$, and suppose $u = u(x, y)$. Then $\nabla \cdot \mathbf{F} = f_x + g_y$, and $\nabla u = \langle u_x, u_y \rangle$. Then we have for this case

$$\begin{aligned} \iint_D u \nabla \cdot \mathbf{F} dS &= \iint_R u (f_x + g_y) dA \\ \iint_S u \mathbf{F} \cdot \mathbf{n} dS &= \oint_C u \mathbf{F} \cdot \mathbf{n} ds \\ \iint_D \mathbf{F} \cdot \nabla u dS &= \iint_R (f u_x + g u_y) dA, \end{aligned}$$

and the result follows. Setting $u = 1$ then gives $\iint_R (f_x + g_y) dA = \oint_C \mathbf{F} \cdot \mathbf{n} ds$, which is the flux form of Green's Theorem.

17.8.50 Apply the Divergence Theorem to the vector field $\mathbf{F} = u\nabla u$. By the product rule (Thm. 14.11), we have $\nabla \cdot (v\nabla v) = \nabla u \cdot \nabla v + u\nabla^2 v$, so the Divergence theorem says that

$$\iiint_D (u\nabla^2 v + \nabla u \cdot \nabla v) dV = \iiint_D (\nabla \cdot (v\nabla v)) dV = \iint_S u\nabla v \cdot \mathbf{n} dS.$$

17.8.51 From Exercise 50, we have both

$$\begin{aligned} \iiint_D (u\nabla^2 v + \nabla u \cdot \nabla v) dV &= \iint_S u\nabla v \cdot \mathbf{n} dS \\ \iiint_D (v\nabla^2 u + \nabla v \cdot \nabla u) dV &= \iint_S v\nabla u \cdot \mathbf{n} dS, \end{aligned}$$

where the second formula is obtained by switching u and v in the first formula. Subtracting the second from the first, and using the fact that the dot product is commutative and integrals are linear, we have the desired result:

$$\iiint_D (u\nabla^2 v - v\nabla^2 u) dV = \iint_S (u\nabla v - v\nabla u) \cdot \mathbf{n} dS.$$

17.8.52 A computation shows that $\nabla\varphi = \frac{-p\mathbf{r}}{|\mathbf{r}|^{p+2}}$. Thus the potential field $\nabla\varphi = 0$ for $p = 0$, so certainly $\nabla^2\varphi = 0$ as well. Otherwise, for $p = 1$, we have $\nabla\varphi = \frac{-\mathbf{r}}{|\mathbf{r}|^3}$; this is an inverse square field, and we have seen many times (e.g. Exercise 36(b)) that the divergence of such a field is zero. Thus $\nabla^2\varphi = 0$ for $p = 1$ as well.

17.8.53 The Divergence theorem applied to the field $\nabla\varphi$ says that

$$\iiint_D (\nabla^2\varphi) dV = \iint_S \nabla\varphi \cdot \mathbf{n} dS$$

and if φ is harmonic, the left side is zero.

17.8.54 Apply Green's First Identity (Exercise 49) to u and u to give

$$\iiint_D (u\nabla^2 u + \nabla u \cdot \nabla u) dV = \iint_S u\nabla u \cdot \mathbf{n} dS.$$

Because $\nabla^2 u = 0$ and $\nabla u \cdot \nabla u = |\nabla u|^2$, the result follows.

17.8.55 If $\mathbf{T}$ is a vector field $\langle t, u, v \rangle$, then by $\iint \mathbf{T}$, we mean $\left\langle \iint t, \iint u, \iint v \right\rangle$.

a. Let $\mathbf{F} = \langle f, g, h \rangle$ and suppose $\mathbf{n} = \langle n_1, n_2, n_3 \rangle$. Then

$$\begin{aligned} \mathbf{n} \times \mathbf{F} &= \langle n_2h - n_3g, n_3f - n_1h, n_1g - n_2f \rangle \\ \nabla \times \mathbf{F} &= \left\langle \frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}, \frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}, \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right\rangle \end{aligned}$$

Considering first the $\mathbf{i}$ component of these vectors, note that for the vector field $\mathbf{F}_1 = \langle 0, h, -g \rangle$, the Divergence theorem says that

$$\begin{aligned} \iint_S (n_2h - n_3g) dS &= \iint_S \langle 0, h, -g \rangle \cdot \langle n_1, n_2, n_3 \rangle dS = \iint_S \mathbf{F}_1 \cdot \mathbf{n} dS = \iiint_D (\nabla \cdot \mathbf{F}_1) dV \\ &= \iint_D \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z} \right) dA \end{aligned}$$

and similarly for the second and third components.

b. Similarly to part (a), note that

$$\mathbf{n} \times \nabla \varphi = \mathbf{n} \times \langle \varphi_x, \varphi_y, \varphi_z \rangle = \langle n_2 \varphi_z - n_3 \varphi_y, n_3 \varphi_x - n_1 \varphi_z, n_1 \varphi_y - n_2 \varphi_x \rangle.$$

Looking first at the x component of this vector, we have

$$n_2 \varphi_z - n_3 \varphi_y = \langle 0, \varphi_z, -\varphi_y \rangle \cdot \langle n_1, n_2, n_3 \rangle = (\nabla \times \langle \varphi, 0, 0 \rangle) \cdot \langle n_1, n_2, n_3 \rangle$$

so that Stokes' theorem says that, writing $\mathbf{F} = \langle \varphi, 0, 0 \rangle$,

$$\iint_S (\nabla \times \langle \varphi, 0, 0 \rangle) \cdot \langle n_1, n_2, n_3 \rangle dS = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} dS = \oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C \langle \varphi, 0, 0 \rangle \cdot d\mathbf{r} = \oint_C \varphi dx.$$

and similarly for the second and third components.

Chapter Seventeen Review

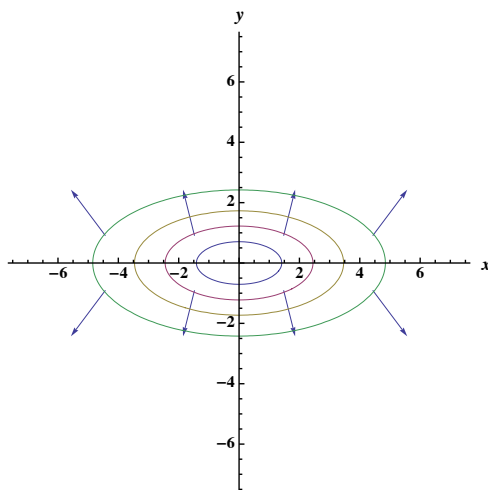
1

- False. The curl is $\frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}(-y) = 2$.
- True. The curl of a conservative vector field is zero.
- False. For example, $\langle -y, x \rangle$ and $\langle 0, 2x \rangle$ both have curl 2.
- False. For example, $\langle x, 0, 0 \rangle$ and $\langle 0, y, 0 \rangle$ both have divergence 1.
- True. By the Divergence theorem, the integral is equal to $\iiint_D \nabla \cdot \mathbf{F} dV = \iiint_D 3 dV$.

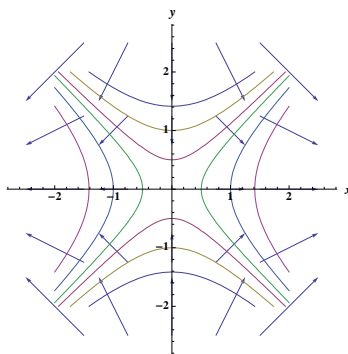
2

- Choice F, because this is a radial vector field whose magnitude increases with distance from the origin.
- Choice E, because this is a rotational field.
- Choice D, because this is also a radial field, but the magnitude is constant.
- Choice C, because this field is vertical along the line $y = x$.
- Choice B, because the magnitude decreases rapidly away from the origin.
- Choice A, because this field is periodic.

3 $\nabla \varphi = \langle 2x, 8y \rangle$.



4 $\nabla\varphi = \langle x, -y \rangle.$



5 $\nabla\varphi = -\frac{\mathbf{r}}{|\mathbf{r}|^3}$

6 $\nabla\varphi = -e^{-x^2-y^2-z^2}\langle x, y, z \rangle.$

7 $\langle x, y \rangle$ is an outward normal; for (x, y) on the circle, $|\langle x, y \rangle| = \sqrt{x^2 + y^2} = 2$, so the unit outward normal is $\frac{1}{2}\langle x, y \rangle.$

8 We have $|\mathbf{r}'(t)| = 5$, so the line integral is

$$\int_C (x^2 - 2xy + y^2) ds = 5 \int_0^\pi (25 \cos^2 t - 10 \cos t \sin t + 25 \sin^2 t) dt = 25 \int_0^\pi (5 - 2 \sin t \cos t) dt = 125\pi.$$

9 Here $|\mathbf{r}'(t)| = \sqrt{4 + 9 + 36} = \sqrt{49} = 7$, so

$$\int_C y e^{-xz} ds = 7 \int_0^2 3te^{12t^2} dt = \frac{21}{24} e^{12t^2} \Big|_0^2 = \frac{7}{8} (e^{48} - 1).$$

10 Parameterize the line by $\langle -3t, 1 + 6t, 2 - 3t \rangle$ for $0 \leq t \leq 1$. Then $|\mathbf{r}'(t)| = \sqrt{9 + 36 + 9} = \sqrt{54} = 3\sqrt{6}$, so

$$\int_C (xz - y^2) ds = 3\sqrt{6} \int_0^1 (3t(3t - 2) - (6t + 1)^2) dt = -57\sqrt{6}.$$

11 For the first parameterization we have $|\mathbf{r}'(t)| = 2$, so

$$\oint_C (x - 2y + 3z) ds = 2 \int_0^{2\pi} (2 \cos t - 4 \sin t) dt = 0.$$

For the second parameterization we have $|\mathbf{r}'(t)| = \sqrt{16t^2 \sin^2(t^2) + 16t^2 \cos^2(t^2)} = 4t$, so

$$\oint_C (x - 2y + 3z) ds = \int_0^{\sqrt{2\pi}} (8t \cos t^2 - 16t \sin t^2) dt = 0.$$

12

a. Parameterize the path from P to Q by $\langle 1 - t, t, 0 \rangle$ for $0 \leq t \leq 1$; then the work done is $\int_C \langle 1, 2y, -4z \rangle \cdot$

$$d\mathbf{r} = \int_0^1 \langle 1, 2t, 0 \rangle \cdot \langle -1, 1, 0 \rangle dt = \int_0^1 (2t - 1) dt = 0.$$

b. Parameterize the path from P to O by $\langle 1 - t, 0, 0 \rangle$, and the path from O to Q by $\langle 0, t, 0 \rangle$, both for $0 \leq t \leq 1$. Then the work done is $\int_C \langle 1, 2y, -4z \rangle \cdot d\mathbf{r} = \int_0^1 (\langle 1, 0, 0 \rangle \cdot \langle -1, 0, 0 \rangle + \langle 1, 2t, 0 \rangle \cdot \langle 0, 1, 0 \rangle) dt =$

$$\int_0^1 (2t - 1) dt = 0.$$

c. Parameterize the quarter circle by $\langle \cos t, \sin t, 0 \rangle$ for $0 \leq t \leq \frac{\pi}{2}$; then the work done is $\int_C \langle 1, 2y, -4z \rangle \cdot$

$$d\mathbf{r} = \int_0^{\pi/2} \langle 1, 2 \sin t, 0 \rangle \cdot \langle -\sin t, \cos t, 0 \rangle dt = \int_0^{\pi/2} (2 \cos t \sin t - \sin t) dt = 0.$$

d. Yes, the work is independent of the path; this is a conservative vector field with potential function $x + y^2 - 2z^2$.

13 Parameterize the first path by $\mathbf{r}_1(t) = \langle 0, t, 0 \rangle$, and the second by $\mathbf{r}_2(t) = \langle 0, 1, 4t \rangle$, both for $0 \leq t \leq 1$. Then $\mathbf{r}'_1(t) = \langle 0, 1, 0 \rangle$ and $\mathbf{r}'_2(t) = \langle 0, 0, 4 \rangle$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 (\langle -t, 0, 0 \rangle \cdot \langle 0, 1, 0 \rangle + \langle -1, 4t, 0 \rangle \cdot \langle 0, 0, 4 \rangle) dt = 0.$$

14 $\mathbf{r}'(t) = \langle 2t, 6t, -2t \rangle$, so

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_1^2 \frac{1}{(11t^4)^{3/2}} \langle t^2, 3t^2, -t^2 \rangle \cdot \langle 2t, 6t, -2t \rangle dt = 11^{-3/2} \int_1^2 t^{-6} \cdot 22t^3 dt = 11^{-3/2} \int_1^2 22t^{-3} dt = \frac{3}{44} \sqrt{11}.$$

15 The circulation is

$$\int_C \mathbf{F} \cdot \mathbf{T} ds = \int_0^{2\pi} \langle 2 \sin t - 2 \cos t, 2 \sin t \rangle \cdot \langle -2 \sin t, 2 \cos t \rangle dt = \int_0^{2\pi} (-4 \sin^2 t + 8 \sin t \cos t) dt = -4\pi.$$

The outward flux is

$$\begin{aligned} \int_C \mathbf{F} \cdot \mathbf{n} ds &= \int_0^{2\pi} ((2 \sin t - 2 \cos t)(2 \cos t) - (2 \sin t)(-2 \sin t)) dt \\ &= 4 \int_0^{2\pi} (\sin t \cos t - \cos^2 t + \sin^2 t) dt = 0. \end{aligned}$$

16 The circulation is

$$\int_C \mathbf{F} \cdot \mathbf{T} ds = \int_0^{2\pi} \langle 2 \cos t, 2 \sin t \rangle \cdot \langle -2 \sin t, 2 \cos t \rangle dt = 0.$$

The outward flux is

$$\int_C \mathbf{F} \cdot \mathbf{n} ds = \int_0^{2\pi} ((2 \cos t)(2 \cos t) - (2 \sin t)(-2 \sin t)) dt = 8\pi.$$

17 The circulation is

$$\int_C \mathbf{F} \cdot \mathbf{T} ds = \frac{1}{4} \int_0^{2\pi} \langle 2 \cos t, 2 \sin t \rangle \cdot \langle -2 \sin t, 2 \cos t \rangle dt = 0.$$

The outward flux is

$$\int_C \mathbf{F} \cdot \mathbf{n} ds = \frac{1}{4} \int_0^{2\pi} ((2 \cos t)(2 \cos t) - (2 \sin t)(-2 \sin t)) dt = 2\pi.$$

18 The circulation is

$$\int_C \mathbf{F} \cdot \mathbf{T} ds = \int_0^{2\pi} \langle 2 \cos t - 2 \sin t, 2 \cos t \rangle \cdot \langle -2 \sin t, 2 \cos t \rangle dt = 4 \int_0^{2\pi} (-\sin t \cos t + 1) dt = 4\pi.$$

The outward flux is

$$\int_C \mathbf{F} \cdot \mathbf{n} ds = \int_0^{2\pi} ((2 \cos t - 2 \sin t)(2 \cos t) - (2 \cos t)(-2 \sin t)) dt = 4 \int_0^{2\pi} (\cos^2 t) dt = 4\pi.$$

19 The normal to the plane $x = 0$ is $\langle 1, 0, 0 \rangle$, so the flux is

$$\int_C \mathbf{F} \cdot \mathbf{n} \, ds = \int_{-1/2}^{1/2} \int_{-L}^L v_0 (L^2 - y^2) \, dy \, dz = \frac{4}{3} v_0 L^3.$$

20 A potential function is xy^2 .

21 A potential function is $xy + yz^2$.

22 A potential function is $e^x \cos y$.

23 A potential function is xye^z .

24

a. $\mathbf{F} = \langle 2xy, x^2 \rangle$, so

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^3 \langle 2(9-t^2)t, (9-t^2)^2 \rangle \cdot \langle -2t, 1 \rangle \, dt \\ &= \int_0^3 (-4t^2(9-t^2) + (9-t^2)^2) \, dt = \int_0^3 (9-5t^2)(9-t^2) \, dt = 0 \end{aligned}$$

b. Because $\mathbf{F} = \nabla \varphi$, where $\varphi(x, y) = x^2 y$, we have $\int_C \mathbf{F} \cdot d\mathbf{r} = \varphi(3) - \varphi(0) = 0 - 0 = 0$.

25

a. $\mathbf{F} = \langle yz, xz, xy \rangle$, so

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^\pi \left\langle \frac{t}{\pi} \sin t, \frac{t}{\pi} \cos t, \sin t \cos t \right\rangle \cdot \left\langle -\sin t, \cos t, \frac{1}{\pi} \right\rangle \, dt \\ &= \int_0^\pi \left(\frac{t}{\pi} (\cos^2 t - \sin^2 t) + \frac{1}{\pi} \sin t \cos t \right) \, dt = 0. \end{aligned}$$

b. $\mathbf{F} = \nabla(xyz) = \nabla \varphi$, so

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \varphi(\cos \pi \sin \pi) - \varphi\left(\cos 0 \sin 0 \cdot \frac{0}{\pi}\right) = 0 - 0 = 0.$$

26

a. Parameterize C by the four paths $\mathbf{r}_1(t) = \langle -1 + 2t, -1 \rangle$, $\mathbf{r}_2(t) = \langle 1, -1 + 2t \rangle$, $\mathbf{r}_3(t) = \langle 1 - 2t, 1 \rangle$, $\mathbf{r}_4(t) = \langle -1, 1 - 2t \rangle$, for $0 \leq t \leq 1$. Then $\mathbf{r}'_1(t) = \langle 2, 0 \rangle$, $\mathbf{r}'_2(t) = \langle 0, 2 \rangle$, $\mathbf{r}'_3(t) = \langle -2, 0 \rangle$, $\mathbf{r}'_4(t) = \langle 0, -2 \rangle$, and

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^1 (\langle -1 + 2t, 1 \rangle \cdot \langle 2, 0 \rangle + \langle 1, 1 - 2t \rangle \cdot \langle 0, 2 \rangle + \langle 1 - 2t, -1 \rangle \cdot \langle -2, 0 \rangle + \langle -1, 2t - 1 \rangle \cdot \langle 0, -2 \rangle) \, dt \\ &= \int_0^1 (4t - 2 + 2 - 4t + 4t - 2 + 2 - 4t) \, dt = \int_0^1 0 \, dt = 0. \end{aligned}$$

b. $\mathbf{F} = \nabla \varphi$, where $\varphi(x, y) = \frac{1}{2}(x^2 - y^2)$, so the integral around any closed curve is zero.

27

$$\text{a. } \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \langle \sin t, 4, -\cos t \rangle \cdot \langle -\sin t, \cos t, 0 \rangle dt = \int_0^{2\pi} (-\sin^2 t + 4 \cos t) dt = -\pi.$$

b. The vector field is not conservative, since for example $\frac{\partial}{\partial y}(y) \neq \frac{\partial}{\partial z}(x)$.

28 For $p \neq 2$, $\mathbf{F} = \nabla \varphi$ where $\varphi = \frac{-1}{(p-2)|\mathbf{r}|^{p-2}}$, while for $p = 2$, $\varphi = \frac{1}{2} \ln(|\mathbf{r}|^2)$, as can be seen by taking the gradient. Thus $\mathbf{F}$ is conservative on all of $\mathbb{R}^2$ for $p < 0$.

29 By the circulation form of Green's Theorem,

$$\oint_C xy^2 dx + x^2y dy = \iint_R \left(\frac{\partial}{\partial x}(x^2y) - \frac{\partial}{\partial y}(xy^2) \right) dA = \iint_R (2xy - 2xy) dA = 0.$$

30 By the circulation form of Green's Theorem,

$$\oint_C (-3y + x^{3/2}) dx + (x - y^{2/3}) dy = \iint_R \left(\frac{\partial}{\partial x}(x - y^{2/3}) - \frac{\partial}{\partial y}(-3y + x^{3/2}) \right) dA = \iint_R 4 dA = 4\pi.$$

31 By the circulation form of Green's Theorem,

$$\begin{aligned} \oint_C (x^3 + xy) dy + (2y^2 - 2x^2y) dx &= \iint_R \left(\frac{\partial}{\partial x}(x^3 + xy) - \frac{\partial}{\partial y}(2y^2 - 2x^2y) \right) dA \\ &= \iint_R (3x^2 + y - 4y + 2x^2) dA = \int_{-1}^1 \int_{-1}^1 (5x^2 - 3y) dy dx = \frac{20}{3}. \end{aligned}$$

32 By the flux form of Green's Theorem, $\oint_C 3x^3 dy - 3y^3 dx = \iint_R (9x^2 + 9y^2) dA = 9 \int_0^4 \int_0^{2\pi} r^3 d\theta dr = 1152\pi$. Because the orientation is clockwise, the answer is -1152π .

33 The ellipse is $\frac{x^2}{16} + \frac{y^2}{4} = 1$; parameterize it by $\mathbf{r}(t) = \langle x, y \rangle = \langle 4 \cos t, 2 \sin t \rangle$, $0 \leq t \leq 2\pi$. Then the area of the region is

$$\frac{1}{2} \oint_C ((4 \cos t)(2 \cos t) - (2 \sin t)(-4 \sin t)) dt = \int_0^{2\pi} 4(\cos^2 t + \sin^2 t) dt = 8\pi.$$

34 $dx = -3 \cos^2 t \sin t dt$, and $dy = 3 \sin^2 t \cos t dt$, so the area of the hypocycloid is

$$\begin{aligned} \frac{1}{2} \oint_C x dy - y dx &= \frac{1}{2} \int_0^{2\pi} ((\cos^3 t)(3 \sin^2 t \cos t) - (\sin^3 t)(-3 \cos^2 t \sin t)) dt \\ &= \frac{3}{2} \int_0^{2\pi} (\cos^4 t \sin^2 t + \sin^4 t \cos^2 t) dt = \frac{3\pi}{8}. \end{aligned}$$

35

a. $\mathbf{F} = (x^2 + y^2)^{-1/2} \langle x, y \rangle$, so the circulation is

$$\begin{aligned} \oint_C \mathbf{F} \cdot d\mathbf{r} &= \iint_R \left(\frac{\partial}{\partial x} \left(\frac{y}{\sqrt{x^2 + y^2}} \right) - \frac{\partial}{\partial y} \left(\frac{x}{\sqrt{x^2 + y^2}} \right) \right) dA \\ &= \iint_R \left(\frac{-xy}{\sqrt{x^2 + y^2}} - \frac{-xy}{\sqrt{x^2 + y^2}} \right) dA = 0. \end{aligned}$$

b. The flux is

$$\begin{aligned}\oint_C \mathbf{F} \cdot \mathbf{n} \, ds &= \iint_R \left(\frac{\partial}{\partial x} \left(\frac{x}{\sqrt{x^2 + y^2}} \right) + \frac{\partial}{\partial y} \left(\frac{y}{\sqrt{x^2 + y^2}} \right) \right) dA \\ &= \iint_R \left(\frac{y^2}{(x^2 + y^2)^{3/2}} + \frac{x^2}{(x^2 + y^2)^{3/2}} \right) dA = \iint_R \left(\frac{1}{\sqrt{x^2 + y^2}} \right) dA = \int_0^\pi \int_1^3 1 \, dr \, d\theta = 2\pi.\end{aligned}$$

36

a. The circulation is $\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R \left(\frac{\partial}{\partial x} (x \cos y) - \frac{\partial}{\partial y} (-\sin y) \right) dA = \int_0^{\pi/2} \int_0^{\pi/2} 2 \cos y \, dy \, dx = \pi$.

b. The flux is $\oint_C \mathbf{F} \cdot \mathbf{n} \, ds = \iint_R \left(\frac{\partial}{\partial x} (-\sin y) + \frac{\partial}{\partial y} (x \cos y) \right) dA = \int_0^{\pi/2} \int_0^{\pi/2} (-x \sin y) \, dy \, dx = -\frac{1}{8}\pi^2$.

37

a. For $\mathbf{F}$ to be conservative, we must have $\frac{\partial}{\partial y} (ax + by) = \frac{\partial}{\partial x} (cx + dy)$, or $b = c$.

b. For $\mathbf{F}$ to be source-free, we must have $\frac{\partial}{\partial x} (ax + by) = -\frac{\partial}{\partial y} (cx + dy)$, or $a = -d$.

c. $\mathbf{F}$ is both conservative and source-free if $b = c$ and $a = -d$, i.e. if $\mathbf{F} = \langle ax + by, bx - ay \rangle$.

38 The divergence is $\frac{\partial}{\partial x} (yz) + \frac{\partial}{\partial y} (xz) + \frac{\partial}{\partial z} (xy) = 0$. The curl is

$$\left\langle \frac{\partial}{\partial y} (xy) - \frac{\partial}{\partial z} (xz), \frac{\partial}{\partial z} (yz) - \frac{\partial}{\partial x} (xy), \frac{\partial}{\partial x} (xz) - \frac{\partial}{\partial y} (yz) \right\rangle = \mathbf{0}.$$

The field is both source-free and irrotational.

39 The divergence is $4|\mathbf{r}|$. The curl is

$$\left\langle \frac{\partial}{\partial y} (z|\mathbf{r}|) - \frac{\partial}{\partial z} (y|\mathbf{r}|), \frac{\partial}{\partial z} (x|\mathbf{r}|) - \frac{\partial}{\partial x} (z|\mathbf{r}|), \frac{\partial}{\partial x} (y|\mathbf{r}|) - \frac{\partial}{\partial y} (x|\mathbf{r}|) \right\rangle = \mathbf{0}.$$

The field is irrotational but not source-free.

40 The divergence is $\frac{\partial}{\partial x} (\sin xy) + \frac{\partial}{\partial y} (\cos yz) + \frac{\partial}{\partial z} (\sin xz) = y \cos xy - z \sin yz + x \cos xz$. The curl is $\langle y \sin yz, -z \cos xz, -x \cos xy \rangle$. The field is neither irrotational nor source-free.

41 The divergence is $\frac{\partial}{\partial x} (2xy + z^4) + \frac{\partial}{\partial y} (x^2) + \frac{\partial}{\partial z} (4xz^3) = 2y + 12xz^2$. The curl is

$$\left\langle \frac{\partial}{\partial y} (4xz^3) - \frac{\partial}{\partial z} (x^2), \frac{\partial}{\partial z} (2xy + z^4) - \frac{\partial}{\partial x} (4xz^3), \frac{\partial}{\partial x} (x^2) - \frac{\partial}{\partial y} (2xy + z^4) \right\rangle = \mathbf{0},$$

so the field is irrotational but not source-free.

42 $|\mathbf{r}|^4 = (x^2 + y^2 + z^2)^2$, so

$$\nabla (x^2 + y^2 + z^2)^{-2} = \langle -4x(x^2 + y^2 + z^2)^{-3}, -4y(x^2 + y^2 + z^2)^{-3}, -4z(x^2 + y^2 + z^2)^{-3} \rangle = -\frac{4\mathbf{r}}{|\mathbf{r}|^6}.$$

Now

$$\frac{\partial}{\partial x} \left(-4x (x^2 + y^2 + z^2)^{-3} \right) = -4 (y^2 + z^2 - 5x^2) (x^2 + y^2 + z^2)^{-4},$$

so that

$$\nabla \cdot \nabla |\mathbf{r}|^{-4} = -4 (x^2 + y^2 + z^2)^{-4} (y^2 + z^2 - 5x^2 + x^2 + z^2 - 5y^2 + x^2 + y^2 - 5z^2) = \frac{12}{|\mathbf{r}|^6}.$$

43

a. The curl is

$$\left\langle \frac{\partial}{\partial y}(-y) - \frac{\partial}{\partial z}(x), \frac{\partial}{\partial z}(z) - \frac{\partial}{\partial x}(-y), \frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}(z) \right\rangle = \langle -1, 1, 1 \rangle.$$

So the scalar component in the direction of $\langle 1, 0, 0 \rangle$ is $\langle -1, 1, 1 \rangle \cdot \langle 1, 0, 0 \rangle = -1$,

b. The scalar component in the direction of $\langle 0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$ is $\langle 0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle \cdot \langle 1, 0, 0 \rangle = 0$.

c. The scalar component of the curl is a maximum in the direction of the curl, i.e. in the direction $\langle -1, 1, 1 \rangle$, whose unit vector is $\frac{1}{\sqrt{3}} \langle -1, 1, 1 \rangle$.

44 The curl of the vector field is $\nabla \times \mathbf{F} = \langle 0, 0, 2 \rangle$, and the component of the curl along a unit vector $\mathbf{n}$ is thus $\langle 0, 0, 2 \rangle \cdot \mathbf{n}$.

a. It does not spin, because $\langle 0, 0, 2 \rangle \cdot \langle 1, 0, 0 \rangle = 0$.

b. The scalar component of the curl in the direction $\langle 0, 0, 1 \rangle$ is 2.

c. It spins the fastest when the paddle wheel is aligned with $\langle 0, 0, 1 \rangle$.

45 Parameterize the sphere by $\langle 3 \sin u \cos v, 3 \sin u \sin v, 3 \cos u \rangle$, $0 \leq u \leq \frac{\pi}{2}$; $0 \leq v \leq 2\pi$. Then $|\mathbf{n}| = 9 \sin u$, so

$$\iint_S 1 \, dS = \iint_R 9 \sin u \, dA = \int_0^{2\pi} \int_0^{\pi/2} 9 \sin u \, du \, dv = 18\pi.$$

46 Parameterize the surface by $\langle v \cos u, v \sin u, v \rangle$ for $2 \leq v \leq 4$, $0 \leq u \leq 2\pi$. Then

$$\iint_S 1 \, dS = \iint_R \sqrt{2} v \, dA = \sqrt{2} \int_2^4 \int_0^{2\pi} v \, du \, dv = 12\pi\sqrt{2}.$$

47 The volume element is $\sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$, so the area is

$$\iint_S 1 \, dS = \sqrt{3} \iint_R 1 \, dA = \sqrt{3} \int_{-1}^1 \int_{-1}^1 1 \, dx \, dy = 4\sqrt{3}.$$

48 The volume element is $\sqrt{z_x^2 + z_y^2 + 1} = \sqrt{2(x^2 + y^2) + 1}$, so the integral is

$$\iint_S 1 \, dS = \iint_R \sqrt{2(x^2 + y^2) + 1} \, dA = \int_0^2 \int_0^{2\pi} r \sqrt{2r^2 + 1} \, d\theta \, dr = \frac{26}{3}\pi.$$

49 The volume element for $z = 2 - x - y$ is $\sqrt{3}$, so the integral is

$$\iint_S (1 + yz) \, dS = \sqrt{3} \iint_R (1 + yz) \, dA = \sqrt{3} \int_0^2 \int_0^{2-x} (1 + y(2 - x - y)) \, dy \, dx = \frac{8\sqrt{3}}{3}.$$

50 The normal to the curved surface of the cylinder at (x, y, z) is $\langle 0, y, z \rangle$, so

$$\iint_S \langle 0, y, z \rangle \cdot \mathbf{n} \, dS = a \iint_R (y^2 + z^2) \, dA = a^2 \cdot \text{area of } R = 32\pi a^3.$$

51 Parameterize the curved surface using spherical coordinates, so that $|\mathbf{n}| = 4 \sin u$; then for the curved surface we have

$$\iint_S (x - y + z) \, dS = 8 \int_0^{2\pi} \int_0^{\pi/2} (\sin u \cos v - \sin u \sin v + \cos u) \sin u \, du \, dv = 8\pi.$$

For the planar surface, $\mathbf{n} = \langle 0, -1, 0 \rangle$ so that $|\mathbf{n}| = 1$ and

$$\iint_S (x - y + z) \, dS = \iint_R (x - y + z) \, dA = \int_0^2 \int_0^{2\pi} r (\cos \theta - \sin \theta) \, d\theta \, dr = 0$$

and the total integral is thus 8π .

52 The normal to the cylinder is $\langle x, y, 0 \rangle$ with magnitude 1, so

$$\iint_S \mathbf{F} \cdot \mathbf{n} \, dS = \iint_R \langle x, y, z \rangle \cdot \langle x, y, 0 \rangle \, dA = \iint_R (x^2 + y^2) \, dA = \iint_R 1 \, dA = 32\pi.$$

53 $\mathbf{F} = (x^2 + y^2 + z^2)^{-1/2} \langle x, y, z \rangle$; using spherical coordinates to parameterize the sphere gives

$$\begin{aligned} \iint_S \mathbf{F} \cdot \mathbf{n} \, dS &= \iint_R \frac{1}{a} \langle a \sin u \cos v, a \sin u \sin v, a \cos u \rangle \cdot \langle a^2 \sin^2 u \cos v, a^2 \sin^2 u \sin v, a^2 \sin u \cos u \rangle \, dA \\ &= \iint_R a^2 \sin u \, dA = \int_0^{2\pi} \int_0^\pi a^2 \sin u \, du \, dv = 4\pi a^2. \end{aligned}$$

54

a. Using the explicit description, we have $\sqrt{z_z^2 + z_y^2 + 1} = \sqrt{4(x^2 + y^2) + 1}$, so

$$\iint_S 1 \, dS = \iint_R \sqrt{4(x^2 + y^2) + 1} \, dA = \int_0^2 \int_0^{2\pi} r \sqrt{4r^2 + 1} \, d\theta \, dr = \frac{\pi}{6} (17\sqrt{17} - 1).$$

b. Using the given parametric description, we have $|\mathbf{n}| = v\sqrt{1 + 4v^2}$, so

$$\iint_S 1 \, dS = \iint_R v \sqrt{1 + 4v^2} \, dA = \int_0^{2\pi} \int_0^2 v \sqrt{1 + 4v^2} \, dv \, d\theta = \frac{\pi}{6} (17\sqrt{17} - 1).$$

c. Using the given parametric description, we have

$$|\mathbf{t}_u \times \mathbf{t}_v| = \left| \langle -\sqrt{v} \sin u, \sqrt{v} \cos u, 0 \rangle \times \left\langle -\frac{1}{2}v^{-1/2} \cos u, \frac{1}{2}v^{-1/2} \sin u, 1 \right\rangle \right| = \frac{1}{2}\sqrt{4v + 1}$$

so that

$$\iint_S 1 \, dS = \frac{1}{2} \iint_R \sqrt{4v + 1} \, dA = \frac{1}{2} \int_0^{2\pi} \int_0^4 \sqrt{4v + 1} \, dv \, d\theta = \frac{\pi}{6} (17\sqrt{17} - 1).$$

55

- a. The base of S is the surface where $z = 0$, or the circle $x^2 + y^2 = a^2$. Similarly, the base of the paraboloid is found by setting $z = 0$; simplifying gives again $x^2 + y^2 = a^2$. The high point of the hemisphere (maximum z -coordinate) occurs when $x = y = 0$; then $z = a$. Similarly, the high point on the paraboloid also occurs when $x = y = 0$ and again this gives $z = a$.
- b. The graph of the paraboloid is inside that of the hemisphere everywhere, so we would expect it to have smaller surface area. We know that the surface area of the hemisphere is $4\pi a^2 \cdot \frac{1}{2} = 2\pi a^2$. For the paraboloid, we have $z_x = -\frac{2x}{a}$, $z_y = -\frac{2y}{a}$, so that $|\mathbf{n}| = \sqrt{\frac{4(x^2 + y^2)}{a^2} + 1}$, so

$$\iint_S 1 \, dS = \frac{1}{a} \iint_R \sqrt{4(x^2 + y^2) + a^2} \, dA = \frac{1}{a} \int_0^a \int_0^{2\pi} r \sqrt{4r^2 + a^2} \, d\theta \, dr = \frac{(5\sqrt{5} - 1)}{6} \pi a^2,$$

which is in fact smaller than the area of the hemisphere.

- c. $\mathbf{n} = \langle a^2 \sin^2 u \cos v, a^2 \sin^2 u \sin v, a^2 \cos u \sin u \rangle$, so

$$\begin{aligned} \iint_S \mathbf{F} \cdot \mathbf{n} \, dS &= \iint_R \langle a \sin u \cos v, a \sin u \sin v, a \cos u \rangle \cdot \langle a^2 \sin^2 u \cos v, a^2 \sin^2 u \sin v, a^2 \cos u \sin u \rangle \, dA \\ &= a^3 \int_0^{2\pi} \int_0^{\pi/2} (\sin^3 u \cos^2 v + \sin^3 u \sin^2 v + \cos^2 u \sin u) \, du \, dv \\ &= a^3 \int_0^{2\pi} \int_0^{\pi/2} (\sin^3 u + \cos^2 u \sin u) \, du \, dv = a^3 \int_0^{2\pi} \int_0^{\pi/2} \sin u \, du \, dv = 2\pi a^3. \end{aligned}$$

- d. For the paraboloid, the parameterization is $\langle v \cos u, v \sin u, a - \frac{v^2}{a} \rangle$, $0 \leq v \leq a$, $0 \leq u \leq 2\pi$, and

$$\mathbf{n} = \langle -v \sin u, v \cos u, 0 \rangle \times \langle \cos u, \sin u, -\frac{2v}{a} \rangle = \langle -\frac{1}{a} 2v^2 \cos u, -\frac{1}{a} 2v^2 \sin u, -v \rangle,$$

so that the outward-pointing normal is $\langle \frac{2v^2}{a} \cos u, \frac{2v^2}{a} \sin u, v \rangle$ and

$$\begin{aligned} \iint_S \mathbf{F} \cdot \mathbf{n} \, dS &= \iint_R \langle v \cos u, v \sin u, a - \frac{v^2}{a} \rangle \cdot \langle \frac{2v^2}{a} \cos u, \frac{2v^2}{a} \sin u, v \rangle \, dA \\ &= \int_0^a \int_0^{2\pi} \left(\frac{2}{a} v^3 + av - \frac{v^3}{a} \right) \, du \, dv = \frac{3}{2} \pi a^3. \end{aligned}$$

56

- a. For the given $\mathbf{r}$, we have $\frac{(a \cos u \sin v)^2}{a^2} + \frac{(b \sin u \sin v)^2}{b^2} + \frac{(c \cos v)^2}{c^2} = \cos^2 u \sin^2 v + \sin^2 u \sin^2 v + \cos^2 v = \sin^2 v + \cos^2 v = 1$
- b. The normal vector is determined by $\mathbf{n} = \mathbf{t}_u \times \mathbf{t}_v = \langle -bc \cos u \sin^2 v, -ac \sin u \sin 2v, -ab \sin v \cos v \rangle$, so that the outward pointing normal is the negative of this vector. Then

$$\begin{aligned} \iint_S 1 \, dS &= \int_0^{2\pi} \int_0^\pi |\langle bc \cos u \sin^2 v, ac \sin u \sin^2 v, ab \sin v \cos v \rangle| \, dv \, du \\ &= \int_0^{2\pi} \int_0^\pi \sqrt{b^2 c^2 \cos^2 u \sin^4 v + a^2 c^2 \sin^2 u \sin^4 v + a^2 b^2 \sin^2 v \cos^2 v} \, dv \, du. \end{aligned}$$

57 Let S be the disk in the xy -plane with radius 2 centered at the origin with upward-pointing normal vector $\mathbf{k}$. Then $\nabla \times \mathbf{F} = \langle x - y, x - y, 0 \rangle$, so

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} dS = \iint_R \langle x - y, x - y, 0 \rangle \cdot \langle 0, 0, 1 \rangle dA = \iint_R 0 dA = 0.$$

58 $\mathbf{F} = \langle u^2 - v^2, u, 2v(6 - 2u - v) \rangle$, $\mathbf{r}(u, v) = \langle u, v, 6 - 2u - v \rangle$, $\mathbf{t}_u \times \mathbf{t}_v = \langle 2, 1, 1 \rangle$

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^3 \int_0^{6-2u} (2u^2 - 4v^2 + 12v - 4uv) dv du = \frac{39\sqrt{6}}{2}.$$

59 The boundary of this region is in the xy -plane, found by setting $z = 0$, so it is $x^2 + y^2 = 99$, the circle of radius $\sqrt{99}$ about the origin. Parameterize the circle in the usual way; then

$$\begin{aligned} \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} dS &= \oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C \langle 0, \sqrt{99} \cos t, \sqrt{99} \sin t \rangle \cdot \langle -\sqrt{99} \sin t, \sqrt{99} \cos t, 0 \rangle dt \\ &= \int_0^{2\pi} 99 \cos^2 t dt = 99\pi. \end{aligned}$$

60 The boundary of this region is the circle $x^2 + z^2 = 4$ for $y = 0$; parameterizing it in the usual way as $\langle 2 \cos t, 0, 2 \sin t \rangle$ gives

$$\begin{aligned} \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} dS &= \oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \langle 4 \cos^2 t - 4 \sin^2 t, 0, 4 \sin t \cos t \rangle \cdot \langle -2 \sin t, 0, 2 \cos t \rangle dt \\ &= 8 \int_0^{2\pi} ((\sin^2 t - \cos^2 t) \sin t + \sin t \cos^2 t) dt = 8 \int_0^{2\pi} \sin^3 t dt = 0. \end{aligned}$$

61 By Stokes' theorem, the circulation around a closed curve C can be found by choosing a surface S of which C is the boundary; then

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} dS.$$

But for $\mathbf{F} = \nabla(10 - x^2 + y^2 + z^2)$, $\nabla \times \mathbf{F} = \mathbf{0}$, so the right-hand side is zero.

62 We have $\nabla \cdot \mathbf{F} = -3$ so that

$$\iiint_S \mathbf{F} \cdot \mathbf{n} dS = \iiint_D \nabla \cdot \mathbf{F} dV = -3 \cdot \text{volume of cube} = -3.$$

63 $\nabla \cdot \mathbf{F} = x^2 + y^2 + z^2$, so

$$\iiint_S \mathbf{F} \cdot \mathbf{n} dS = \iiint_D (x^2 + y^2 + z^2) dV = \int_0^3 \int_0^{2\pi} \int_0^\pi r^2 \cdot r^2 \sin u du dv dr = \frac{972}{5}\pi.$$

64 $\nabla \cdot \mathbf{F} = 2(x + y + z)$, so

$$\iiint_S \mathbf{F} \cdot \mathbf{n} dS = \iiint_D 2(x + y + z) dV = \int_0^8 \int_0^2 \int_0^{2\pi} 2r(r \cos \theta + r \sin \theta + t) d\theta dr dt = 256\pi.$$

65 $\nabla \cdot \mathbf{F} = 3(x^2 + y^2)$, so the outward flux across the boundary S of a hemisphere D of radius a is

$$\iiint_S \mathbf{F} \cdot \mathbf{n} dS = \iiint_D 3(x^2 + y^2) dV = \int_0^a \int_0^{2\pi} \int_0^{\pi/2} 3r^2 \sin^2 u \cdot r^2 \sin u du dv dr = \frac{4}{5}\pi a^5, \text{ so that the net}$$

flux across the region bounded by the hemispheres of radii 1 and 2 is $\frac{4}{5}\pi(32 - 1) = \frac{124}{5}\pi$.

66 $\nabla \cdot \mathbf{F} = 0$, so the flux is zero across any surface that bounds a region where $\mathbf{F}$ is defined and differentiable; the given region does not include zero, so is one of these. Thus the net outward flux is zero.

67 Using the Divergence theorem, $\nabla \cdot \mathbf{F} = 2x + \sin y + 2y - 2 \sin y + 2z + \sin y = 2(x + y + z)$, so that

$$\iint_S \mathbf{F} \cdot \mathbf{n} dS = \iiint_D 2(x + y + z) dV = \int_0^4 \int_0^1 \int_0^{1-x} 2(x + y + z) dy dx dz = \frac{32}{3}.$$

68

- a. The normal vectors point outwards everywhere on S ; that is, on the curved surface, they point upwards, and on the flat surface they point in the direction of negative x .
- b. Parameterize C by two paths: $\mathbf{r}_1(t) = \langle a \cos t, a \sin t, 0 \rangle$ for $-\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$ and $\mathbf{r}_2(t) = \langle 0, a - 2t, 0 \rangle$ for $0 \leq t \leq a$. Then $\mathbf{r}'_1(t) = \langle -a \sin t, a \cos t, 0 \rangle$ and $\mathbf{r}'_2(t) = \langle 0, -2, 0 \rangle$. So

$$\begin{aligned} \oint_C \mathbf{F} \cdot d\mathbf{r} &= \int_{-\pi/2}^{\pi/2} \langle -a \sin t, a \cos t, a \sin t - 2a \cos t \rangle \cdot \langle -a \sin t, a \cos t, 0 \rangle dt \\ &\quad + \int_0^a \langle 2t - a, 0, a - 2t \rangle \cdot \langle 0, -2, 0 \rangle dt = \int_{-\pi/2}^{\pi/2} a^2 dt = \pi a^2. \end{aligned}$$

- c. $\nabla \times \mathbf{F} = \langle 2, 4, 2 \rangle$. For the curved portion of S , using spherical coordinates, the normal vector is $\langle a^2 \sin^2 u \cos v, a^2 \sin^2 u \sin v, a^2 \cos u \sin u \rangle$, and for the flat portion, the normal vector is $\langle -1, 0, 0 \rangle$. Then
- $$\begin{aligned} \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} dS &= \iint_{S_1} (2a^2 \sin^2 u \cos v + 4a^2 \sin^2 u \sin v + 2a^2 \cos u \sin u) dS + \iint_{S_2} (-2) dS = \\ &= 2a^2 \int_{-\pi/2}^{\pi/2} \int_0^{\pi/2} (\sin^2 u \cos v + 2 \sin^2 u \sin v + \sin u \cos u) du dv - \pi a^2 = 2\pi a^2 - \pi a^2 = \pi a^2. \end{aligned}$$

Second-Order Differential Equations

Basic Ideas

D2.1.1 The *order* of a differential equation is the highest-order derivative that appears in the equation. Thus for example $y'(t) + y(t) = 0$ is a first-order equation, while $y''(t) + y(t) = 0$ is a second-order equation.

D2.1.2 A differential equation is *linear* if each additive term of the equation either does not depend on the unknown function y (so that it is either constant or depends only on t), or is a multiple of y or one of its derivatives by a constant or by a function of t only. In other words, a differential equation is linear if it is of the form

$$y^{(n)}(t) + p_{n-1}(t)y^{(n-1)}(t) + \cdots + p_1(t)y'(t) + p_0(t)y(t) = f(t).$$

A *nonlinear* differential equation is one that is not linear.

D2.1.3 A differential equation $y''(t) + p(t)y'(t) + q(t)y(t) = f(t)$ is *homogeneous* if $f(t) = 0$ for t in the domain we are interested in. It is *nonhomogeneous* if this is not the case. Thus for example $y''(t) + 3ty(t) = 0$ is homogeneous, while $y''(t) + 3ty(t) = t^2$ is nonhomogeneous.

D2.1.4 The general form is $y''(t) + p(t)y'(t) + q(t)y(t) = f(t)$. If the original form of the equation has a coefficient on $y''(t)$ other than 1, simply divide through by it to get an equation in this general form.

D2.1.5 Two functions f and g are linearly dependent on an interval I if there is some nonzero constant c such that for each $x \in I$ we have $f(x) = cg(x)$. That is, they are linearly dependent if one is a nonzero constant multiple of another.

D2.1.6 By Theorem D2.2, there are two linearly independent solutions to a second-order linear homogeneous differential equation.

D2.1.7 The general solution of a second-order linear nonhomogeneous differential equation is the sum of (a) any single particular solution of the nonhomogeneous equation, and (b) the general solution of the homogeneous equation derived by setting $f(t) = 0$ in the nonhomogeneous equation. See Theorems D2.3 and D2.4.

D2.1.8 If $y''(t) + p(t)y'(t) + q(t)y(t) = f(t)$ is a second-order linear nonhomogeneous differential equation with initial conditions $y(0) = A$ and $y'(0) = B$, we solve it as follows: first find the general solution of the corresponding homogeneous differential equation $y''(t) + p(t)y'(t) + q(t)y(t) = 0$; this will have the form $c_1y_1(t) + c_2y_2(t)$ where $y_1(t)$ and $y_2(t)$ are linearly independent solutions to the homogeneous equation. Next, find any particular solution, say $y_3(t)$, to the original nonhomogeneous equation. By Theorem D2.4, the general solution to the nonhomogeneous equation is then $c_1y_1(t) + c_2y_2(t) + y_3(t)$. Now use the initial conditions to construct the two equations

$$\begin{aligned}c_1y_1(0) + c_2y_2(0) &= A - y_3(0) \\c_1y_1'(0) + c_2y_2'(0) &= B - y_3'(0)\end{aligned}$$

and solve these for c_1 and c_2 .

D2.1.9 Because the highest order derivative appearing is the second derivative, this is a second-order differential equation. Because y and its derivatives only appear in terms by themselves, not with other derivatives of y , it is linear. Finally, since there is a nonzero term ($10t^2$) that does not depend on y , it is nonhomogeneous.

D2.1.10 Because the highest order derivative appearing is the first derivative, this is a first-order differential equation. Because there is a y^3 term, it is nonlinear. Finally, since the term $-4t$ is a nonzero term not depending on y , it is nonhomogeneous.

D2.1.11 Because the highest order derivative appearing is the second derivative, this is a second-order differential equation. Because there is a term involving yy' , it is nonlinear. Finally, since there is a nonzero term (e^t) that does not depend on y , it is nonhomogeneous.

D2.1.12 Because the highest order derivative appearing is the second derivative, this is a second-order differential equation. Because z and its derivatives only appear in terms by themselves, not with other derivatives of z , it is linear. Finally, since every nonzero term depends on z , it is homogeneous.

D2.1.13 Because $\frac{d^2}{dt^2}e^{kt} = \frac{d}{dt}(ke^{kt}) = k^2e^{kt}$, we have

$$y''(t) - 4y(t) = (3e^{2t} - 5e^{-2t})'' - 4(3e^{2t} - 5e^{-2t}) = 12e^{2t} - 20e^{-2t} - (12e^{2t} - 20e^{-2t}) = 0.$$

D2.1.14 Because $(\sin at)'' = (a \cos at)' = -a^2 \sin at$ and $(\cos at)'' = (-a \sin at)' = -a^2 \cos at$, we have

$$\begin{aligned} y''(t) + 16y(t) &= (10 \sin 4t - 20 \cos 4t)'' + 16(10 \sin 4t - 20 \cos 4t) \\ &= -160 \sin 4t + 320 \cos 4t + (160 \sin 4t - 320 \cos 4t) \\ &= 0. \end{aligned}$$

D2.1.15 Because $\frac{d^2}{dt^2}e^{kt} = \frac{d}{dt}(ke^{kt}) = k^2e^{kt}$, we have

$$\begin{aligned} y''(t) - 9y(t) &= (4e^{3t} + 3e^{-3t} - 2t)'' - 9(4e^{3t} + 3e^{-3t} - 2t) \\ &= (36e^{3t} + 27e^{-3t}) - (36e^{3t} + 27e^{-3t} - 18t) \\ &= 18t. \end{aligned}$$

D2.1.16 Using the derivatives of $\sin at$ and $\cos at$ from Exercise 14, we have

$$\begin{aligned} y''(t) + 25y(t) &= (2 \sin 5t - 6 \cos 5t + \frac{1}{2} \cos t)'' + 25(2 \sin 5t - 6 \cos 5t + \frac{1}{2} \cos t) \\ &= (-50 \sin 5t + 150 \cos 5t - \frac{1}{2} \cos t) + (50 \sin 5t - 150 \cos 5t + \frac{25}{2} \cos t) \\ &= 12 \cos t. \end{aligned}$$

D2.1.17 We have

$$\begin{aligned} y''(t) - y'(t) - 2y(t) &= (C_1e^{-t} + C_2e^{2t})'' - (C_1e^{-t} + C_2e^{2t})' - 2(C_1e^{-t} + C_2e^{2t}) \\ &= (C_1e^{-t} + 4C_2e^{2t}) - (-C_1e^{-t} + 2C_2e^{2t}) - (2C_1e^{-t} + 2C_2e^{2t}) \\ &= 0. \end{aligned}$$

D2.1.18 We have

$$\begin{aligned} y''(t) + 2y'(t) - 3y(t) &= (C_1e^{-3t} + C_2e^t + e^{2t})'' + 2(C_1e^{-3t} + C_2e^t + e^{2t})' - 3(C_1e^{-3t} + C_2e^t + e^{2t}) \\ &= (9C_1e^{-3t} + C_2e^t + 4e^{2t}) + (-6C_1e^{-3t} + 2C_2e^t + 4e^{2t}) \\ &\quad - (3C_1e^{-3t} + 3C_2e^t + 3e^{2t}) \\ &= 5e^{2t}. \end{aligned}$$

D2.1.19 We have

$$\begin{aligned}
 y''(t) + 6y'(t) + 25y(t) &= (e^{-3t}(C_1 \sin 4t + C_2 \cos 4t))'' + 6(e^{-3t}(C_1 \sin 4t + C_2 \cos 4t))' \\
 &\quad + 25(e^{-3t}(C_1 \sin 4t + C_2 \cos 4t)) \\
 &= (-3e^{-3t}(C_1 \sin 4t + C_2 \cos 4t) + e^{-3t}(4C_1 \cos 4t - 4C_2 \sin 4t))' \\
 &\quad + 6(-3e^{-3t}(C_1 \sin 4t + C_2 \cos 4t) + e^{-3t}(4C_1 \cos 4t - 4C_2 \sin 4t)) \\
 &\quad + 25(e^{-3t}(C_1 \sin 4t + C_2 \cos 4t)) \\
 &= (e^{-3t}((-3C_1 - 4C_2) \sin 4t + (4C_1 - 3C_2) \cos 4t))' \\
 &\quad + 6(-3e^{-3t}(C_1 \sin 4t + C_2 \cos 4t) + e^{-3t}(4C_1 \cos 4t - 4C_2 \sin 4t)) \\
 &\quad + 25(e^{-3t}(C_1 \sin 4t + C_2 \cos 4t)) \\
 &= -3e^{-3t}((-3C_1 - 4C_2) \sin 4t + (4C_1 - 3C_2) \cos 4t) \\
 &\quad + e^{-3t}((-12C_1 - 16C_2) \cos 4t + (-16C_1 + 12C_2) \sin 4t) \\
 &\quad + 6(-3e^{-3t}(C_1 \sin 4t + C_2 \cos 4t) + e^{-3t}(4C_1 \cos 4t - 4C_2 \sin 4t)) \\
 &\quad + 25(e^{-3t}(C_1 \sin 4t + C_2 \cos 4t)) \\
 &= e^{-3t}((9C_1 + 12C_2) \sin 4t + (-12C_1 + 9C_2) \cos 4t) \\
 &\quad + e^{-3t}((-16C_1 + 12C_2) \sin 4t + (-12C_1 - 16C_2) \cos 4t) \\
 &\quad + e^{-3t}((-18C_1 - 24C_2) \sin 4t + (24C_1 - 18C_2) \cos 4t) \\
 &\quad + e^{-3t}(25C_1 \sin 4t + 25C_2 \cos 4t) \\
 &= 0.
 \end{aligned}$$

D2.1.20 We have

$$\begin{aligned}
 y''(t) + 8y'(t) + 25y(t) &= (e^{-4t}(C_1 \sin 3t + C_2 \cos 3t) + 2)'' + 8(e^{-4t}(C_1 \sin 3t + C_2 \cos 3t) + 2)' \\
 &\quad + 25(e^{-4t}(C_1 \sin 3t + C_2 \cos 3t) + 2) \\
 &= (-4e^{-4t}(C_1 \sin 3t + C_2 \cos 3t) + e^{-4t}(3C_1 \cos 3t - 3C_2 \sin 3t))' \\
 &\quad + 8(-4e^{-4t}(C_1 \sin 3t + C_2 \cos 3t) + e^{-4t}(3C_1 \cos 3t - 3C_2 \sin 3t)) \\
 &\quad + 25(e^{-4t}(C_1 \sin 3t + C_2 \cos 3t) + 2) \\
 &= (e^{-4t}((-4C_1 - 3C_2) \sin 3t + (3C_1 - 4C_2) \cos 3t))' \\
 &\quad + e^{-4t}(-32C_1 \sin 3t - 32C_2 \cos 3t) + e^{-4t}(24C_1 \cos 3t - 24C_2 \sin 3t) \\
 &\quad + e^{-4t}(25C_1 \sin 3t + 25C_2 \cos 3t) + 50 \\
 &= (-4e^{-4t}((-4C_1 - 3C_2) \sin 3t + (3C_1 - 4C_2) \cos 3t)) \\
 &\quad + e^{-4t}((-12C_1 - 9C_2) \cos 3t + (-9C_1 + 12C_2) \sin 3t) \\
 &\quad + e^{-4t}((-32C_1 - 24C_2) \sin 3t + (24C_1 - 32C_2) \cos 3t) \\
 &\quad + e^{-4t}(25C_1 \sin 3t + 25C_2 \cos 3t) + 50 \\
 &= e^{-4t}((16C_1 + 12C_2) \sin 3t + (-12C_1 + 16C_2) \cos 3t) \\
 &\quad + e^{-4t}((-9C_1 + 12C_2) \sin 3t + (-12C_1 - 9C_2) \cos 3t) \\
 &\quad + e^{-4t}((-32C_1 - 24C_2) \sin 3t + (24C_1 - 32C_2) \cos 3t) \\
 &\quad + e^{-4t}(25C_1 \sin 3t + 25C_2 \cos 3t) + 50 \\
 &= 50.
 \end{aligned}$$

D2.1.21 We have

$$\begin{aligned}
 ty''(t) - (t+1)y'(t) + y(t) &= t(C_1e^t + C_2(t+1))'' - (t+1)(C_1e^t + C_2(t+1))' \\
 &\quad + (C_1e^t + C_2(t+1)) \\
 &= t(C_1e^t + C_2)' - (t+1)(C_1e^t + C_2) + (C_1e^t + C_2(t+1)) \\
 &= tC_1e^t - tC_1e^t - tC_2 - C_1e^t - C_2 + C_1e^t + tC_2 + C_2 \\
 &= 0.
 \end{aligned}$$

D2.1.22 We have

$$\begin{aligned}
 t^2y''(t) + 2ty'(t) - 2y(t) &= t^2(C_1t^{-2} + C_2t + \frac{1}{2}t^3)'' + 2t(C_1t^{-2} + C_2t + \frac{1}{2}t^3)' \\
 &\quad - 2(C_1t^{-2} + C_2t + \frac{1}{2}t^3) \\
 &= t^2(-2C_1t^{-3} + C_2 + \frac{3}{2}t^2)' + 2t(-2C_1t^{-3} + C_2 + \frac{3}{2}t^2) \\
 &\quad - 2(C_1t^{-2} + C_2t + \frac{1}{2}t^3) \\
 &= t^2(6C_1t^{-4} + 3t) - 4C_1t^{-2} + 2C_2t + 3t^3 - 2C_1t^{-2} - 2C_2t - t^3 \\
 &= 6C_1t^{-2} + 3t^3 - 4C_1t^{-2} + 2C_2t + 3t^3 - 2C_1t^{-2} - 2C_2t - t^3 \\
 &= 5t^3.
 \end{aligned}$$

D2.1.23 The two given solutions are linearly independent, since for example at $t = 0$, $\frac{1}{5} \cdot 5e^{-6 \cdot 0} = e^{6 \cdot 0}$ while at $t = 1$ we see that $\frac{1}{5} \cdot 5e^{-6} = e^{-6} \neq e^6$, so that the two solutions do not differ by a constant multiple. Because the two given solutions are linearly independent, the general solution is $y(t) = C_1e^{6t} + C_2e^{-6t}$. Note that the coefficient of 5 in the second solution has been subsumed into the constant C_2 .

D2.1.24 The two given solutions are linearly independent, since for example at $t = 0$, $\sin \sqrt{5}t = 0 \cdot \cos \sqrt{5}t$, but this is not true at $t = \frac{\pi}{2}$, so that the two solutions do not differ by a constant multiple. Because the two given solutions are linearly independent, the general solution is $y(t) = C_1 \cos \sqrt{5}t + C_2 \sin \sqrt{5}t$.

D2.1.25 The two solutions are linearly independent, since for example at $t = 0$, $te^{-t} = 0 \cdot e^{-t}$, but this is not true at $t = 1$, so that the two solutions do not differ by a constant multiple. Because the two solutions are linearly independent, the general solution is $y(t) = C_1e^{-t} + C_2te^{-t}$.

D2.1.26 The two solutions are linearly independent, since for example at $t = 1$, $t = 1 \cdot t^{-1}$, but this is not true at $t = 2$, so that the two solutions do not differ by a constant multiple. Because the two solutions are linearly independent, the general solution is $y(t) = C_1t + C_2t^{-1}$.

D2.1.27 $y''(t) - y(t) = (e^{-3t})'' - e^{-3t} = 9e^{-3t} - e^{-3t} = 8e^{-3t}$.

D2.1.28 Substituting gives

$$\begin{aligned}
 y''(t) + y(t) &= (2 \sin t - \cos 2t)'' + (2 \sin t - \cos 2t) \\
 &= (2 \cos t + 2 \sin 2t)' + 2 \sin t - \cos 2t \\
 &= -2 \sin t + 4 \cos 2t + 2 \sin t - \cos 2t \\
 &= 3 \cos 2t.
 \end{aligned}$$

D2.1.29 Substituting gives

$$\begin{aligned}
 y''(t) - 4y'(t) + 4y(t) &= (t^2e^{2t})'' - 4(t^2e^{2t})' + 4(t^2e^{2t}) \\
 &= (2te^{2t} + 2t^2e^{2t})' - 4(2te^{2t} + 2t^2e^{2t}) + 4t^2e^{2t} \\
 &= 2e^{2t} + 4te^{2t} + 4te^{2t} + 4t^2e^{2t} - 8te^{2t} - 8t^2e^{2t} + 4t^2e^{2t} \\
 &= 2e^{2t}.
 \end{aligned}$$

D2.1.30 Substituting gives

$$\begin{aligned} t^2 y''(t) + t y'(t) - 4y(t) &= t^2(-2t + t^2)'' + t(-2t + t^2)' - 4(-2t + t^2) \\ &= t^2(-2 + 2t)' + t(-2 + 2t) + 8t - 4t^2 \\ &= 2t^2 - 2t + 2t^2 + 8t - 4t^2 = 6t. \end{aligned}$$

D2.1.31 Substituting $\frac{1}{2}e^{-t}$ for $y(t)$ gives

$$y''(t) - 49y(t) = \left(\frac{1}{2}e^{-t}\right)'' - 49\left(\frac{1}{2}e^{-t}\right) = \frac{1}{2}e^{-t} - \frac{49}{2}e^{-t} = -24e^{-t}.$$

Substituting $\frac{1}{2}e^{-t} + 3e^{7t}$ for $y(t)$ gives

$$y''(t) - 49y(t) = \left(\frac{1}{2}e^{-t} + 3e^{7t}\right)'' - 49\left(\frac{1}{2}e^{-t} + 3e^{7t}\right) = \frac{1}{2}e^{-t} + 147e^{7t} - \frac{49}{2}e^{-t} - 147e^{7t} = -24e^{-t}.$$

Thus both of the functions given are in fact particular solutions. Their difference is $3e^{7t}$; substituting this into the equation gives

$$y''(t) - 49y(t) = (3e^{7t})'' - 49(3e^{7t}) = 147e^{7t} - 147e^{7t} = 0,$$

so that the two particular solutions differ by a solution of the homogeneous equation.

D2.1.32 Substituting $2\sin t$ for $y(t)$ gives

$$y''(t) + 16y(t) = (2\sin t)'' + 16(2\sin t) = -2\sin t + 32\sin t = 30\sin t.$$

Substituting $2\sin t - 8\cos 4t$ for $y(t)$ gives

$$\begin{aligned} y''(t) + 16y(t) &= (2\sin t - 8\cos 4t)'' + 16(2\sin t - 8\cos 4t) \\ &= -2\sin t + 128\cos 4t + 32\sin t - 128\cos 4t \\ &= 30\sin t. \end{aligned}$$

Thus both of the functions given are in fact particular solutions. Their difference is $8\cos 4t$; substituting this into the equation gives

$$y''(t) + 16y(t) = (8\cos 4t)'' + 16(8\cos 4t) = -128\cos 4t + 128\cos 4t = 0,$$

so that the two particular solutions differ by a solution of the homogeneous equation.

D2.1.33 Substituting $-e^t$ for $y(t)$ gives

$$y''(t) - y'(t) - 12y(t) = (-e^t)'' - (-e^t)' - 12(-e^t) = -e^t + e^t + 12e^t = 12e^t.$$

Substituting $6e^{4t} - e^t$ for $y(t)$ gives

$$\begin{aligned} y''(t) - y'(t) - 12y(t) &= (6e^{4t} - e^t)'' - (6e^{4t} - e^t)' - 12(6e^{4t} - e^t) \\ &= 96e^{4t} - e^t - (24e^{4t} - e^t) - 72e^{4t} + 12e^t \\ &= 12e^t. \end{aligned}$$

Thus both of the functions given are in fact particular solutions. Their difference is $6e^{4t}$; substituting this into the equation gives

$$y''(t) - y'(t) - 12y(t) = (6e^{4t})'' - (6e^{4t})' - 12(6e^{4t}) = 96e^{4t} - 24e^{4t} - 72e^{4t} = 0,$$

so that the two particular solutions differ by a solution of the homogeneous equation.

D2.1.34 Substituting $-\frac{t^2}{2}$ for $y(t)$ gives

$$\begin{aligned} t^2 y''(t) + 2ty'(t) - 30y(t) &= t^2 \left(-\frac{t^2}{2}\right)'' + 2t \left(-\frac{t^2}{2}\right)' - 30 \left(-\frac{t^2}{2}\right) \\ &= -t^2 - 2t^2 + 15t^2 \\ &= 12t^2. \end{aligned}$$

Substituting $3t^5 - \frac{t^2}{2}$ for $y(t)$ gives

$$\begin{aligned} t^2 y''(t) + 2ty'(t) - 30y(t) &= t^2 \left(3t^5 - \frac{t^2}{2}\right)'' + 2t \left(3t^5 - \frac{t^2}{2}\right)' - 30 \left(3t^5 - \frac{t^2}{2}\right) \\ &= t^2(60t^3 - 1) + 2t(15t^4 - t) - 90t^5 + 15t^2 \\ &= 60t^5 - t^2 + 30t^5 - 2t^2 - 90t^5 + 15t^2 \\ &= 12t^2. \end{aligned}$$

Thus both of the functions given are in fact particular solutions. Their difference is $3t^5$; substituting this into the equation gives

$$t^2 y''(t) + 2ty'(t) - 30y(t) = t^2(3t^5)'' + 2t(3t^5)' - 30(3t^5) = 60t^5 + 30t^5 - 90t^5 = 0,$$

so that the two particular solutions differ by a solution of the homogeneous equation.

D2.1.35 Evaluating the differential expression $y''(t) + 2y(t)$ for the three values given, we get:

$$\begin{aligned} (\sin \sqrt{2}t)'' + 2 \sin \sqrt{2}t &= -2 \sin \sqrt{2}t + 2 \sin \sqrt{2}t = 0 \\ (e^t)'' + 2e^t &= e^t + 2e^t = 3e^t \\ (\cos \sqrt{2}t)'' + 2 \cos \sqrt{2}t &= -2 \cos \sqrt{2}t + 2 \cos \sqrt{2}t = 0. \end{aligned}$$

Thus $\sin \sqrt{2}t$ and $\cos \sqrt{2}t$ are solutions of the homogeneous equation and e^t is a solution of the nonhomogeneous equation. Because $\sin \sqrt{2}t$ and $\cos \sqrt{2}t$ are linearly independent, the general solution of the nonhomogeneous equation is

$$y(t) = c_1 \sin \sqrt{2}t + c_2 \cos \sqrt{2}t + e^t.$$

D2.1.36 Evaluating the differential expression $y''(t) - 4y(t)$ for the three values given, we get:

$$\begin{aligned} (5e^{2t})'' - 4(5e^{2t}) &= 20e^{2t} - 20e^{2t} = 0 \\ (e^{-2t})'' - 4(e^{-2t}) &= 4e^{-2t} - 4e^{-2t} = 0 \\ (-\cos t)'' - 4(-\cos t) &= \cos t + 4 \cos t = 5 \cos t. \end{aligned}$$

This $5e^{2t}$ and e^{-2t} are solutions of the homogeneous equation and $-\cos t$ is a solution of the nonhomogeneous equation. Because $5e^{2t}$ and e^{-2t} are linearly independent, the general solution of the nonhomogeneous equation is

$$y(t) = c_1 e^{2t} + c_2 e^{-2t} - \cos t.$$

Notice that the coefficient 5 of e^{2t} was subsumed into the constant c_1 . Writing $y(t) = 5c_1 e^{2t} + c_2 e^{-2t} - \cos t$ is equally correct but unnecessarily complicated.

D2.1.37 Evaluating the differential expression $y''(t) - 3y'(t) + \frac{25}{4}y(t)$ for the three values given, we get

$$\begin{aligned}
 & (e^{3t/2} \cos 2t)'' - 3(e^{3t/2} \cos 2t)' + \frac{25}{4}(e^{3t/2} \cos 2t) \\
 &= \left(\frac{3}{2}e^{3t/2} \cos 2t - 2e^{3t/2} \sin 2t\right)' - 3\left(\frac{3}{2}e^{3t/2} \cos 2t - 2e^{3t/2} \sin 2t\right) + \frac{25}{4}(e^{3t/2} \cos 2t) \\
 &= (e^{3t/2}(\frac{3}{2} \cos 2t - 2 \sin 2t))' - 3(\frac{3}{2}e^{3t/2} \cos 2t - 2e^{3t/2} \sin 2t) + \frac{25}{4}(e^{3t/2} \cos 2t) \\
 &= \frac{3}{2}e^{3t/2}(\frac{3}{2} \cos 2t - 2 \sin 2t) + e^{3t/2}(-3 \sin 2t - 4 \cos 2t) \\
 &\quad - \frac{9}{2}e^{3t/2} \cos 2t + 6e^{3t/2} \sin 2t + \frac{25}{4}e^{3t/2} \cos 2t \\
 &= e^{3t/2}(\frac{9}{4} \cos 2t - 3 \sin 2t - 3 \sin 2t - 4 \cos 2t - \frac{9}{2} \cos 2t + 6 \sin 2t + \frac{25}{4} \cos 2t) \\
 &= 0
 \end{aligned}$$

$$\begin{aligned}
 & (e^{3t/2} \sin 2t)'' - 3(e^{3t/2} \sin 2t)' + \frac{25}{4}(e^{3t/2} \sin 2t) \\
 &= \left(\frac{3}{2}e^{3t/2} \sin 2t + 2e^{3t/2} \cos 2t\right)' - 3\left(\frac{3}{2}e^{3t/2} \sin 2t + 2e^{3t/2} \cos 2t\right) + \frac{25}{4}(e^{3t/2} \sin 2t) \\
 &= (e^{3t/2}(\frac{3}{2} \sin 2t + 2 \cos 2t))' - 3(\frac{3}{2}e^{3t/2} \sin 2t + 2e^{3t/2} \cos 2t) + \frac{25}{4}(e^{3t/2} \sin 2t) \\
 &= \frac{3}{2}e^{3t/2}(\frac{3}{2} \sin 2t + 2 \cos 2t) + e^{3t/2}(3 \cos 2t - 4 \sin 2t) \\
 &\quad - \frac{9}{2}e^{3t/2} \sin 2t - 6e^{3t/2} \cos 2t + \frac{25}{4}e^{3t/2} \sin 2t \\
 &= e^{3t/2}(\frac{9}{4} \sin 2t + 3 \cos 2t + 3 \cos 2t - 4 \sin 2t - \frac{9}{2} \sin 2t - 6 \cos 2t + \frac{25}{4} \sin 2t) \\
 &= 0
 \end{aligned}$$

$$(48 + 100t)'' - 3(48 + 100t)' + \frac{25}{4}(48 + 100t) = 0 - 300 + 300 + 625t = 625t.$$

Thus $e^{3t/2} \cos 2t$ and $e^{3t/2} \sin 2t$ are linearly independent solutions of the homogeneous equation, and $48 + 100t$ is a particular solution of the nonhomogeneous equation. Thus the general solution of the nonhomogeneous equation is

$$y(t) = c_1 e^{3t/2} \cos 2t + c_2 e^{3t/2} \sin 2t + 48 + 100t.$$

D2.1.38 Evaluating the differential expression $t^2 y''(t) + 2t y'(t) - 6y(t)$ for the three values given, we get

$$\begin{aligned}
 t^2(t^{-3})'' + 2t(t^{-3})' - 6(t^{-3}) &= 12t^{-3} - 6t^{-3} - 6t^{-3} = 0 \\
 t^2\left(\frac{t^4}{2}\right)'' + 2t\left(\frac{t^4}{2}\right)' - 6\left(\frac{t^4}{2}\right) &= 6t^4 + 4t^4 - 3t^4 = 7t^4 \\
 t^2(t^2)'' + 2t(t^2)' - 6(t^2) &= 2t^2 + 4t^2 - 6t^2 = 0.
 \end{aligned}$$

Thus t^{-3} and t^2 are linearly independent solutions of the homogeneous equation, and $\frac{t^4}{2}$ is a particular solution of the nonhomogeneous equation. Thus the general solution of the nonhomogeneous equation is

$$y(t) = c_1 t^{-3} + c_2 t^2 + \frac{t^4}{2}.$$

D2.1.39 Substituting the initial conditions into $y(t)$ gives the system of simultaneous equations

$$\begin{aligned}
 c_1 \sin 0 + c_2 \cos 0 &= y(0) = 4 & \text{so that} & & c_2 &= 4 \\
 3c_1 \cos 0 - 3c_2 \sin 0 &= y'(0) = 0 & & & 3c_1 &= 0.
 \end{aligned}$$

Thus $c_1 = 0$ and $c_2 = 4$, and the solution to the initial value problem is $y(t) = 4 \cos 3t$.

D2.1.40 Substituting the initial conditions into $y(t)$ gives the system of simultaneous equations

$$\begin{aligned} c_1 e^0 + c_2 e^{-0} &= y(0) = 2 & \text{so that} & & c_1 + c_2 &= 2 \\ c_1 e^0 - c_2 e^{-0} &= y'(0) = -2 & & & c_1 - c_2 &= -2. \end{aligned}$$

Thus $c_1 = 0$ and $c_2 = 2$, and the solution to the initial value problem is $y(t) = 2e^{-t}$.

D2.1.41 Substituting the initial conditions into $y(t)$ gives the system of simultaneous equations

$$\begin{aligned} c_1 e^{5 \cdot 0} + c_2 e^{-4 \cdot 0} &= y(0) = -3 & \text{so that} & & c_1 + c_2 &= -3 \\ 5c_1 e^{5 \cdot 0} - 4c_2 e^{-4 \cdot 0} &= y'(0) = 3 & & & 5c_1 - 4c_2 &= 3. \end{aligned}$$

Thus $c_1 = -1$ and $c_2 = -2$, and the solution to the initial value problem is $y(t) = -e^{5t} - 2e^{-4t}$.

D2.1.42 Substituting the initial conditions into $y(t)$ gives the system of simultaneous equations

$$\begin{aligned} c_1 \sin 0 + c_2 \cos 0 - \cos 0 &= y(0) = 4 & \text{so that} & & c_2 &= 5 \\ 2c_1 \cos 0 - 2c_2 \sin 0 + 3 \sin 0 &= y'(0) = 2 & & & 2c_1 &= 2. \end{aligned}$$

Thus $c_1 = 1$ and $c_2 = 5$, and the solution to the initial value problem is $y(t) = \sin 2t + 5 \cos 2t - \cos 3t$.

D2.1.43 Substituting the initial conditions into $y(t)$ gives the system of simultaneous equations

$$\begin{aligned} c_1 e^{4 \cdot 0} + c_2 e^{-4 \cdot 0} - 0^2 - \frac{1}{8} &= y(0) = 0 & \text{so that} & & c_1 + c_2 &= \frac{1}{8} \\ 4c_1 e^{4 \cdot 0} - 4c_2 e^{-4 \cdot 0} - 2 \cdot 0 &= y'(0) = 0 & & & 4c_1 - 4c_2 &= 0. \end{aligned}$$

Thus $c_1 = c_2 = \frac{1}{16}$, and the solution to the initial value problem is $y(t) = \frac{1}{16}e^{4t} + \frac{1}{16}e^{-4t} - t^2 - \frac{1}{8}$.

D2.1.44 Substituting the initial conditions into $y(t)$ gives the system of simultaneous equations

$$\begin{aligned} c_1 \cdot 1^{-2} + c_2 \cdot 1 &= y(1) = 3 & \text{so that} & & c_1 + c_2 &= 3 \\ -2c_1 \cdot 1^{-3} + c_2 &= y'(1) = 0 & & & -2c_1 + c_2 &= 0. \end{aligned}$$

Thus $c_1 = 1$ and $c_2 = 2$, and the solution to the initial value problem is $y(t) = t^{-2} + 2t$.

D2.1.45 Substituting the initial conditions into $y(t)$ gives the system of simultaneous equations

$$\begin{aligned} c_1 \cdot 1^{-2} + c_2 \cdot 1^2 &= y(1) = 1 & \text{so that} & & c_1 + c_2 &= 1 \\ -2c_1 \cdot 1^{-3} + 2c_2 \cdot 1 &= y'(1) = -1 & & & -2c_1 + 2c_2 &= -1. \end{aligned}$$

Thus $c_1 = \frac{3}{4}$ and $c_2 = \frac{1}{4}$, and the solution to the initial value problem is $y(t) = \frac{3}{4}t^{-2} + \frac{1}{4}t^2$.

D2.1.46 Using the facts that

$$\begin{aligned} (e^{-4t} \sin 3t)' &= -4e^{-4t} \sin 3t + 3e^{-4t} \cos 3t = e^{-4t}(3 \cos 3t - 4 \sin 3t) \\ (e^{-4t} \cos 3t)' &= -4e^{-4t} \cos 3t - 3e^{-4t} \sin 3t = -e^{-4t}(4 \cos 3t + 3 \sin 3t), \end{aligned}$$

substitute the initial conditions into $y(t)$ to get the system of simultaneous equations

$$\begin{aligned} c_1 e^{-4 \cdot 0} \sin 0 + c_2 e^{-4 \cdot 0} \cos 0 &= y(0) = 1 \\ c_1 (e^{-4 \cdot 0} (3 \cos 0 - 4 \sin 0)) - c_2 (e^{-4 \cdot 0} (4 \cos 0 + 3 \sin 0)) &= y'(0) = -1 \end{aligned}$$

so that

$$\begin{aligned} c_2 &= 1 \\ 3c_1 - 4c_2 &= -1. \end{aligned}$$

Thus $c_1 = c_2 = 1$, and the solution to the initial value problem is $y(t) = e^{-4t}(\sin 3t + \cos 3t)$.

D2.1.47

a. False. By Theorems D2.2 and D2.4, a second-order linear differential equation has two linearly independent solutions, so that the general solution must involve two terms with arbitrary constants. Note that 0 is linearly dependent with any nonzero function, so that these theorems imply that neither linearly independent solution is everywhere zero.

b. True. Substituting $y_p + cy_h$ into the nonhomogeneous equation gives

$$\begin{aligned} y'' + py' + qy &= (y_p + cy_h)'' + p(y_p + cy_h)' + q(y_p + cy_h) \\ &= (y_p'' + py_p' + qy_p) + c(y_h'' + py_h' + qy_h) \\ &= f + 0 = f, \end{aligned}$$

so that $y_p + cy_h$ satisfies the nonhomogeneous equation. This is the content of Theorem D2.4.

c. False. Because $1 - \cos^2 x = \sin^2 x$, this pair of function is $\{\sin^2 x, 5\sin^2 x\}$, which are obviously constant multiples of one another and thus linearly dependent.

d. False. Substitute $y_1 + y_2$ into the formula to get

$$\begin{aligned} y'' + yy' &= (y_1 + y_2)'' + (y_1 + y_2)(y_1 + y_2)' \\ &= y_1'' + y_2'' + y_1y_1' + y_2y_2' + y_1y_2' + y_2y_1' \\ &= (y_1'' + y_1y_1') + (y_2'' + y_2y_2') + y_1y_2' + y_2y_1' \\ &= y_1y_2' + y_2y_1' \end{aligned}$$

since both y_1 and y_2 satisfy the differential equation. Because there is no reason to expect $y_1y_2' + y_2y_1'$ to be zero, we see that $y_1 + y_2$ need not be a solution of the equation. This does not violate Theorem D2.1 since the given equation is not linear.

e. False. The general solution of this equation is $y(t) = c_1 \sin \sqrt{2}t + c_2 \cos \sqrt{2}t$. The condition $y(0) = 4$ means that $c_2 = 4$. We need a second condition in order to get a value for c_1 . Thus there are multiple solutions.

D2.1.48 Substitution gives

$$\begin{aligned} y''(t) - 12y'(t) + 36y(t) &= (C_1e^{6t} + C_2te^{6t})'' - 12(C_1e^{6t} + C_2te^{6t})' + 36(C_1e^{6t} + C_2te^{6t}) \\ &= 36C_1e^{6t} + C_2(e^{6t} + 6te^{6t})' - (72C_1e^{6t} + 12C_2e^{6t} + 72C_2te^{6t}) \\ &\quad + 36C_1e^{6t} + 36C_2te^{6t} \\ &= 36C_1e^{6t} + 6C_2e^{6t} + 6C_2e^{6t} + 36C_2te^{6t} - 72C_1e^{6t} - 12C_2e^{6t} - 72C_2te^{6t} \\ &\quad + 36C_1e^{6t} + 36C_2te^{6t} \\ &= 0. \end{aligned}$$

D2.1.49 Substitution gives

$$\begin{aligned}
 y''(t) - 12y'(t) + 36y(t) &= (C_1e^{6t} + C_2te^{6t} + t^2e^{6t})'' - 12(C_1e^{6t} + C_2te^{6t} + t^2e^{6t})' \\
 &\quad + 36(C_1e^{6t} + C_2te^{6t} + t^2e^{6t}) \\
 &= 36C_1e^{6t} + (C_2e^{6t} + 6C_2te^{6t} + 2te^{6t} + 6t^2e^{6t})' \\
 &\quad - 72C_1e^{6t} - 12C_2e^{6t} - 72C_2te^{6t} - 24te^{6t} - 72t^2e^{6t} \\
 &\quad + 36C_1e^{6t} + 36C_2te^{6t} + 36t^2e^{6t} \\
 &= 36C_1e^{6t} + 6C_2e^{6t} + 6C_2e^{6t} + 36C_2te^{6t} + 2e^{6t} + 12te^{6t} + 12te^{6t} + 36t^2e^{6t} \\
 &\quad - 72C_1e^{6t} - 12C_2e^{6t} - 72C_2te^{6t} - 24te^{6t} - 72t^2e^{6t} \\
 &\quad + 36C_1e^{6t} + 36C_2te^{6t} + 36t^2e^{6t} \\
 &= 36C_1e^{6t} + 12C_2e^{6t} + 36C_2te^{6t} + 2e^{6t} + 24te^{6t} + 36t^2e^{6t} \\
 &\quad - 72C_1e^{6t} - 12C_2e^{6t} - 72C_2te^{6t} - 24te^{6t} - 72t^2e^{6t} \\
 &\quad + 36C_1e^{6t} + 36C_2te^{6t} + 36t^2e^{6t} \\
 &= 2e^{6t}.
 \end{aligned}$$

D2.1.50 Substitution gives

$$\begin{aligned}
 y''(t) + 4y(t) &= (C_1 \sin 2t + C_2 \cos 2t - 2t \cos 2t)'' + 4(C_1 \sin 2t + C_2 \cos 2t - 2t \cos 2t) \\
 &= -4C_1 \sin 2t - 4C_2 \cos 2t - 2(\cos 2t - 2t \sin 2t)' + 4C_1 \sin 2t + 4C_2 \cos 2t - 8t \cos 2t \\
 &= -2(-2 \sin 2t - 2 \sin 2t - 4t \cos 2t) - 8t \cos 2t \\
 &= 8 \sin 2t
 \end{aligned}$$

D2.1.51 Substitution gives

$$\begin{aligned}
 t^2y''(t) - 3ty'(t) + 4y(t) &= t^2(C_1t^2 + C_2t^2 \ln t)'' - 3t(C_1t^2 + C_2t^2 \ln t)' + 4(C_1t^2 + C_2t^2 \ln t) \\
 &= t^2(2C_1) + t^2(2C_2t \ln t + C_2t)' - 6C_1t^2 - 3t(2C_2t \ln t + C_2t) \\
 &\quad + 4(C_1t^2 + C_2t^2 \ln t) \\
 &= 2C_1t^2 + 2C_2t^2 \ln t + 2C_2t^2 + C_2t^2 - 6C_1t^2 - 6C_2t^2 \ln t - 3C_2t^2 \\
 &\quad + 4C_1t^2 + 4C_2t^2 \ln t \\
 &= 0.
 \end{aligned}$$

D2.1.52 Substitution gives

$$\begin{aligned}
 t^2y''(t) - 3ty'(t) + 4y(t) &= t^2(C_1t^2 + C_2t^2 \ln t + t^2 \ln^2 t)'' - 3t(C_1t^2 + C_2t^2 \ln t + t^2 \ln^2 t)' \\
 &\quad + 4(C_1t^2 + C_2t^2 \ln t + t^2 \ln^2 t) \\
 &= t^2(2C_1t + 2C_2t \ln t + C_2t + 2t \ln^2 t + 2t \ln t)' \\
 &\quad - 3t(2tC_1 + 2C_2t \ln t + C_2t + 2t \ln^2 t + 2t \ln t) \\
 &\quad + 4C_1t^2 + 4C_2t^2 \ln t + 4t^2 \ln^2 t \\
 &= t^2(2C_1 + 2C_2 \ln t + 2C_2 + C_2 + 2 \ln^2 t + 4 \ln t + 2 \ln t + 2) \\
 &\quad - 6C_1t^2 - 6C_2t^2 \ln t - 3C_2t^2 - 6t^2 \ln^2 t - 6t^2 \ln t \\
 &\quad + 4C_1t^2 + 4C_2t^2 \ln t + 4t^2 \ln^2 t \\
 &= 2C_1t^2 + 2C_2t^2 \ln t + 2C_2t^2 + C_2t^2 + 2t^2 \ln^2 t + 4t^2 \ln t + 2t^2 \ln t + 2t^2 \\
 &\quad - 6C_1t^2 - 6C_2t^2 \ln t - 3C_2t^2 - 6t^2 \ln^2 t - 6t^2 \ln t \\
 &\quad + 4C_1t^2 + 4C_2t^2 \ln t + 4t^2 \ln^2 t \\
 &= 2t^2.
 \end{aligned}$$

D2.1.53 Substitution gives

$$\begin{aligned}
 t^2 y''(t) + t y'(t) + (t^2 - \frac{1}{4})y(t) &= t^2(t^{-1/2}(C_1 \cos t + C_2 \sin t))'' \\
 &\quad + t(t^{-1/2}(C_1 \cos t + C_2 \sin t))' + (t^2 - \frac{1}{4})(t^{-1/2}(C_1 \cos t + C_2 \sin t)) \\
 &= t^2(-\frac{1}{2}t^{-3/2}(C_1 \cos t + C_2 \sin t) + t^{-1/2}(-C_1 \sin t + C_2 \cos t))' \\
 &\quad + t(-\frac{1}{2}t^{-3/2}(C_1 \cos t + C_2 \sin t) + t^{-1/2}(-C_1 \sin t + C_2 \cos t)) \\
 &\quad + C_1 t^{3/2} \cos t + C_2 t^{3/2} \sin t - \frac{1}{4}C_1 t^{-1/2} \cos t - \frac{1}{4}C_2 t^{-1/2} \sin t \\
 &= t^2(\frac{3}{4}t^{-5/2}(C_1 \cos t + C_2 \sin t) - \frac{1}{2}t^{-3/2}(-C_1 \sin t + C_2 \cos t)) \\
 &\quad + t^2(-\frac{1}{2}t^{-3/2}(-C_1 \sin t + C_2 \cos t) + t^{-1/2}(-C_1 \cos t - C_2 \sin t)) \\
 &\quad - \frac{1}{2}C_1 t^{-1/2} \cos t - \frac{1}{2}C_2 t^{-1/2} \sin t - C_1 t^{1/2} \sin t + C_2 t^{1/2} \cos t \\
 &\quad + C_1 t^{3/2} \cos t + C_2 t^{3/2} \sin t - \frac{1}{4}C_1 t^{-1/2} \cos t - \frac{1}{4}C_2 t^{-1/2} \sin t \\
 &= \frac{3}{4}t^{-1/2}(C_1 \cos t + C_2 \sin t) - \frac{1}{2}t^{1/2}(-C_1 \sin t + C_2 \cos t) \\
 &\quad - \frac{1}{2}t^{1/2}(-C_1 \sin t + C_2 \cos t) + t^{3/2}(-C_1 \cos t - C_2 \sin t) \\
 &\quad - \frac{1}{2}C_1 t^{-1/2} \cos t - \frac{1}{2}C_2 t^{-1/2} \sin t - C_1 t^{1/2} \sin t + C_2 t^{1/2} \cos t \\
 &\quad + C_1 t^{3/2} \cos t + C_2 t^{3/2} \sin t - \frac{1}{4}C_1 t^{-1/2} \cos t - \frac{1}{4}C_2 t^{-1/2} \sin t \\
 &= 0.
 \end{aligned}$$

D2.1.54

a. Substituting $y = \sin t$ and $y = \cos t$ into $y'' + y = 0$ gives

$$\begin{aligned}
 y'' + y &= (\sin t)'' + \sin t = (\cos t)' + \sin t = -\sin t + \sin t = 0 \\
 y'' + y &= (\cos t)'' + \cos t = (-\sin t)' + \cos t = -\cos t + \cos t = 0.
 \end{aligned}$$

b. Because we have found two linearly independent solutions, the general solution, by Theorem D2.2, is $y = C_1 \sin t + C_2 \cos t$.

c. Substituting $y = \sin 2t$ and $y = \cos 2t$ into $y'' + 4y = 0$ gives

$$\begin{aligned}
 y'' + 4y &= (\sin 2t)'' + 4 \sin 2t = (2 \cos 2t)' + 4 \sin 2t = -4 \sin 2t + 4 \sin 2t = 0 \\
 y'' + 4y &= (\cos 2t)'' + 4 \cos 2t = (-2 \sin 2t)' + 4 \cos 2t = -4 \cos 2t + 4 \cos 2t = 0.
 \end{aligned}$$

d. Because we have found two linearly independent solutions, the general solution, by Theorem D2.2, is $y = C_1 \sin 2t + C_2 \cos 2t$.

e. Both $y = \sin kt$ and $y = \cos kt$ are solutions, since substitution gives

$$\begin{aligned}
 y'' + ky &= (\sin kt)'' + k^2 \sin kt = (k \cos kt)' + k^2 \sin kt = -k^2 \sin kt + k^2 \sin kt = 0 \\
 y'' + ky &= (\cos kt)'' + k^2 \cos kt = (-k \sin kt)' + k^2 \cos kt = -k^2 \cos kt + k^2 \cos kt = 0.
 \end{aligned}$$

Because those solutions are linearly independent, the general solution, by Theorem D2.2, is $y = C_1 \sin kt + C_2 \cos kt$.

D2.1.55

a. Substitution gives

$$\begin{aligned}y'' - y &= (e^t)'' - e^t = (e^t)' - e^t = e^t - e^t = 0 \\y'' - y &= (e^{-t})'' - e^{-t} = (-e^{-t})' - e^{-t} = e^{-t} - e^{-t} = 0.\end{aligned}$$

b. $\sinh t$ and $\cosh t$ are each linear combinations of the solutions e^t and e^{-t} , so they are both solutions. They are linearly independent since if $a \sinh t + b \cosh t = 0$, then

$$a\left(\frac{e^t - e^{-t}}{2}\right) + b\left(\frac{e^t + e^{-t}}{2}\right) = \frac{a+b}{2}e^t + \frac{b-a}{2}e^{-t} = 0.$$

Because e^t and e^{-t} are linearly independent, we must have $a+b = b-a = 0$, so that $a = b = 0$. This proves that $\sinh t$ and $\cosh t$ are linearly independent as well.

c. Because $\sinh' = \cosh$ and $\cosh' = \sinh$, substitution gives

$$\begin{aligned}y'' - y &= (\sinh t)'' - \sinh t = (\cosh t)' - \sinh t = \sinh t - \sinh t = 0 \\y'' - y &= (\cosh t)'' - \cosh t = (\sinh t)' - \cosh t = \cosh t - \cosh t = 0.\end{aligned}$$

d. From part (a), the general solution is $C_1 e^t + C_2 e^{-t}$. From part (c), the general solution is $C_1 \sinh t + C_2 \cosh t$.

e. Substitution gives

$$\begin{aligned}y'' - k^2 y &= (e^{kt})'' - k^2 e^{kt} = (k e^{kt})' - k^2 e^{kt} = k^2 e^{kt} - k^2 e^{kt} = 0 \\y'' - k^2 y &= (e^{-kt})'' - k^2 e^{-kt} = (-k e^{-kt})' - k^2 e^{-kt} = k^2 e^{-kt} - k^2 e^{-kt} = 0.\end{aligned}$$

e. In terms of exponentials, from part (e), the general solution is $C_1 e^{kt} + C_2 e^{-kt}$. Because $\cosh kt = \frac{e^{kt} + e^{-kt}}{2}$ and $\sinh kt = \frac{e^{kt} - e^{-kt}}{2}$, an identical argument to that in part (b) shows that $\cosh(kt)$ and $\sinh(kt)$ are also solutions to $y'' - k^2 y$ and that they are linearly independent. So in terms of hyperbolic functions, the general solution is $C_1 \sinh kt + C_2 \cosh kt$.

D2.1.56 With $y = C_1 e^{-2t} + C_2 e^{-t} + C_3 e^t$, we have

$$\begin{aligned}y' &= -2C_1 e^{-2t} - C_2 e^{-t} + C_3 e^t \\y'' &= 4C_1 e^{-2t} + C_2 e^{-t} + C_3 e^t \\y''' &= -8C_1 e^{-2t} - C_2 e^{-t} + C_3 e^t\end{aligned}$$

so that

$$\begin{aligned}y'''(t) + 2y''(t) - y'(t) - 2y(t) &= -8C_1 e^{-2t} - C_2 e^{-t} + C_3 e^t + 2(4C_1 e^{-2t} + C_2 e^{-t} + C_3 e^t) \\&\quad - (-2C_1 e^{-2t} - C_2 e^{-t} + C_3 e^t) - 2(C_1 e^{-2t} + C_2 e^{-t} + C_3 e^t) \\&= 0.\end{aligned}$$

D2.1.57 Note that $(e^{kt})^{(\text{iv})} = k^4 e^{kt}$, $(\sin kt)^{(\text{iv})} = k^4 \sin kt$, and $(\cos kt)^{(\text{iv})} = k^4 \cos kt$. Thus with $y = C_1 e^{-2t} + C_2 e^{2t} + C_3 \sin 2t + C_4 \cos 2t$, we have

$$y^{(\text{iv})} = 16C_1 e^{-2t} + 16C_2 e^{2t} + 16C_3 \sin 2t + 16C_4 \cos 2t = 16y(t).$$

D2.1.58

a. $\frac{d}{dt}(y(t)^2) = 2y(t) \frac{d}{dt}(y(t)) = 2y(t)y'(t)$.

- b. Because $2yy' = \frac{d}{dt}(y^2)$, we can substitute in $y''(t) - 2y(t)y'(t) = 0$ to get $y''(t) - (y(t)^2)' = 0$.
- c. Integrating with respect to t gives $\int (y''(t) - (y(t)^2)') dt = \int 0 dt$, or $y'(t) - y(t)^2 = C$. This is a separable equation; rearranging gives $\frac{y'(t)}{y(t)^2 + C} = 1$, or $\frac{dy}{y^2 + C} = dt$.
- d. There are now three cases.
- If $C > 0$ then integrating gives $\frac{1}{\sqrt{C}} \tan^{-1}(\frac{y}{\sqrt{C}}) = t + C'$, so that $\frac{y}{\sqrt{C}} = \tan(\sqrt{C}t + C'\sqrt{C})$ and $y = \sqrt{C} \tan(\sqrt{C}t + C'\sqrt{C})$. Renaming constants gives $y = C_1 \tan(C_2 + C_1 t)$.
 - If $C = 0$ then integrating $\frac{y'}{y^2} = dt$ gives $-y^{-1} = t + C_1$ so that $y = -\frac{1}{t+C_1}$.
 - If $C < 0$ then integrating gives

$$\frac{1}{2\sqrt{-C}} \ln \left| \frac{y - \sqrt{-C}}{y + \sqrt{-C}} \right| = t + C',$$

so that, writing $D = \sqrt{-C}$,

$$\frac{y - D}{y + D} = \pm e^{2D(t+C')}, \quad \text{and solving for } y \text{ gives } y = D \frac{1 \pm e^{2D(t+C')}}{1 \mp e^{2D(t+C')}}.$$

Finally, rename constants to get

$$y = D \frac{1 \pm e^{2Dt+2DC'}}{1 \mp e^{2Dt+2DC'}} = C_1 \frac{1 - C_2 e^{2C_1 t}}{1 + C_2 e^{2C_1 t}}.$$

(Note that in the final equation, the sign of C_2 may be chosen appropriately so that we can force the $-$ sign in the numerator and the $+$ sign in the denominator).

D2.1.59

- $\frac{d}{dt}(y'(t)^2) = 2y'(t) \frac{d}{dt}(y'(t)) = 2y'(t)y''(t)$.
- From part (a), $y''(t)y'(t) = \frac{1}{2} \cdot \frac{d}{dt}(y'(t)^2) = 1$, so that $(y'(t)^2)' = 2$.
- Integrating both sides with respect to t gives $\int (y'(t)^2)' dt = \int 2 dt$, or $y'(t)^2 = 2t + C_1$ where C_1 is an arbitrary constant. Thus $y'(t) = \pm\sqrt{2t + C_1}$.
- Solving this equation simply involves integrating the right-hand side:

$$y(t) = \int \pm\sqrt{2t + C_1} dt = \int \pm(2t + C_1)^{1/2} dt = \pm\frac{1}{3}(2t + C_1)^{3/2} + C_2.$$

Thus there are two families of solutions.

D2.1.60

- With $v = y'$, we have $v' = 2v$, or $v' - 2v = 0$. The integrating factor is $e^{\int -2 dt} = e^{-2t}$; this gives $e^{-2t}v' - 2e^{-2t}v = 0$, or $(e^{-2t}v)' = 0$. Thus $e^{-2t}v = C_1$, so that $v = y' = C_1 e^{2t}$.
- Integrating once again gives $y = \frac{C_1}{2} e^{2t} + C_3 = C_2 e^{2t} + C_3$. As a check, note that

$$y'' = (C_2 e^{2t} + C_3)'' = (2C_2 e^{2t})' = 4C_2 e^{2t} = 2(2C_2 e^{2t}) = 2y'.$$

D2.1.61

- With $v = y'$, we have $v' = 3v + 4$, or $v' - 3v = 4$. The integrating factor is $e^{\int -3 dt} = e^{-3t}$; this gives $e^{-3t}v' - 3e^{-3t}v = (e^{-3t}v)' = 4e^{-3t}$. Integrate both sides to get $e^{-3t}v = -\frac{4}{3}e^{-3t} + C_1$, so that $v = -\frac{4}{3} + C_1 e^{3t}$. This is the same as $y' = -\frac{4}{3} + C_1 e^{3t}$.

- b. Integrating once again gives $y = -\frac{4}{3}t + \frac{C_1}{3}e^{3t} + C_2 = C_2 - \frac{4}{3}t + C_3e^{3t}$. As a check, note that

$$\begin{aligned}y'' &= (C_2 - \frac{4}{3}t + C_3e^{3t})'' = 9C_3e^{3t} \\ 3y' + 4 &= 3(C_2 - \frac{4}{3}t + C_3e^{3t})' + 4 = -4 + 9C_3e^{3t} + 4 = 9C_3e^{3t}.\end{aligned}$$

D2.1.62

- a. With $v = y'$, we have $v' = e^{-v}$. This is separable: $e^v v' = 1$; integrating both sides gives $e^v = t + C$, so that $v = \ln(t + C_1)$. Substituting back gives $y' = \ln(t + C_1)$.
- b. Integrating once again gives $y = (t + C_1) \ln(t + C_1) - t + C_2$. As a check, note that

$$\begin{aligned}y'' &= ((t + C_1) \ln(t + C_1))'' + (-t + C_2)'' = (\ln(t + C_1) + 1)' = \frac{1}{t + C_1} \\ e^{-y'} &= e^{-((t+C_1) \ln(t+C_1) - t + C_2)'} = e^{-1 - \ln(t+C_1) + 1} = e^{-\ln(t+C_1)} = \frac{1}{t + C_1}.\end{aligned}$$

D2.1.63

- a. With $v = y'$ we get $v' = 2tv^2$, so that $v^{-2}v' = 2t$. Integrating both sides gives $-v^{-1} = t^2 + C_1$, so that $v = -\frac{1}{t^2 + C_1}$. Substituting back gives $y' = -\frac{1}{t^2 + C_1}$.
- b. Integrating once again gives if $C_1 > 0$

$$y = -\frac{1}{\sqrt{C_1}} \arctan\left(\frac{t}{\sqrt{C_1}}\right) + C_2.$$

and if $C_1 < 0$

$$y = \frac{1}{2\sqrt{|C_1|}} \ln \left| \frac{t + \sqrt{|C_1|}}{t - \sqrt{|C_1|}} \right| + C_2.$$

As a check, note that for $C_1 > 0$,

$$\begin{aligned}y' &= -\frac{1}{\sqrt{C_1}} \cdot \frac{\sqrt{C_1}}{t^2 + C_1} = -(t^2 + C_1)^{-1} \\ y'' &= 2t(t^2 + C_1)^{-2}\end{aligned}$$

so that indeed $y'' = 2t(y')^2$, and for $C_1 < 0$, regardless of the sign of $\frac{t + \sqrt{|C_1|}}{t - \sqrt{|C_1|}}$,

$$\begin{aligned}y' &= \frac{-1}{t^2 - |C_1|} = \frac{-1}{t^2 + C_1} \\ y'' &= \frac{2t}{(t^2 + C_1)^2}\end{aligned}$$

and again $y'' = 2t(y')^2$.

D2.1.64

- a. Because

$$\begin{aligned}(\sin 4t)'' + 16 \sin 4t &= (4 \cos 4t)' + 16 \sin 4t = -16 \sin 4t + 16 \sin 4t = 0 \\ (\cos 4t)'' + 16 \cos 4t &= (-4 \sin 4t)' + 16 \cos 4t = -16 \cos 4t + 16 \cos 4t = 0\end{aligned}$$

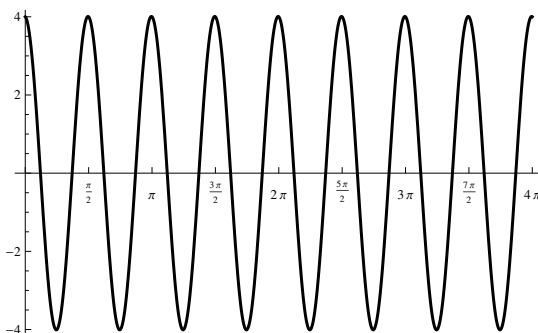
we see that $\sin 4t$ and $\cos 4t$ are linearly independent solutions, so that $y(t) = C_1 \sin 4t + C_2 \cos 4t$ is the general solution.

b. Substituting the initial conditions into $y(t)$ gives the system of simultaneous equations

$$\begin{aligned} C_1 \cdot \sin(4 \cdot 0) + C_2 \cdot \cos(4 \cdot 0) &= y(0) = 4 \\ 4C_1 \cdot \cos(4 \cdot 0) - 4C_2 \cdot \sin(4 \cdot 0) &= y'(0) = -1 \end{aligned} \quad \text{so that} \quad \begin{aligned} C_2 &= 4 \\ 4C_1 &= -1. \end{aligned}$$

Thus $C_1 = -\frac{1}{4}$ and $C_2 = 4$, and the solution to the initial value problem is $y(t) = -\frac{1}{4} \sin 4t + 4 \cos 4t$.

c. A graph of the solution for $0 \leq t \leq 4\pi$ is



D2.1.65

a. Computing derivatives gives

$$\begin{aligned} (e^{-3t/2} \sin 2t)' &= -\frac{3}{2}e^{-3t/2} \sin 2t + 2e^{-3t/2} \cos 2t = e^{-3t/2} \left(2 \cos 2t - \frac{3}{2} \sin 2t \right) \\ (e^{-3t/2} \sin 2t)'' &= \left(-\frac{3}{2}e^{-3t/2} \sin 2t + 2e^{-3t/2} \cos 2t \right)' \\ &= \frac{9}{4}e^{-3t/2} \sin 2t - 3e^{-3t/2} \cos 2t - 3e^{-3t/2} \cos 2t - 4e^{-3t/2} \sin 2t \\ &= e^{-3t/2} \left(-6 \cos 2t - \frac{7}{4} \sin 2t \right) \\ (e^{-3t/2} \cos 2t)' &= -\frac{3}{2}e^{-3t/2} \cos 2t - 2e^{-3t/2} \sin 2t = e^{-3t/2} \left(-\frac{3}{2} \cos 2t - 2 \sin 2t \right) \\ (e^{-3t/2} \cos 2t)'' &= \left(-\frac{3}{2}e^{-3t/2} \cos 2t - 2e^{-3t/2} \sin 2t \right)' \\ &= \frac{9}{4}e^{-3t/2} \cos 2t + 3e^{-3t/2} \sin 2t + 3e^{-3t/2} \sin 2t - 4e^{-3t/2} \cos 2t \\ &= e^{-3t/2} \left(-\frac{7}{4} \cos 2t + 6 \sin 2t \right) \end{aligned}$$

Substituting $e^{-3t/2} \sin 2t$ and $e^{-3t/2} \cos 2t$ gives

$$\begin{aligned} y'' + 3y' + \frac{25}{4}y &= (e^{-3t/2} \sin 2t)'' + 3(e^{-3t/2} \sin 2t)' + \frac{25}{4}(e^{-3t/2} \sin 2t) \\ &= e^{-3t/2} \left(-6 \cos 2t - \frac{7}{4} \sin 2t \right) + 3e^{-3t/2} \left(2 \cos 2t - \frac{3}{2} \sin 2t \right) \\ &\quad + \frac{25}{4}(e^{-3t/2} \sin 2t) \\ &= e^{-3t/2} \left(-6 \cos 2t - \frac{7}{4} \sin 2t + 6 \cos 2t - \frac{9}{2} \sin 2t + \frac{25}{4} \sin 2t \right) \\ &= 0 \end{aligned}$$

$$\begin{aligned}
y'' + 3y' + \frac{25}{4}y &= (e^{-3t/2} \cos 2t)'' + 3(e^{-3t/2} \cos 2t)' + \frac{25}{4}(e^{-3t/2} \cos 2t) \\
&= e^{-3t/2} \left(-\frac{7}{4} \cos 2t + 6 \sin 2t \right) + 3e^{-3t/2} \left(-\frac{3}{2} \cos 2t - 2 \sin 2t \right) \\
&\quad + \frac{25}{4}(e^{-3t/2} \cos 2t) \\
&= e^{-3t/2} \left(-\frac{7}{4} \cos 2t + 6 \sin 2t - \frac{9}{2} \cos 2t - 6 \sin 2t + \frac{25}{4} \cos 2t \right) \\
&= 0
\end{aligned}$$

so that $e^{-3t/2} \sin 2t$ and $e^{-3t/2} \cos 2t$ are linearly independent solutions, so that the general solution is $y(t) = e^{-3t/2}(C_1 \sin 2t + C_2 \cos 2t)$.

b. Because

$$\begin{aligned}
y'(t) &= -\frac{3}{2}e^{-3t/2}(C_1 \sin 2t + C_2 \cos 2t) + e^{-3t/2}(2C_1 \cos 2t - 2C_2 \sin 2t) \\
&= e^{-3t/2} \left((2 \cos 2t - \frac{3}{2} \sin 2t)C_1 + (-2 \sin 2t - \frac{3}{2} \cos 2t)C_2 \right),
\end{aligned}$$

substituting the initial conditions into $y(t)$ gives the system of simultaneous equations

$$\begin{aligned}
e^{-3 \cdot 0/2}(\sin(2 \cdot 0)C_1 + \cos(2 \cdot 0)C_2) &= y(0) = 4 \\
e^{-3 \cdot 0/2}((2 \cos 0 - \frac{3}{2} \sin 0)C_1 + (-2 \sin 0 - \frac{3}{2} \cos 0)C_2) &= y'(0) = 0
\end{aligned}$$

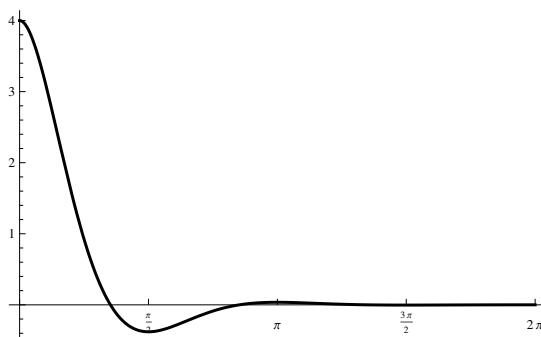
so that

$$\begin{aligned}
C_2 &= 4 \\
2C_1 - \frac{3}{2}C_2 &= 0.
\end{aligned}$$

Thus $C_1 = 3$ and $C_2 = 4$, and the solution to the initial value problem is

$$y(t) = e^{-3t/2}(3 \sin 2t + 4 \cos 2t).$$

c. A plot of the solution for $0 \leq t \leq 2\pi$ is



D2.1.66

a. Because $(\sin 3t)' = 3 \cos 3t$, $(\sin 3t)'' = -9 \sin 3t$, $(\cos 3t)' = -3 \sin 3t$, and $(\cos 3t)'' = -9 \cos 3t$, have, substituting $y = \sin 3t$ and $y = \cos 3t$,

$$\begin{aligned}
y'' + 9y &= -9 \sin 3t + 9 \sin 3t = 0 \\
y'' + 9y &= -9 \cos 3t + 9 \cos 3t = 0,
\end{aligned}$$

so that $\sin 3t$ and $\cos 3t$ are two linearly independent solutions to the homogeneous problem. Also, substituting $y = \sin t$, we get

$$y'' + 9y = -\sin t + 9\sin t = 8\sin t,$$

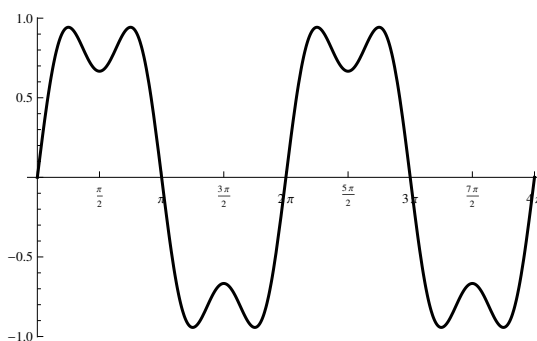
so that $\sin t$ is a solution to the nonhomogeneous problem. Thus the general solution to the nonhomogeneous problem is $y(t) = C_1 \sin 3t + C_2 \cos 3t + \sin t$.

b. Substituting the initial conditions into $y(t)$ gives the system of simultaneous equations

$$\begin{aligned} C_1 \sin(3 \cdot 0) + C_2 \cos(3 \cdot 0) + \sin 0 &= y(0) = 0 \\ 3C_1 \cos(3 \cdot 0) - 3C_2 \sin(3 \cdot 0) + \cos 0 &= y'(0) = 2 \end{aligned} \quad \text{so that} \quad \begin{aligned} C_2 &= 0 \\ 3C_1 &= 1. \end{aligned}$$

Thus $C_1 = \frac{1}{3}$ and $C_2 = 0$, and the solution to the initial value problem is $y(t) = \frac{1}{3} \sin 3t + \sin t$.

c. A plot of the solution for $0 \leq t \leq 4\pi$ is



D2.1.67

a. Computing derivatives gives

$$\begin{aligned} (e^{-3t} \sin 4t)' &= -3e^{-3t} \sin 4t + 4e^{-3t} \cos 4t = e^{-3t}(-3 \sin 4t + 4 \cos 4t) \\ (e^{-3t} \sin 4t)'' &= (e^{-3t}(-3 \sin 4t + 4 \cos 4t))' \\ &= -3e^{-3t}(-3 \sin 4t + 4 \cos 4t) + e^{-3t}(-12 \cos 4t - 16 \sin 4t) \\ &= e^{-3t}(-7 \sin 4t - 24 \cos 4t) \\ (e^{-3t} \cos 4t)' &= -3e^{-3t} \cos 4t - 4e^{-3t} \sin 4t = e^{-3t}(-3 \cos 4t - 4 \sin 4t) \\ (e^{-3t} \cos 4t)'' &= (e^{-3t}(-3 \cos 4t - 4 \sin 4t))' \\ &= -3e^{-3t}(-3 \cos 4t - 4 \sin 4t) + e^{-3t}(12 \sin 4t - 16 \cos 4t) \\ &= e^{-3t}(24 \sin 4t - 7 \cos 4t) \end{aligned}$$

Then, substituting $y = e^{-3t} \sin 4t$ and $y = e^{-3t} \cos 4t$ gives

$$\begin{aligned} y'' + 6y' + 25y &= (e^{-3t} \sin 4t)'' + 6(e^{-3t} \sin 4t)' + 25(e^{-3t} \sin 4t) \\ &= e^{-3t}(-7 \sin 4t - 24 \cos 4t) + 6e^{-3t}(-3 \sin 4t + 4 \cos 4t) + 25e^{-3t} \sin 4t \\ &= 0 \\ y'' + 6y' + 25y &= (e^{-3t} \cos 4t)'' + 6(e^{-3t} \cos 4t)' + 25(e^{-3t} \cos 4t) \\ &= e^{-3t}(24 \sin 4t - 7 \cos 4t) + 6e^{-3t}(-3 \cos 4t - 4 \sin 4t) + 25e^{-3t} \cos 4t \\ &= 0 \end{aligned}$$

so that $e^{-3t} \sin 4t$ and $e^{-3t} \cos 4t$ are two linearly independent solutions to the homogeneous problem. Also, substituting $y = e^{-t}$, we get

$$y'' + 6y' + 25y = e^{-t} - 6e^{-t} + 25e^{-t} = 20e^{-t},$$

so that e^{-t} is a solution to the nonhomogeneous problem. Thus the general solution to the nonhomogeneous problem is $y(t) = e^{-3t}(C_1 \sin 4t + C_2 \cos 4t) + e^{-t}$.

b. Because

$$\begin{aligned} y'(t) &= -3e^{-3t}(C_1 \sin 4t + C_2 \cos 4t) + e^{-3t}(4C_1 \cos 4t - 4C_2 \sin 4t) - e^{-t} \\ &= e^{-3t}((-3 \sin 4t + 4 \cos 4t)C_1 + (-4 \sin 4t - 3 \cos 4t)C_2) - e^{-t}, \end{aligned}$$

substituting the initial conditions into $y(t)$ gives the system of simultaneous equations

$$\begin{aligned} e^{-3 \cdot 0}(\sin(0)C_1 + \cos(0)C_2) + e^{-0} &= y(0) = 2 \\ e^{-3 \cdot 0}((-3 \sin 0 + 4 \cos 0)C_1 + (-4 \sin 0 - 3 \cos 0)C_2) - e^{-0} &= y'(0) = 0 \end{aligned}$$

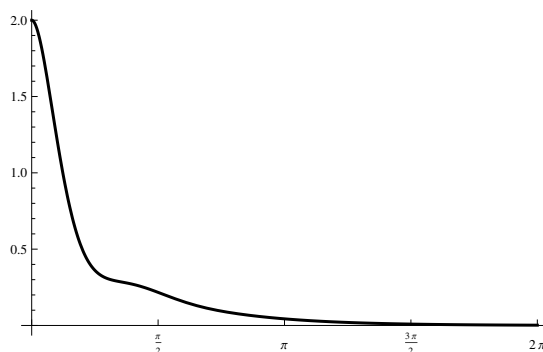
so that

$$\begin{aligned} C_2 &= 1 \\ 4C_1 - 3C_2 &= 1. \end{aligned}$$

Thus $C_1 = C_2 = 1$, and the solution to the initial value problem is

$$y(t) = e^{-3t}(\sin 4t + \cos 4t) + e^{-t}.$$

c. A plot of the solution for $0 \leq t \leq 2\pi$ is



D2.1.68

a. Let $u = y'$; then the differential equation becomes

$$u'(x) = \frac{\sqrt{1 + u(x)^2}}{sx}.$$

The initial condition $y(1) = 0$ is irrelevant to this problem, while $y'(1) = 0$ becomes the initial condition $u(1) = 0$.

b. This equation is separable: divide both sides by $\sqrt{1 + u^2}$ to obtain

$$\frac{du}{\sqrt{1 + u^2}} = \frac{dx}{sx}.$$

Integrating both sides gives

$$\ln |u + \sqrt{1 + u^2}| = \frac{1}{s} \ln(sx) + C,$$

so that exponentiating both sides gives

$$u + \sqrt{1 + u^2} = \pm e^C \cdot e^{(1/s) \ln(sx)} = \pm e^C \cdot (sx)^{1/s} = \pm e^C s^{1/s} x^{1/s} = C_1 x^{1/s}.$$

Because $u(1) = y'(1) = 0$, we get

$$0 + \sqrt{1 + 0^2} = C_1 \cdot 1^{1/s}, \quad \text{so that} \quad C_1 = 1.$$

Thus $u + \sqrt{1 + u^2} = x^{1/s}$. To simplify, subtract u from both sides and square to get

$$1 + u^2 = x^{2/s} - 2ux^{1/s} + u^2, \quad \text{so that} \quad u = \frac{1}{2}x^{-1/s}(x^{2/s} - 1) = \frac{1}{2}(x^{1/s} - x^{-1/s}).$$

- c. Because $u = y'$, we have the equation $y' = \frac{1}{2}(x^{1/s} - x^{-1/s})$; solve this by integrating both sides with respect to x . (Recall that $s > 1$, so that we need not worry about integrating $x^{\pm 1/s}$ and getting logarithmic functions):

$$y = \frac{1}{2} \left(\frac{s}{s+1} x^{(s+1)/s} - \frac{s}{s-1} x^{(s-1)/s} \right) + C_2. \quad (17.1)$$

Because $y(1) = 0$, we get

$$0 = \frac{1}{2} \left(\frac{s}{s+1} - \frac{s}{s-1} \right) + C_2 = -\frac{s}{s^2-1} + C_2,$$

so that $C_2 = \frac{s}{s^2-1}$.

- d. Replacing C_2 in (17.1) by its value, and factoring out sx , we get

$$y = \frac{sx}{2} \left(\frac{x^{1/s}}{s+1} - \frac{x^{-1/s}}{s-1} \right) + \frac{s}{s^2-1}.$$

D2.1.69

- a. Multiplying $mx''(t) = F(x)$ by $x'(t)$ gives

$$mx''(t)x'(t) = F(x)x'(t). \quad (17.2)$$

Now, by the Chain Rule (or, see Exercise 59(a)), $2x''(t)x'(t) = \frac{d}{dt}(x'(t)^2)$, so that $x''(t)x'(t) = \frac{1}{2} \frac{d}{dt}(x'(t)^2)$. Further, with $\varphi(x) = -F(x)$, differentiating with respect to time gives, again by the Chain Rule, $\frac{d}{dt}(\varphi(x)) = \varphi'(x)x'(t) = -F(x)x'(t)$. Making these substitutions in (17.2) gives

$$\frac{d}{dt} \left(m \cdot \frac{1}{2} (x'(t))^2 \right) = -\frac{d}{dt}(\varphi(x)), \quad \text{so} \quad \frac{d}{dt} \left[\frac{1}{2} m (x'(t))^2 + \varphi(x) \right] = 0.$$

- b. With $E = \frac{1}{2}mv^2 + \varphi$, since the time derivative of E is zero, it follows that E is conserved in time.

D2.1.70

- a. With $y = t$, we have $y' = 1$ and $y'' = 0$. Substituting gives

$$0 - \frac{1}{t} \cdot 1 + \frac{1}{t^2}t = 0,$$

so that $y_1 = t$ is a solution.

- b. Let $y_2 = v(t)y_1(t) = tv(t)$ be any other solution. Now, $y_2' = v(t) + tv'(t)$ and $y_2'' = 2v'(t) + tv''(t)$, substituting y_2 in the equation gives

$$\begin{aligned} y_2'' - \frac{1}{t}y_2' + \frac{1}{t^2}y_2 &= 2v'(t) + tv''(t) - \frac{1}{t}(v(t) + tv'(t)) + \frac{1}{t^2}(tv(t)) \\ &= 2v'(t) + tv''(t) - \frac{1}{t}v(t) - v'(t) + \frac{1}{t}v(t) \\ &= tv''(t) + v'(t). \end{aligned}$$

Because y_2 is a solution, we have $tv''(t) + v'(t) = 0$, or $v'' = -\frac{v'}{t}$.

- c. Letting $w = v'$ gives the differential equation $w' = -\frac{w}{t}$.
- d. This is a separable equation; rearranging gives $\frac{w'}{w} = -\frac{1}{t}$, and integrating both sides gives $\ln|w| = C - \ln t = C + \ln \frac{1}{t}$. Thus $w = \pm e^{C + \ln(1/t)} = \frac{C_1}{t}$. Note that since $t > 0$, we do not need absolute value signs around t .
- e. Substituting back gives $v' = \frac{C_1}{t}$. Integrating both sides gives $v = C_1 \ln t + C_2$. Note that since $t > 0$, we do not need absolute value signs here.
- f. Because $y_2(t) = tv(t)$, we get for a general solution $C_2t + C_1t \ln t$.

Linear Homogeneous Equations

D2.2.1 We assume that all derivatives of the function are multiples of the function itself, so we start with e^{rt} .

D2.2.2 The characteristic polynomial is $r^2 - 3r + 10 = 0$; see the discussion in the first few paragraphs of this section.

D2.2.3 The characteristic polynomial is a quadratic equation, so it can have two distinct real roots, one repeated real root, or two conjugate complex roots.

D2.2.4 The general solution when the characteristic polynomial has distinct real roots is $y(t) = c_1e^{r_1t} + c_2e^{r_2t}$, where $r_1 \neq r_2$ are the real roots.

D2.2.5 When the characteristic polynomial has a repeated real root r , the general solution is $y(t) = c_1e^{rt} + c_2te^{rt}$.

D2.2.6 If the characteristic polynomial $r^2 + pr + q = 0$ has two conjugate complex roots, the general solution is $y(t) = c_1e^{at} \cos bt + c_2e^{at} \sin bt$, where $a = -\frac{p}{2}$ and $b = 4q - p^2$.

D2.2.7 Because the roots are $-2 \pm 3i$, we have $a = -2$ and $b = 3$ in Case 3, so that the general solution is $y(t) = c_1e^{-2t} \cos 3t + c_2e^{-2t} \sin 3t$.

D2.2.8 The trial solution for a second-order Cauchy-Euler equation is $y(t) = t^p$.

D2.2.9 The characteristic polynomial is $r^2 - 25 = (r + 5)(r - 5) = 0$, with roots ± 5 . Because the roots are real and distinct, the general solution is $y(t) = c_1e^{5t} + c_2e^{-5t}$.

D2.2.10 The characteristic polynomial is $r^2 - 2r - 15 = (r - 5)(r + 3) = 0$, with roots 5 and -3 . Because the roots are real and distinct, the general solution is $y(t) = c_1e^{5t} + c_2e^{-3t}$.

D2.2.11 The characteristic polynomial is $r^2 - 3r = r(r - 3) = 0$, with roots 0 and 3. Because the roots are real and distinct, the general solution is $y(t) = c_1e^{0t} + c_2e^{3t} = c_1 + c_2e^{3t}$.

D2.2.12 The characteristic polynomial is $r^2 - r - \frac{3}{4} = (r - \frac{3}{2})(r + \frac{1}{2}) = 0$, with roots $\frac{3}{2}$ and $-\frac{1}{2}$. Because the roots are real and distinct, the general solution is $y(t) = c_1e^{3t/2} + c_2e^{-t/2}$.

D2.2.13 The characteristic polynomial is $2r^2 + 6r - 20 = (2r - 4)(r + 5)$, with roots 2 and -5 . Because the roots are real and distinct, the general solution is $y(t) = c_1e^{2t} + c_2e^{-5t}$.

D2.2.14 The characteristic polynomial is $r^2 - \frac{5}{2}r + 1 = \frac{1}{2}(x - 2)(2x - 1)$, with roots 2 and $\frac{1}{2}$. Because the roots are real and distinct, the general solution is $y(t) = c_1e^{2t} + c_2e^{t/2}$.

D2.2.15 The characteristic polynomial $r^2 - 36 = (r + 6)(r - 6)$ has distinct real roots ± 6 , so the general solution is $y(t) = c_1 e^{6t} + c_2 e^{-6t}$. Then $y'(t) = 6c_1 e^{6t} - 6c_2 e^{-6t}$. Substituting the initial conditions gives

$$\begin{array}{rcl} c_1 e^{6 \cdot 0} + c_2 e^{-6 \cdot 0} = y(0) = 3 & & c_1 + c_2 = 3 \\ 6c_1 e^{6 \cdot 0} - 6c_2 e^{-6 \cdot 0} = y'(0) = 0 & \text{so that} & 6c_1 - 6c_2 = 0. \end{array}$$

Thus $c_1 = c_2 = \frac{3}{2}$, and the solution is $y(t) = \frac{3}{2}(e^{6t} + e^{-6t})$.

D2.2.16 The characteristic polynomial $r^2 - 6r = r(r - 6)$ has distinct real roots 0 and 6, so the general solution is $y(t) = c_1 e^{0t} + c_2 e^{6t} = c_1 + c_2 e^{6t}$. Then $y'(t) = 6c_2 e^{6t}$. Substituting the initial conditions gives

$$\begin{array}{rcl} c_1 + c_2 e^{6 \cdot 0} = y(0) = -1 & & c_1 + c_2 = -1 \\ 6c_2 e^{6 \cdot 0} = y'(0) = 2 & \text{so that} & 6c_2 = 2. \end{array}$$

Thus $c_1 = -\frac{4}{3}$ and $c_2 = \frac{1}{3}$, so that the solution is $y(t) = -\frac{4}{3} + \frac{1}{3}e^{6t}$.

D2.2.17 The characteristic polynomial $r^2 - 3r - 18 = (r - 6)(r + 3)$ has distinct real roots -3 and 6 , so the general solution is $y(t) = c_1 e^{6t} + c_2 e^{-3t}$. Then $y'(t) = 6c_1 e^{6t} - 3c_2 e^{-3t}$. Substituting the initial conditions gives

$$\begin{array}{rcl} c_1 e^{6 \cdot 0} + c_2 e^{-3 \cdot 0} = y(0) = 0 & & c_1 + c_2 = 0 \\ 6c_1 e^{6 \cdot 0} - 3c_2 e^{-3 \cdot 0} = y'(0) = 4 & \text{so that} & 6c_1 - 3c_2 = 4. \end{array}$$

Thus $c_1 = \frac{4}{9}$ and $c_2 = -\frac{4}{9}$. The general solution is $y(t) = \frac{4}{9}(e^{6t} - e^{-3t})$.

D2.2.18 The characteristic polynomial $r^2 + 8r + 15 = (r + 3)(r + 5)$ has distinct real roots -3 and -5 , so the general solution is $y(t) = c_1 e^{-3t} + c_2 e^{-5t}$. Then $y'(t) = -3c_1 e^{-3t} - 5c_2 e^{-5t}$. Substituting the initial conditions gives

$$\begin{array}{rcl} c_1 e^{-3 \cdot 0} + c_2 e^{-5 \cdot 0} = y(0) = 2 & & c_1 + c_2 = 2 \\ -3c_1 e^{-3 \cdot 0} - 5c_2 e^{-5 \cdot 0} = y'(0) = 4 & \text{so that} & -3c_1 - 5c_2 = 4. \end{array}$$

Thus $c_1 = 7$ and $c_2 = -5$. The solution is $y(t) = 7e^{-3t} - 5e^{-5t}$.

D2.2.19 The characteristic polynomial $r^2 - 2r - \frac{5}{4} = \frac{1}{4}(2r - 5)(2r + 1)$ has distinct real roots $-\frac{1}{2}$ and $\frac{5}{2}$, so the general solution is $y(t) = c_1 e^{5t/2} + c_2 e^{-t/2}$. Then $y'(t) = \frac{5}{2}c_1 e^{5t/2} - \frac{1}{2}c_2 e^{-t/2}$. Substituting the initial conditions gives

$$\begin{array}{rcl} c_1 e^{5 \cdot 0/2} + c_2 e^{-0/2} = y(0) = 3 & & c_1 + c_2 = 3 \\ \frac{5}{2}c_1 e^{5 \cdot 0/2} - \frac{1}{2}c_2 e^{-0/2} = y'(0) = 0 & \text{so that} & \frac{5}{2}c_1 - \frac{1}{2}c_2 = 0. \end{array}$$

Thus $c_1 = \frac{1}{2}$ and $c_2 = \frac{5}{2}$. The solution is $y(t) = \frac{1}{2}e^{5t/2} + \frac{5}{2}e^{-t/2}$.

D2.2.20 The characteristic polynomial $r^2 - 10r + 21 = (r - 3)(r - 7)$ has distinct real roots 3 and 7, so the general solution is $y(t) = c_1 e^{3t} + c_2 e^{7t}$. Then $y'(t) = 3c_1 e^{3t} + 7c_2 e^{7t}$. Substituting the initial conditions gives

$$\begin{array}{rcl} c_1 e^{3 \cdot 0} + c_2 e^{7 \cdot 0} = y(0) = -3 & & c_1 + c_2 = -3 \\ 3c_1 e^{3 \cdot 0} + 7c_2 e^{7 \cdot 0} = y'(0) = -1 & \text{so that} & 3c_1 + 7c_2 = -1. \end{array}$$

Thus $c_1 = -5$ and $c_2 = 2$. The solution is $y(t) = -5e^{3t} + 2e^{7t}$.

D2.2.21 The characteristic polynomial $r^2 - 2r + 1 = (r - 1)^2$ has the repeated real root 1, so the general solution is $y(t) = c_1 e^t + c_2 t e^t$. Then $y'(t) = c_1 e^t + c_2(e^t + t e^t)$. Substituting the initial conditions gives

$$\begin{array}{rcl} c_1 e^0 + c_2 \cdot 0 \cdot e^0 = y(0) = 4 & & c_1 = 4 \\ c_1 e^0 + c_2(e^0 + 0 \cdot e^0) = y'(0) = 0 & \text{so that} & c_1 + c_2 = 0. \end{array}$$

Thus $c_1 = 4$ and $c_2 = -4$, and the solution is $y(t) = 4e^t - 4te^t$.

D2.2.22 The characteristic polynomial $r^2 + 6r + 9 = (r + 3)^2$ has the repeated real root -3 , so the general solution is $y(t) = c_1 e^{-3t} + c_2 t e^{-3t}$. Then $y'(t) = -3c_1 e^{-3t} + c_2(e^{-3t} - 3t e^{-3t})$. Substituting for the initial conditions gives

$$\begin{array}{lcl} c_1 e^{-3 \cdot 0} + c_2 \cdot 0 \cdot e^{-3 \cdot 0} = y(0) = 0 & \text{so that} & c_1 = 0 \\ -3c_1 e^{-3 \cdot 0} + c_2(e^{-3 \cdot 0} - 3 \cdot 0 \cdot e^{-3 \cdot 0}) = y'(0) = -1 & & -3c_1 + c_2 = -1. \end{array}$$

Thus $c_1 = 0$ and $c_2 = -1$, so that the solution is $y(t) = -t e^{-3t}$.

D2.2.23 The characteristic polynomial $r^2 - r + \frac{1}{4} = (r - \frac{1}{2})^2$ has the repeated real root $\frac{1}{2}$, so the general solution is $y(t) = c_1 e^{t/2} + c_2 t e^{t/2}$. Then $y'(t) = \frac{1}{2}c_1 e^{t/2} + c_2(e^{t/2} + \frac{1}{2}t e^{t/2})$. Substituting for the initial conditions gives

$$\begin{array}{lcl} c_1 e^{0/2} + c_2 \cdot 0 \cdot e^{0/2} = y(0) = 1 & \text{so that} & c_1 = 1 \\ \frac{1}{2}c_1 e^{0/2} + c_2(e^{0/2} + \frac{1}{2} \cdot 0 \cdot e^{0/2}) = y'(0) = 2 & & \frac{1}{2}c_1 + c_2 = 2. \end{array}$$

Thus $c_1 = 1$ and $c_2 = \frac{3}{2}$, so that the solution is $y(t) = e^{t/2} + \frac{3}{2}t e^{t/2}$.

D2.2.24 The characteristic polynomial $r^2 - 4\sqrt{2}r + 8 = (r - 2\sqrt{2})^2$ has the repeated real root $2\sqrt{2}$, so the general solution is $y(t) = c_1 e^{2t\sqrt{2}} + c_2 t e^{2t\sqrt{2}}$. Then $y'(t) = 2\sqrt{2}c_1 e^{2t\sqrt{2}} + c_2(e^{2t\sqrt{2}} + 2\sqrt{2}t e^{2t\sqrt{2}})$. Substituting for the initial conditions gives

$$\begin{array}{lcl} c_1 e^{2\sqrt{2} \cdot 0} + c_2 \cdot 0 \cdot e^{2\sqrt{2} \cdot 0} = y(0) = 1 & \text{so that} & c_1 = 1 \\ 2\sqrt{2}c_1 e^{2\sqrt{2} \cdot 0} + c_2(e^{2\sqrt{2} \cdot 0} + 2\sqrt{2} \cdot 0 \cdot e^{2\sqrt{2} \cdot 0}) = y'(0) = 0 & & 2\sqrt{2}c_1 + c_2 = 0. \end{array}$$

Thus $c_1 = 1$ and $c_2 = -2\sqrt{2}$, so that the solution is $y(t) = e^{2t\sqrt{2}} - 2\sqrt{2}t e^{2t\sqrt{2}}$.

D2.2.25 The characteristic polynomial $r^2 + 4r + 4 = (r + 2)^2$ has the repeated real root -2 , so the general solution is $y(t) = c_1 e^{-2t} + c_2 t e^{-2t}$. Then $y'(t) = -2c_1 e^{-2t} + c_2(e^{-2t} - 2c_2 t e^{-2t})$. Substituting for the initial conditions gives

$$\begin{array}{lcl} c_1 e^{-2 \cdot 0} + c_2 \cdot 0 \cdot e^{-2 \cdot 0} = y(0) = 1 & \text{so that} & c_1 = 1 \\ -2c_1 e^{-2 \cdot 0} + c_2(e^{-2 \cdot 0} - 2 \cdot 0 \cdot e^{-2 \cdot 0}) = y'(0) = 0 & & -2c_1 + c_2 = 0. \end{array}$$

Thus $c_1 = 1$ and $c_2 = 2$, so that the solution is $y(t) = e^{-2t} + 2t e^{-2t} = e^{-2t}(1 + 2t)$.

D2.2.26 The characteristic polynomial $r^2 + 3r + \frac{9}{4} = (r + \frac{3}{2})^2$ has the repeated real root $-\frac{3}{2}$, so the general solution is $y(t) = c_1 e^{-3t/2} + c_2 t e^{-3t/2}$. Then $y'(t) = -\frac{3}{2}c_1 e^{-3t/2} + c_2(e^{-3t/2} - \frac{3}{2}t e^{-3t/2})$. Substituting for the initial conditions gives

$$\begin{array}{lcl} c_1 e^{-3 \cdot 0/2} + c_2 \cdot 0 \cdot e^{-3 \cdot 0/2} = y(0) = 0 & \text{so that} & c_1 = 0 \\ -\frac{3}{2}c_1 e^{-3 \cdot 0/2} + c_2(e^{-3 \cdot 0/2} - \frac{3}{2} \cdot 0 \cdot e^{-3 \cdot 0/2}) = y'(0) = 3 & & -\frac{3}{2}c_1 + c_2 = 3. \end{array}$$

Thus $c_1 = 0$ and $c_2 = 3$, so that the solution is $y(t) = 3t e^{-3t/2}$.

D2.2.27 The characteristic polynomial $r^2 + 9 = (r + 3i)(r - 3i)$ has the complex conjugate roots $\pm 3i$. By Table D2.1, the general solution is $y(t) = e^{0t}(c_1 \cos 3t + c_2 \sin 3t) = c_1 \cos 3t + c_2 \sin 3t$. Then $y'(t) = -3c_1 \sin 3t + 3c_2 \cos 3t$. Substituting for the initial conditions gives

$$\begin{array}{lcl} c_1 \cos(3 \cdot 0) + c_2 \sin(3 \cdot 0) = y(0) = 8 & \text{so that} & c_1 = 8 \\ -3c_1 \sin(3 \cdot 0) + 3c_2 \cos(3 \cdot 0) = y'(0) = -8 & & 3c_2 = -8. \end{array}$$

Thus $c_1 = 8$ and $c_2 = -\frac{8}{3}$, so that the solution is $y(t) = 8 \cos 3t - \frac{8}{3} \sin 3t$.

D2.2.28 The characteristic polynomial $r^2 + 6r + 25$ has the complex conjugate roots $\frac{-6 \pm \sqrt{36-100}}{2} = -3 \pm 4i$. By Table D2.1, the general solution is $y(t) = e^{-3t}(c_1 \cos 4t + c_2 \sin 4t)$, and then $y(0) = c_1$. Now,

$$\begin{aligned} y'(t) &= e^{-3t}((-3c_1 + 4c_2) \cos 4t - (4c_1 + 3c_2) \sin 4t) \\ &= e^{-3t}(-3 \cos 4t - 4 \sin 4t)c_1 + e^{-3t}(4 \cos 4t - 3 \sin 4t)c_2. \end{aligned}$$

so that $y'(0) = -3c_1 + 4c_2$. Substituting for the initial conditions gives

$$\begin{aligned} c_1 &= 4 \\ -3c_1 + 4c_2 &= 0. \end{aligned}$$

Thus $c_1 = 4$ and $c_2 = 3$, so that the solution is $y(t) = e^{-3t}(4 \cos 4t + 3 \sin 4t)$.

D2.2.29 The characteristic polynomial $r^2 - 2r + 5$ has the complex conjugate roots $\frac{2 \pm \sqrt{4-20}}{2} = 1 \pm 2i$. By Table D2.1, the general solution is $y(t) = e^t(c_1 \cos 2t + c_2 \sin 2t)$, and then $y(0) = c_1$. Now,

$$\begin{aligned} y'(t) &= e^t((c_1 + 2c_2) \cos 2t + (-2c_1 + c_2) \sin 2t) \\ &= e^t(\cos 2t - 2 \sin 2t)c_1 + e^{3t}(2 \cos 2t + \sin 2t)c_2. \end{aligned}$$

so that $y'(0) = c_1 + 2c_2$. Substituting for the initial conditions gives

$$\begin{aligned} c_1 &= 1 \\ c_1 + 2c_2 &= -1. \end{aligned}$$

Thus $c_1 = 1$ and $c_2 = -1$, so that the solution is $y(t) = e^t(\cos 2t - \sin 2t)$.

D2.2.30 The characteristic polynomial $r^2 + 4r + 5$ has the complex conjugate roots $\frac{-4 \pm \sqrt{16-20}}{2} = -2 \pm i$. By Table D2.1, the general solution is $y(t) = e^{-2t}(c_1 \cos t + c_2 \sin t)$, and then $y(0) = c_1$. Now,

$$\begin{aligned} y'(t) &= e^{-2t}((-2c_1 + c_2) \cos t - (c_1 + 2c_2) \sin t) \\ &= e^{-2t}(-2 \cos t - \sin t)c_1 + e^{-2t}(\cos t - 2 \sin t)c_2 \end{aligned}$$

so that $y'(0) = -2c_1 + c_2$. Substituting for the initial conditions gives

$$\begin{aligned} c_1 &= 2 \\ -2c_1 + c_2 &= -2. \end{aligned}$$

Thus $c_1 = 2$ and $c_2 = 2$, so that the solution is $y(t) = 2e^{-2t}(\cos t + \sin t)$.

D2.2.31 The characteristic polynomial $r^2 + 6r + 10$ has the complex conjugate roots $\frac{-6 \pm \sqrt{36-40}}{2} = -3 \pm i$. By Table D2.1, the general solution is $y(t) = e^{-3t}(c_1 \cos t + c_2 \sin t)$, and then $y(0) = c_1$. Now,

$$\begin{aligned} y'(t) &= e^{-3t}((-3c_1 + c_2) \cos t - (c_1 + 3c_2) \sin t) \\ &= e^{-3t}(-3 \cos t - \sin t)c_1 + e^{-3t}(\cos t - 3 \sin t)c_2 \end{aligned}$$

so that $y'(0) = -3c_1 + c_2$. Substituting for the initial conditions gives

$$\begin{aligned} c_1 &= 0 \\ -3c_1 + c_2 &= 6. \end{aligned}$$

Thus $c_1 = 0$ and $c_2 = 6$, so that the solution is $y(t) = 6e^{-3t} \sin t$.

D2.2.32 The characteristic polynomial $r^2 - r + \frac{1}{2}$ has the complex conjugate roots $\frac{1 \pm \sqrt{1-2}}{2} = \frac{1}{2} \pm \frac{1}{2}i$. By Table D2.1, the general solution is $y(t) = e^{t/2}(c_1 \cos \frac{1}{2}t + c_2 \sin \frac{1}{2}t)$, and then $y(0) = c_1$. Now,

$$\begin{aligned} y'(t) &= \frac{1}{2}e^{t/2}((c_1 + c_2) \cos \frac{t}{2} + (-c_1 + c_2) \sin \frac{t}{2}) \\ &= \frac{1}{2}e^{-t/2}((\cos \frac{t}{2} - \sin \frac{t}{2})c_1 + \frac{1}{2}(\cos \frac{t}{2} + \sin \frac{t}{2})c_2) \end{aligned}$$

so that $y'(0) = \frac{1}{2}c_1 + \frac{1}{2}c_2$. Substituting for the initial conditions gives

$$\begin{aligned}c_1 &= 3 \\ \frac{1}{2}c_1 + \frac{1}{2}c_2 &= -2.\end{aligned}$$

Thus $c_1 = 3$ and $c_2 = -7$, so that the solution is $y(t) = e^{t/2}(3 \cos \frac{t}{2} - 7 \sin \frac{t}{2})$.

D2.2.33 The characteristic polynomial $r^2 + (1-1)r - 1 = r^2 - 1$ has roots $r = \pm 1$, so that the general solution is $y(t) = c_1t + c_2t^{-1}$. Thus $y(1) = c_1 + c_2$. Now, $y'(t) = c_1 - c_2t^{-2}$, so that $y'(1) = c_1 - c_2$. Substituting initial values gives

$$\begin{aligned}c_1 + c_2 &= 2 \\ c_1 - c_2 &= 0.\end{aligned}$$

Thus $c_1 = c_2 = 1$, and the solution is $y(t) = t + t^{-1}$.

D2.2.34 The characteristic polynomial $r^2 + (2-1)r - 12 = r^2 + r - 12 = (r+4)(r-3)$ has roots 3 and -4, so that the general solution is $y(t) = c_1t^3 + c_2t^{-4}$. Thus $y(1) = c_1 + c_2$. Now, $y'(t) = 3c_1t^2 - 4c_2t^{-5}$, so that $y'(1) = 3c_1 - 4c_2$. Substituting initial values gives

$$\begin{aligned}c_1 + c_2 &= 0 \\ 3c_1 - 4c_2 &= 6.\end{aligned}$$

Thus $c_1 = \frac{6}{7}$ and $c_2 = -\frac{6}{7}$, so the solution is $y(t) = \frac{6}{7}(t^3 - t^{-4})$.

D2.2.35 The characteristic polynomial $r^2 + (-1-1)r - 15 = r^2 - 2r - 15 = (r-5)(r+3)$ has roots -3 and 5, so that the general solution is $y(t) = c_1t^5 + c_2t^{-3}$. Thus $y(1) = c_1 + c_2$. Now, $y'(t) = 5c_1t^4 - 3c_2t^{-4}$, so that $y'(1) = 5c_1 - 3c_2$. Substituting initial values gives

$$\begin{aligned}c_1 + c_2 &= 6 \\ 5c_1 - 3c_2 &= -1.\end{aligned}$$

Thus $c_1 = \frac{17}{8}$ and $c_2 = \frac{31}{8}$, so the solution is $y(t) = \frac{1}{8}(17t^5 + 31t^{-3})$.

D2.2.36 The characteristic polynomial $r^2 + (4-1)t - 4 = r^2 + 3r - 4 = (r+4)(r-1)$ has roots 1 and -4, so that the general solution is $y(t) = c_1t + c_2t^{-4}$. Thus $y(1) = c_1 + c_2$. Now, $y'(t) = c_1 - 4c_2t^{-5}$, so that $y'(1) = c_1 - 4c_2$. Substituting initial values gives

$$\begin{aligned}c_1 + c_2 &= 5 \\ c_1 - 4c_2 &= -3.\end{aligned}$$

Thus $c_1 = \frac{17}{5}$ and $c_2 = \frac{8}{5}$, so the solution is $y(t) = \frac{1}{5}(17t + 8t^{-4})$.

D2.2.37 The characteristic polynomial $r^2 + (6-1)r + 6 = r^2 + 5r + 6 = (r+2)(r+3)$ has roots -2 and -3, so that the general solution is $y(t) = c_1t^{-2} + c_2t^{-3}$. Thus $y(1) = c_1 + c_2$. Now, $y'(t) = -2c_1t^{-3} - 3c_2t^{-4}$, so that $y'(1) = -2c_1 - 3c_2$. Substituting initial values gives

$$\begin{aligned}c_1 + c_2 &= 0 \\ -2c_1 - 3c_2 &= -4.\end{aligned}$$

Thus $c_1 = -4$ and $c_2 = 4$, so the solution is $y(t) = 4(t^{-3} - t^{-2})$.

D2.2.38 The characteristic polynomial $r^2 + (1-1)r - 2 = r^2 - 2$ has roots $\pm\sqrt{2}$, so that the general solution is $y(t) = c_1t^{\sqrt{2}} + c_2t^{-\sqrt{2}}$. Thus $y(1) = c_1 + c_2$. Now, $y'(t) = \sqrt{2}c_1t^{\sqrt{2}-1} - \sqrt{2}c_2t^{-\sqrt{2}-1}$, so that $y'(1) = \sqrt{2}c_1 - \sqrt{2}c_2$. Substituting initial values gives

$$\begin{aligned}c_1 + c_2 &= 8 \\ \sqrt{2}c_1 - \sqrt{2}c_2 &= -12.\end{aligned}$$

Thus $c_1 = 4 - 3\sqrt{2}$ and $c_2 = 4 + 3\sqrt{2}$, so the solution is $y(t) = (4 - 3\sqrt{2})t^{\sqrt{2}} + (4 + 3\sqrt{2})t^{-\sqrt{2}}$.

D2.2.39

- a. False. This does not fit the pattern, since it is not a constant-coefficient differential equation. Trying e^{rt} gives

$$y'' + ty' + 4y = r^2 e^{rt} + r t e^{rt} + 4e^{rt} = (r^2 + 4)e^{rt} + r t e^{rt},$$

which will never be identically zero.

- b. False. In a Cauchy-Euler equation, in each term the power of t matches the order of the derivative. That is not the case here, since for example the coefficient of the zeroth order derivative, y itself, is t^2 .
- c. False. The characteristic polynomial is a quadratic, and a quadratic with complex roots has a pair of complex conjugate roots. $2 + 3i$ and $-2 + 3i$ are not complex conjugates.
- d. True. If the characteristic polynomial has roots $\pm 2i$, then the general solution is $e^{0t}(c_1 \cos 2t + c_2 \sin 2t) = c_1 \cos 2t + C \sin 2t$. But by the double-angle formula, $\sin 2t = 2 \sin t \cos t$, so that this is equal to $c_1 \cos 2t + 2C \sin t \cos t = c_1 \cos 2t + c_2 \sin t \cos t$. The differential equation $y'' + 4y = 0$ has these roots.
- e. False. For any such differential equation, the multiples of t appearing in the cos and sin terms must be equal.

D2.2.40 We show that in fact we can write it in that form with $A > 0$.

- a.

$$y = A \sin(\omega t + \varphi) = A(\sin \omega t \cos \varphi + \cos \omega t \sin \varphi)$$

- b. Rewrite the result of (a) as

$$y = (A \cos \varphi) \sin \omega t + (A \sin \varphi) \cos \omega t.$$

Then if this is equal to $c_1 \sin \omega t + c_2 \cos \omega t$, we see that $c_1 = A \cos \varphi$ and $c_2 = A \sin \varphi$.

- c. First, note that

$$\sqrt{c_1^2 + c_2^2} = \sqrt{A^2 \cos^2 \varphi + A^2 \sin^2 \varphi} = \sqrt{A^2(\cos^2 \varphi + \sin^2 \varphi)} = \sqrt{A^2} = A.$$

Then

$$\frac{c_2}{c_1} = \frac{A \sin \varphi}{A \cos \varphi} = \tan \varphi.$$

D2.2.41 Again we assume $A > 0$.

- a. $y = A \cos(\omega t + \varphi) = A(\cos \omega t \cos \varphi - \sin \omega t \sin \varphi)$.

- b. Rewrite the result of (a) as

$$y = (-A \sin \varphi) \sin \omega t + (A \cos \varphi) \cos \omega t.$$

Then if this is equal to $c_1 \sin \omega t + c_2 \cos \omega t$, we see that $c_1 = -A \sin \varphi$ and $c_2 = A \cos \varphi$.

- c. First, note that

$$\sqrt{c_1^2 + c_2^2} = \sqrt{(-A)^2 \sin^2 \varphi + A^2 \cos^2 \varphi} = \sqrt{A^2(\sin^2 \varphi + \cos^2 \varphi)} = \sqrt{A^2} = A.$$

Then

$$\frac{c_1}{c_2} = \frac{-A \sin \varphi}{A \cos \varphi} = -\tan \varphi.$$

D2.2.42 Here $c_1 = 2$ and $c_2 = -2$, so that $A = \sqrt{8} = 2\sqrt{2}$. Also, $\tan \varphi = \frac{c_2}{c_1} = -1$, so that $\varphi = \frac{3\pi}{4}$ or $\varphi = \frac{7\pi}{4}$. Because the numerator (sin) is negative, the angle is a fourth quadrant angle, so $\varphi = \frac{7\pi}{4}$ and

$$2 \sin 3t - 2 \cos 3t = 2\sqrt{2} \sin(3t + \frac{7\pi}{4}).$$

D2.2.43 Here $c_1 = -3$ and $c_2 = 3$, so that $A = \sqrt{18} = 3\sqrt{2}$. Also, $\tan \varphi = \frac{c_2}{c_1} = -1$, so that $\varphi = \frac{3\pi}{4}$ or $\varphi = \frac{7\pi}{4}$. Because the numerator (sin) is positive, the angle is a second quadrant angle, so $\varphi = \frac{3\pi}{4}$ and

$$-3 \sin 4t + 3 \cos 4t = 3\sqrt{2} \sin(4t + \frac{3\pi}{4}).$$

D2.2.44 Here $c_1 = \sqrt{3}$ and $c_2 = 1$, so that $A = \sqrt{3+1} = 2$. Also, $\tan \varphi = \frac{1}{\sqrt{3}}$; since both c_1 and c_2 are positive, φ is a first quadrant angle, so is equal to $\frac{\pi}{6}$. Thus

$$\sqrt{3} \sin t + \cos t = 2 \sin(t + \frac{\pi}{6}).$$

D2.2.45 Here $c_1 = -1$ and $c_2 = \sqrt{3}$, so that $A = \sqrt{1+3} = 2$. Also, $\tan \varphi = \frac{c_2}{c_1} = -\sqrt{3}$; since c_1 is negative and c_2 is positive, this is the second quadrant angle $\frac{2\pi}{3}$. Thus

$$-\sin 2t + \sqrt{3} \cos 2t = 2 \sin(2t + \frac{2\pi}{3}).$$

D2.2.46 The characteristic polynomial is $r^3 - 4r = r(r+2)(r-2) = 0$, with roots 0, 2, and -2. Thus e^0 , e^{2t} , and e^{-2t} should all be solutions to the equation, so that the general solution is thus $y(t) = c_1 + c_2 e^{2t} + c_3 e^{-2t}$.

D2.2.47 The characteristic polynomial is $r^3 - r^2 - 6r = r(r-3)(r+2)$, with roots -2, 0, and 3. Thus e^{-2t} , $e^0 = 1$, and e^{3t} should all be solutions to the equation, so that the general solution is $y(t) = c_1 + c_2 e^{-2t} + c_3 e^{3t}$.

D2.2.48 The characteristic polynomial is $r^3 + r = r(r+i)(r-i)$, with roots 0 and $\pm i$. Thus $e^0 = 1$ should be a solution. From the latter two factors, we should also have $e^{0 \cdot t}(c_1 \cos t + c_2 \sin t)$ as a solution. Thus the general solution is

$$y(t) = c_1 \cos t + c_2 \sin t + c_3.$$

D2.2.49 The characteristic polynomial is $r^3 - 6r^2 + 8r = r(r-4)(r-2)$, with roots 0, 2, and 4. Thus $e^0 = 1$, e^{2t} , and e^{4t} should all be solutions to the equation, so that the general solution is thus $y(t) = c_1 + c_2 e^{2t} + c_3 e^{4t}$.

D2.2.50 The characteristic polynomial is

$$r^4 - 5r^2 + 4 = (r^2 - 4)(r^2 - 1) = (r+1)(r-1)(r+2)(r-2),$$

so that $e^{\pm t}$ and $e^{\pm 2t}$ should all be solutions to the equation. The general solution is thus

$$y(t) = c_1 e^t + c_2 e^{-t} + c_3 e^{2t} + c_4 e^{-2t}.$$

D2.2.51 The characteristic polynomial is

$$r^4 + 5r^2 + 4 = (r^2 + 4)(r^2 + 1) = (r+i)(r-i)(r+2i)(r-2i).$$

From the first pair of roots, we get the solution $y(t) = c_1 \cos t + c_2 \sin t$. From the second pair, we get the solution $y(t) = c_3 \cos 2t + c_4 \sin 2t$. The general solution is thus

$$y(t) = c_1 \cos t + c_2 \sin t + c_3 \cos 2t + c_4 \sin 2t.$$

D2.2.52 The characteristic polynomial is $r^2 + (-1-1)r + 1 = r^2 - 2r + 1 = (r-1)^2$. Because this has the repeated root 1, the general solution is $y(t) = c_1 t^1 + c_2 t^1 \ln t = c_1 t + c_2 t \ln t$.

D2.2.53 The characteristic polynomial is $r^2 + (3 - 1)r + 1 = r^2 + 2r + 1 = (r + 1)^2$. Because this has the repeated root -1 , the general solution is $y(t) = c_1 t^{-1} + c_2 t^{-1} \ln t = \frac{c_1 + c_2 \ln t}{t}$.

D2.2.54 The characteristic polynomial is $r^2 + (-3 - 1)r + 4 = r^2 - 4r + 4 = (r - 2)^2$. Because this has the repeated root 2 , the general solution is $y(t) = c_1 t^2 + c_2 t^2 \ln t$.

D2.2.55 The characteristic polynomial is $r^2 + (7 - 1)r + 9 = r^2 + 6r + 9 = (r + 3)^2$. Because this has the repeated root -3 , the general solution is $y(t) = c_1 t^{-3} + c_2 t^{-3} \ln t = \frac{c_1 + c_2 \ln t}{t^3}$.

D2.2.56 The characteristic polynomial is $r^2 + (1 - 1)r + 1 = r^2 + 1 = (r + i)(r - i)$. Thus $\alpha = 0$ and $\beta = 1$, so the general solution is

$$y(t) = c_1 t^0 \cos(\ln t) + c_2 t^0 \sin(\ln t) = c_1 \cos \ln t + c_2 \sin \ln t.$$

D2.2.57 The characteristic polynomial is $r^2 + (7 - 1)r + 25 = r^2 + 6r + 25$, with roots $\frac{-6 \pm \sqrt{36 - 100}}{2} = -3 \pm 4i$. Thus $\alpha = -3$ and $\beta = 4$, so the general solution is

$$y(t) = t^{-3}(c_1 \cos(4 \ln t) + c_2 \sin(4 \ln t)).$$

D2.2.58 The characteristic polynomial is $r^2 + (-1 - 1)r + 5 = r^2 - 2r + 5$, with roots $\frac{2 \pm \sqrt{4 - 20}}{2} = 1 \pm 2i$. Thus $\alpha = 1$ and $\beta = 2$, so the general solution is

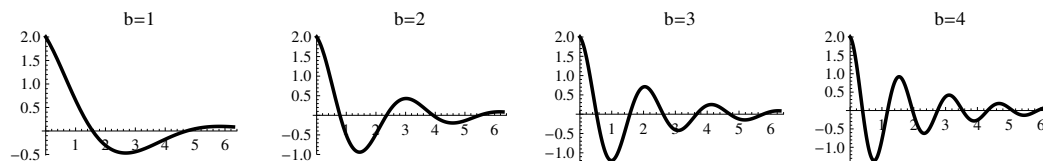
$$y(t) = t(c_1 \cos(2 \ln t) + c_2 \sin(2 \ln t)).$$

D2.2.59 The characteristic polynomial is $r^2 + (0 - 1)r + \frac{1}{2} = r^2 - r + \frac{1}{2}$, with roots $\frac{1 \pm \sqrt{1 - 2}}{2} = \frac{1}{2} \pm \frac{1}{2}i$. Thus $\alpha = \beta = \frac{1}{2}$, so the general solution is

$$y(t) = \sqrt{t}(c_1 \cos(\frac{1}{2} \ln t) + c_2 \sin(\frac{1}{2} \ln t)).$$

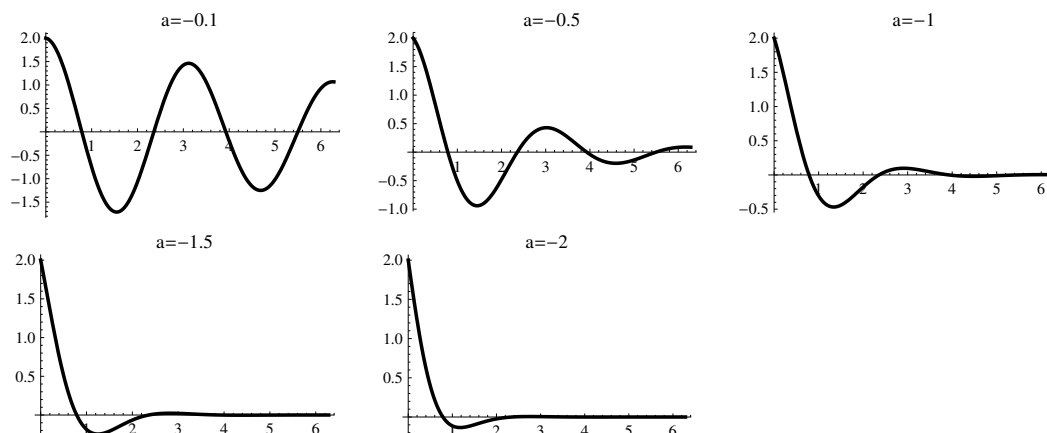
D2.2.60

- a. With the values given, plots of the solution are



The function is $e^{-t/2}(2 \cos bt)$, so successive maxima occur when $bt = 2n\pi$, so for $t = \frac{2n\pi}{b}$.

- b. The frequency is the reciprocal of the wavelength, so it is $\frac{b}{2\pi}$ cycles per unit time. Thus it is b cycles per 2π units of time.
- c. With $b = 2$, $c_1 = 2$ and $c_2 = 0$, the general solution becomes $2e^{at} \cos 2t$. Plots for the given values of a are:

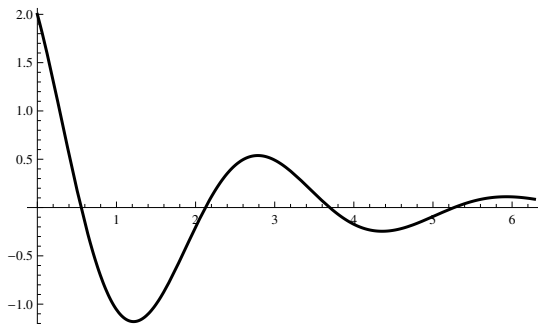


The initial value of $2e^{at} \cos 2t$ is obviously 2. We want to find the first value for which $2e^{at} \cos 2t = \frac{2}{3}$, so that $e^{at} \cos 2t = \frac{1}{3}$. Solving numerically for each of the given values of a produces

a	-0.1	-0.5	-1	-1.5	-2
t	0.604	0.557	0.496	0.437	0.384

The decreasing times reflect the stronger contribution from the exponential decay term as t gets more and more negative.

d. A plot of this function is



Extrema of this function occur when the derivative vanishes. We have

$$\begin{aligned} y'(t) &= e^{-t/2}(-4 \sin 2t - 2 \cos 2t) - \frac{1}{2}e^{-t/2}(2 \cos 2t - \sin 2t) \\ &= -\frac{1}{2}e^{-t/2}(6 \cos 2t + 7 \sin 2t). \end{aligned}$$

Thus the derivative vanishes for $2t = \tan^{-1} \frac{6}{7}$, or $2t \approx 2.43297 + n\pi$. Thus $t \approx 1.2165 + \frac{\pi}{2}n$. Because the graph oscillates, successive extrema alternate between maxima and minima, so that successive peaks are separated by π . Finally, the amplitude of the function at time t is

$$\sqrt{(2e^{-t/2})^2 + (-e^{-t/2})^2} = \sqrt{5}e^{-t/2} = e^{-t/2}\sqrt{5}.$$

The amplitude has decayed to $1/3$ of its original value when $e^{-t/2} = \frac{1}{3}$, so when $t = 2 \ln 3 \approx 2.197$.

D2.2.61 Noting the comment before exercises 46-51, the characteristic polynomial of this differential equation is $r^4 + k^2r^2 = r^2(r + ki)(r - ki)$. From the repeated roots of 0 we get a solution $c_1e^0 + c_2xe^0 = c_1 + c_2x$, and from the complex conjugate roots we get a solution $e^{0x}(c_3 \cos kx + c_4 \sin kx) = c_3 \cos kx + c_4 \sin kx$. Putting these together gives for a general solution

$$y(x) = c_1 + c_2x + c_3 \cos kx + c_4 \sin kx.$$

D2.2.62

- The characteristic polynomial associated with this equation is $r^2 + (a-1)r + b$.
- Suppose $y = y(t)$, and let $t = e^x$, so that $x = \ln t$. Then

$$y'(t) = y'(x) x'(t) = \frac{1}{t} y'(x) = e^{-x} y'(x).$$

Differentiating the first equality above a second time gives

$$y''(t) = \frac{d}{dt}(y'(x) x'(t)) = \frac{d}{dt}(y'(x)) \cdot x'(t) + y'(x) \cdot x''(t).$$

Now apply the chain rule: $\frac{d}{dt}(y'(x)) = \frac{d}{dx}(y'(x)) \cdot x'(t) = \frac{1}{t}y''(x)$. Plugging this in we get

$$\begin{aligned} y''(t) &= \frac{1}{t}y''(x) \cdot x'(t) + y'(x) \cdot x''(t) \\ &= \frac{1}{t^2} \cdot y''(x) + x''(t)y'(x) = \frac{1}{t^2}(y''(x) - y'(x)) \\ &= e^{-2x}(y''(x) - y'(x)). \end{aligned}$$

Now substitute the expressions above into $t^2y''(t) + aty'(t) + by(t) = 0$ to get

$$\begin{aligned} t^2y''(t) + tay'(t) + by(t) &= e^{2x}(y''(x) - y'(x))e^{-2x} + e^xae^{-x}y'(x) + by(x) \\ &= y''(x) - y'(x) + ay'(x) + by(x) \\ &= y''(x) + (a-1)y'(x) + by(x). \end{aligned}$$

- c. The characteristic polynomial for the above equation is $r^2 + (a-1)r + b$, which is the same as the polynomial in part (a).
- d. If this characteristic polynomial has a repeated root, it is $\frac{1-a}{2}$, so that the general solution is $y(x) = c_1e^{(1-a)x/2} + c_2xe^{(1-a)x/2}$.
- e. Substituting $x = \ln t$ gives

$$\begin{aligned} y(t) &= c_1e^{(1-a)(\ln t)/2} + c_2(\ln t)e^{(1-a)(\ln t)/2} \\ &= c_1t^{(1-a)/2} + c_2 \cdot t^{(1-a)/2} \ln t \end{aligned}$$

for the general solution. (Note that $(1-a)/2$ is the repeated root of the characteristic polynomial, so that this general solution may be rewritten $y(t) = c_1t^r + c_2t^r \ln t$).

D2.2.63

- a. The equation associated with this Cauchy-Euler equation is $r^2 + (a-1)r + b = 0$, with roots

$$\frac{-(a-1) \pm \sqrt{(a-1)^2 - 4b}}{2}.$$

Assuming it has repeated roots, we have $(a-1)^2 - 4b = 0$, so that the repeated root is $p = \frac{1-a}{2}$. Rearranging terms gives $2p + a = 1$.

- b. Substitute $t^pv(t)$ into $t^2y'' + aty' + by$ to get

$$\begin{aligned} &t^2(t^pv(t))'' + at(t^pv(t))' + b(t^pv(t)) \\ &= t^2(pt^{p-1}v(t) + t^pv'(t))' + at(pt^{p-1}v(t) + t^pv'(t)) + bt^pv(t) \\ &= pt^2(t^{p-1}v(t))' + t^2(t^pv'(t))' + apt^pv(t) + at^{p+1}v'(t) + bt^pv(t) \\ &= pt^2((p-1)t^{p-2}v(t) + t^{p-1}v'(t)) + t^2(pt^{p-1}v'(t) + t^pv''(t)) \\ &\quad + at^{p+1}v'(t) + (ap+b)t^pv(t) \\ &= p(p-1)t^pv(t) + pt^{p+1}v'(t) + pt^{p+1}v'(t) + t^{p+2}v''(t) + at^{p+1}v'(t) \\ &\quad + (ap+b)t^pv(t) \\ &= t^{p+2}v''(t) + (a+2p)t^{p+1}v'(t) + (p(p-1) + ap+b)t^pv(t) \\ &= t^p(t^2v''(t) + tv'(t) + (p^2 - p + ap+b)v(t)) \end{aligned}$$

Because $p = \frac{1-a}{2}$ and $b = \frac{(a-1)^2}{4}$, we have

$$\begin{aligned} p^2 - p + ap + b &= \frac{(1-a)^2}{4} - \frac{1-a}{2} + \frac{a(1-a)}{2} + \frac{(a-1)^2}{4} \\ &= \frac{(a-1)^2}{2} + \frac{a-a^2+a-1}{2} \\ &= \frac{(a-1)^2}{2} - \frac{(a-1)^2}{2} = 0, \end{aligned}$$

so that

$$t^2(t^p v(t))'' + at(t^p v(t))' + b(t^p v(t)) = t^p(t^2 v''(t) + tv'(t)).$$

Thus if $t^p v(t)$ satisfies the left-hand side, so that it is zero, then the right-hand side is zero as well. Because $t^p = 0$ only for $t = 0$ (and we are assuming $t > 0$), we must have $t^2 v''(t) + tv'(t) = 0$.

- c. Let $w = v'$; then $t^2 w'(t) + tw(t) = 0$. This is separable: $\frac{w'(t)}{w(t)} = -\frac{1}{t}$, so that $\frac{dw}{w} = -\frac{dt}{t}$. Integrating gives $\ln |w| = -\ln t + C$; exponentiating both sides gives $w = \frac{C_1}{t}$ (where $C_1 = \pm e^C$).
- d. Because $w = v'$, we have the equation $v' = \frac{C_1}{t}$; integrating gives $v(t) = C_1 \ln t$. Thus $C_1 t^p \ln t$ is a solution to the original equation as well. Now, t^p and $t^p \ln t$ are linearly independent, since at $t = 1$, $0 \cdot 1^p = 1^p \ln 1$, but $t^p \ln t$ is not everywhere zero. Thus we have two linearly independent solutions, and the general solution is $c_1 t^p + c_2 t^p \ln t$.

D2.2.64

- a. As in Problem 62(a), making the indicated substitution $t = e^x$ converts $t^2 y''(t) + aty'(t) + by(t) = 0$ to $y''(x) + (a-1)y'(x) + by(x) = 0$.
- b. The roots of this quadratic are

$$r_1, r_2 = \frac{1-a \pm \sqrt{(a-1)^2 - 4b}}{2}.$$

Assuming the roots are complex, it follows that (following the notation in the text)

$$\alpha = \frac{1-a}{2}, \quad \beta = \frac{4b - (a-1)^2}{2},$$

and that the general solution is

$$y(t) = c_1 t^\alpha \cos(\beta \ln t) + c_2 t^\alpha \sin(\beta \ln t).$$

D2.2.65

- a. Because r_i is a root of $r^2 + pr + q$ for $i = 1, 2$, we have for $y = e^{r_i t}$

$$y'' + py' + qy = r_i^2 e^{r_i t} + pr_i e^{r_i t} + qe^{r_i t} = e^{r_i t}(r_i^2 + pr_i + q) = 0,$$

so that $e^{r_i t}$ is a solution of the equation for $i = 1, 2$. Thus linear combinations are also solutions, so that in particular

$$u(t) = \frac{e^{r_1 t} - e^{r_2 t}}{r_1 - r_2}$$

is a solution since we are assuming the roots are distinct so that $r_1 \neq r_2$.

- b. As $r_1 \rightarrow r_2$, both numerator and denominator go to zero, so we need to use L'Hôpital's rule. In this application, the variable is r_1 , so

$$\lim_{r_1 \rightarrow r_2} u(t) = \lim_{r_1 \rightarrow r_2} \frac{e^{r_1 t} - e^{r_2 t}}{r_1 - r_2} = \lim_{r_1 \rightarrow r_2} \frac{te^{r_1 t}}{1} = te^{r_2 t}.$$

Thus as the distinct roots come close together, this solution $u(t)$ approaches $te^{r_2 t}$. By continuity, this is also a solution to the equation in the situation where r_2 is a repeated root.

D2.2.66 Differentiating gives

$$\begin{aligned} \frac{d}{dr}((e^{rt})'' + p(e^{rt})' + q(e^{rt})) &= \frac{d}{dr}\left(\frac{d^2}{dt^2}e^{rt}\right) + p\frac{d}{dr}\left(\frac{d}{dt}e^{rt}\right) + \frac{d}{dr}(q(e^{rt})) \\ &= \frac{d^2}{dt^2}\frac{d}{dr}e^{rt} + p\frac{d}{dt}\frac{d}{dr}e^{rt} + q\frac{d}{dr}e^{rt} \\ &= \frac{d^2}{dt^2}(te^{rt}) + p\frac{d}{dt}(te^{rt}) + qte^{rt} \\ &= (te^{rt})'' + p(te^{rt})' + q(te^{rt}). \end{aligned}$$

Thus te^{rt} satisfies the equation if e^{rt} does.

Linear Nonhomogeneous Equations

D2.3.1 To find the general solution of the nonhomogeneous equation, first find the general solution of the corresponding homogeneous equation, $c_1 y_1(t) + c_2 y_2(t)$, and then find a particular solution $y_p(t)$ of the nonhomogeneous equation (perhaps using the method of undetermined coefficients). The general solution of the nonhomogeneous equation is then $c_1 y_1(t) + c_2 y_2(t) + y_p(t)$.

D2.3.2 Because the right-hand side is a cubic polynomial, the trial solution must be a polynomial of degree no higher than 3, so try $At^3 + Bt^2 + Ct + D$.

D2.3.3 Because the right-hand side is an exponential, we want to use an exponential. Checking first that -4 is not a root of the characteristic polynomial, we conclude that we should start with Ae^{-4t} .

D2.3.4 Because the right-hand side is sinusoidal, we choose $A \sin 3t + B \cos 3t$ for a trial solution. Note that the general solution uses exponentials, so there is no risk of resonance here.

D2.3.5 Because the right-hand side is damped sinusoidal, we choose $Ae^{-t} \sin 4t + Be^{-t} \cos 4t$ for a trial solution. Note that the general solution is a pure exponential solution, so there is no risk of resonance.

D2.3.6 The general solution is an exponential solution. For the particular solution, use a mixed trial solution consisting of a linear polynomial $A + Bt$ (corresponding to $1 + t$) and $C \sin 3t + D \cos 3t$ (corresponding to $\cos 3t$).

D2.3.7 The general solution is an exponential solution. For the particular solution, use a mixed trial solution consisting of a quadratic polynomial multiplied by the appropriate exponential: $(At^2 + Bt + C)e^{-t}$.

D2.3.8 The characteristic polynomial is $r^2 - 4r + 4 = (r - 2)^2$, so that e^{2t} and te^{2t} are the two linearly independent solutions of the homogeneous equation. Thus we must use $At^2 e^{2t}$ as a trial solution.

D2.3.9 The general solution is an exponential solution, so we can use a trial solution $At + B$, and compute

$$(At + B)'' - 9(At + B) = -9At - 9B = 2t + 1.$$

Thus $-9A = 2$ and $-9B = 1$, so that $A = -\frac{2}{9}$ and $B = -\frac{1}{9}$. The particular solution is $-\frac{1}{9}(2t + 1)$.

D2.3.10 The general solution is an exponential solution, so we can use a trial solution consisting of a cubic polynomial, and compute

$$\begin{aligned} (At^3 + Bt^2 + Ct + D)'' - 2(At^3 + Bt^2 + Ct + D)' - 15(At^3 + Bt^2 + Ct + D) \\ = 6At + 2B - 6At^2 - 4Bt - 2C - 15At^3 - 15Bt^2 - 15Ct - 15D \\ = -15At^3 + (-6A - 15B)t^2 + (6A - 4B - 16C)t + (2B - 2C - 15D) \\ = 3t^3 - 2t - 4. \end{aligned}$$

This gives us the system of equations

$$\begin{array}{rcl} -15A & = & 3 \\ -6A - 15B & = & 0 \\ 6A - 4B - 16C & = & -2 \\ 2B - 2C - 15D & = & -4 \end{array}.$$

Solving gives $A = -\frac{1}{5}$, $B = \frac{2}{25}$, $C = \frac{3}{100}$, and $D = \frac{41}{50}$, so that the particular solution is

$$y_p(t) = -\frac{1}{5}t^3 + \frac{2}{25}t^2 + \frac{3}{100}t + \frac{41}{50}.$$

D2.3.11 The general solution is an exponential solution, so we can use a quartic polynomial as the trial solution, and compute

$$\begin{aligned}(At^4 + Bt^3 + Ct^2 + Dt + E)'' + 3(At^4 + Bt^3 + Ct^2 + Dt + E)' - 18(At^4 + Bt^3 + Ct^2 + Dt + E) \\= 12At^2 + 6Bt + 2C + 12At^3 + 9Bt^2 + 6Ct + 3D - 18At^4 - 18Bt^3 - 18Ct^2 - 18Dt - 18E \\= -18At^4 + (12A - 18B)t^3 + (12A + 9B - 18C)t^2 + (6B + 6C - 18D)t + (2C + 3D - 18E) \\= 2t^4 - t^2.\end{aligned}$$

This gives us the system of equations

$$\begin{aligned}-18A &= 2 \\12A - 18B &= 0 \\12A + 9B - 18C &= -1 \\6B + 6C - 18D &= 0 \\2C + 3D - 18E &= 0.\end{aligned}$$

Solving gives the particular solution

$$y_p(t) = -\frac{1}{9}t^4 - \frac{2}{27}t^3 - \frac{1}{18}t^2 - \frac{7}{162}t - \frac{13}{972}.$$

D2.3.12 The general solution is sinusoidal, so we can use a cubic polynomial as the trial solution, and compute

$$\begin{aligned}(At^3 + Bt^2 + Ct + D)'' + 5(At^3 + Bt^2 + Ct + D) = 6At + 2B + 5At^3 + 5Bt^2 + 5Ct + 5D \\= 5At^3 + 5Bt^2 + (6A + 5C)t + (2B + 5D) = 6t^3 - t^2.\end{aligned}$$

This gives the system of equations

$$\begin{aligned}5A &= 6 \\5B &= -1 \\6A + 5C &= 0 \\2B + 5D &= 0.\end{aligned}$$

Solving gives for the particular solution

$$y_p(t) = \frac{6}{5}t^3 - \frac{1}{5}t^2 - \frac{36}{25}t + \frac{2}{25}.$$

D2.3.13 The general solutions are e^t and e^{-t} , so we can use Ae^{-2t} as a trial solution, and compute

$$(Ae^{-2t})'' - (Ae^{-2t}) = 4Ae^{-2t} - Ae^{-2t} = 3Ae^{-2t} = e^{-2t},$$

so that $A = \frac{1}{3}$, and the particular solution is

$$y_p(t) = \frac{1}{3}e^{-2t}.$$

D2.3.14 The general solutions are sinusoidal, so we can use Ae^{-3t} as a trial solution, and compute

$$(Ae^{-3t})'' + 3(Ae^{-3t}) = 9Ae^{-3t} + 3Ae^{-3t} = 12Ae^{-3t} = 5e^{-3t},$$

so that $A = \frac{5}{12}$, and the particular solution is

$$y_p(t) = \frac{5}{12}e^{-3t}.$$

D2.3.15 The characteristic polynomial is $r^2 - 4y - 32 = (r - 8)(r + 4)$, so that e^{-3t} is not a general solution. Thus we can use Ae^{-3t} as a trial solution, and compute

$$\begin{aligned}(Ae^{-3t})'' - 4(Ae^{-3t})' - 32(Ae^{-3t}) &= 9Ae^{-3t} + 12Ae^{-3t} - 32Ae^{-3t} \\ &= -11Ae^{-3t} = 6e^{-3t},\end{aligned}$$

so that $A = -\frac{6}{11}$, and the particular solution is

$$y_p(t) = -\frac{6}{11}e^{-3t}.$$

D2.3.16 The characteristic polynomial is $r^2 + 2y - 8 = (r + 4)(r - 2)$, so that $e^{\pm t}$ is not a general solution. Thus we can use $Ae^{-t} + Be^t$ as a trial solution, and compute

$$\begin{aligned}(Ae^{-t} + Be^t)'' + 2(Ae^{-t} + Be^t)' - 8(Ae^{-t} + Be^t) &= Ae^{-t} + Be^t - 2Ae^{-t} + 2Be^t - 8Ae^{-t} - 8Be^t \\ &= -9Ae^{-t} - 5Be^t = e^{-t} - 2e^t,\end{aligned}$$

so that $A = -\frac{1}{9}$ and $B = \frac{2}{5}$, and the particular solution is

$$y_p(t) = -\frac{1}{9}e^{-t} + \frac{2}{5}e^t.$$

D2.3.17 The general solution is exponential, so we can use a trial solution of the form $A \sin 2t + B \cos 2t$, and compute

$$\begin{aligned}(A \sin 2t + B \cos 2t)'' - (A \sin 2t + B \cos 2t) &= -4A \sin 2t - 4B \cos 2t - A \sin 2t - B \cos 2t \\ &= -5A \sin 2t - 5B \cos 2t = 3 \sin 2t,\end{aligned}$$

so that $A = -\frac{3}{5}$ and $B = 0$, and the particular solution is

$$y_p(t) = -\frac{3}{5} \sin 2t.$$

D2.3.18 The general solution involves $\sin 2t$ and $\cos 2t$, so we may use $A \sin 3t + B \cos 3t$ as a trial solution, and compute

$$\begin{aligned}(A \sin 3t + B \cos 3t)'' + 4(A \sin 3t + B \cos 3t) &= -9A \sin 3t - 9B \cos 3t + 4A \sin 3t + 4B \cos 3t \\ &= -5A \sin 3t - 5B \cos 3t = 5 \cos 3t,\end{aligned}$$

so that $A = 0$ and $B = -1$, and the particular solution is

$$y_p(t) = -\cos 3t.$$

D2.3.19 The characteristic polynomial is $r^2 - r - 6 = (r - 3)(r + 2)$, so the general solution does not involve $\sin t$ or $\cos t$. Thus we may use $A \sin t + B \cos t$ as a trial solution, and compute

$$\begin{aligned}(A \sin t + B \cos t)'' - (A \sin t + B \cos t)' - 6(A \sin t + B \cos t) \\ &= -A \sin t - B \cos t - A \cos t + B \sin t - 6A \sin t - 6B \cos t \\ &= (-7A + B) \sin t + (-A - 7B) \cos t = \sin t + 3 \cos t.\end{aligned}$$

This gives the system of equations

$$\begin{aligned}-7A + B &= 1 \\ -A - 7B &= 3.\end{aligned}$$

Solving gives $A = -\frac{1}{5}$ and $B = -\frac{2}{5}$, so that the particular solution is

$$y_p(t) = -\frac{1}{5}(\sin t + 2 \cos t).$$

D2.3.20 The characteristic polynomial is $r^2 - 3r - 4 = (r - 4)(r + 1)$, so the general solution does not involve $\sin 2t$ or $\cos 2t$. Thus we may use $A \sin 2t + B \cos 2t$ as the trial solution, and compute

$$\begin{aligned}(A \sin 2t + B \cos 2t)'' - 3(A \sin 2t + B \cos 2t)' - 4(A \sin 2t + B \cos 2t) \\&= -4A \sin 2t - 4B \cos 2t - 6A \cos 2t + 6B \sin 2t - 4A \sin 2t - 4B \cos 2t \\&= (-6A - 8B) \cos 2t + (-8A + 6B) \sin 2t = 2 \cos 2t - 3 \sin 2t.\end{aligned}$$

This gives the system of equations

$$\begin{aligned}-8A + 6B &= -3 \\ -6A - 8B &= 2.\end{aligned}$$

Solving gives $A = \frac{3}{25}$ and $B = -\frac{17}{50}$, so that the particular solution is

$$y_p(t) = \frac{1}{50}(6 \sin 2t - 17 \cos 2t).$$

D2.3.21 The general solution involves $e^{\pm 2t}$, so we may use as a trial solution for the exponential piece Ae^t , and for the polynomial piece a constant polynomial B . Computing gives

$$(Ae^t + B)'' - 4(Ae^t + B) = -3Ae^t - 4B = 2e^t - 1,$$

so that $A = -\frac{2}{3}$ and $B = \frac{1}{4}$, so that the particular solution is

$$y_p(t) = -\frac{2}{3}e^t + \frac{1}{4}.$$

D2.3.22 The general solution involves $\sin t$ and $\cos t$, so we may use as a trial solution for the sinusoidal piece $A \cos 2t + B \sin 2t$, and for the polynomial piece a cubic equation $Ct^3 + Dt^2 + Et + F$. Then

$$\begin{aligned}y''(t) &= -4A \cos 2t - 4B \sin 2t + 6Ct + 2D \\ y(t) &= A \cos 2t + B \sin 2t + Ct^3 + Dt^2 + Et + F,\end{aligned}$$

so that

$$y''(t) + y(t) = -3A \cos 2t - 3B \sin 2t + Ct^3 + Dt^2 + (6C + E)t + (2D + F) = \cos 2t + t^3.$$

Thus $A = -\frac{1}{3}$, $B = 0$, $C = 1$, $D = 0$, $E = -6$, and $F = 0$, so that the particular solution is

$$y_p(t) = -\frac{1}{3} \cos t + t^3 - 6t.$$

D2.3.23 The characteristic polynomial is $r^2 + r - 12 = (r + 4)(r - 3)$, so that the solution involves e^{3t} and e^{-4t} . Thus we may use $Ate^{-t} + Be^{-t}$ as a trial solution for the sinusoidal piece. Computing gives

$$\begin{aligned}(Ate^{-t} + Be^{-t})'' + (Ate^{-t} + Be^{-t})' - 12(Ate^{-t} + Be^{-t}) \\&= (Ae^{-t} - Ate^{-t} - Be^{-t})' + (Ae^{-t} - Ate^{-t} - Be^{-t}) - 12(Ate^{-t} + Be^{-t}) \\&= -Ae^{-t} - Ae^{-t} + Ate^{-t} + Be^{-t} + Ae^{-t} - Ate^{-t} - Be^{-t} - 12(Ate^{-t} + Be^{-t}) \\&= (-A - 12B)e^{-t} + (-12A)te^{-t} = 2te^{-t}.\end{aligned}$$

Thus $A = -\frac{1}{6}$ and $B = \frac{1}{72}$, so the particular solution is

$$y_p(t) = -\frac{1}{6}te^{-t} + \frac{1}{72}e^{-t}.$$

D2.3.24 The general solution involves $\cos 2t$ and $\sin 2t$, so we may use $Ae^{-t} \cos t + Be^{-t} \sin t$ as a trial solution. Then

$$\begin{aligned} y(t) &= Ae^{-t} \cos t + Be^{-t} \sin t \\ y'(t) &= -Ae^{-t} \cos t - Ae^{-t} \sin t - Be^{-t} \sin t + Be^{-t} \cos t \\ &= (B - A)e^{-t} \cos t + (-A - B)e^{-t} \sin t \\ y''(t) &= -(B - A)e^{-t} \cos t - (B - A)e^{-t} \sin t - (-A - B)e^{-t} \sin t + (-A - B)e^{-t} \cos t \\ &= -2Be^{-t} \cos t + 2Ae^{-t} \sin t \end{aligned}$$

Thus

$$y''(t) + 4y(t) = (2A + 4B)e^{-t} \sin t + (4A - 2B)e^{-t} \cos t = e^{-t} \cos t,$$

so that $A = \frac{1}{5}$ and $B = -\frac{1}{10}$, and the particular solution is

$$y_p(t) = \frac{1}{10}e^{-t}(2 \cos t - \sin t).$$

D2.3.25 The general solution involves $\sin 4t$ and $\cos 4t$, so that we can use $(At + B) \cos 2t + (Ct + D) \sin 2t$ as a trial solution. Then

$$\begin{aligned} y(t) &= (At + B) \cos 2t + (Ct + D) \sin 2t \\ y'(t) &= A \cos 2t - 2(At + B) \sin 2t + C \sin 2t + 2(Ct + D) \cos 2t \\ &= (2Ct + A + 2D) \cos 2t + (-2At + C - 2B) \sin 2t \\ y''(t) &= 2C \cos 2t - 2(2Ct + A + 2D) \sin 2t - 2A \sin 2t + 2(-2At + C - 2B) \cos 2t \\ &= (-4At - 4B + 4C) \cos 2t + (-4Ct - 4A - 4D) \sin 2t. \end{aligned}$$

Thus

$$y''(t) + 16y(t) = (12At + 12B + 4C) \cos 2t + (12Ct - 4A + 12D) \sin 2t = t \cos 2t.$$

This gives the system of equations

$$\begin{aligned} 12A &= 1 \\ 12B + 4C &= 0 \\ 12C &= 0 \\ -4A &+ 12D = 0. \end{aligned}$$

Solving gives $A = \frac{1}{12}$, $B = 0$, $C = 0$, and $D = \frac{1}{36}$, so that the particular solution is

$$y_p(t) = \frac{1}{36}(\sin 2t + 3t \cos 2t).$$

D2.3.26 The general solution involves $e^{\pm 3t}$, so we can use $(At^2 + Bt + C)e^{-t}$ as a trial solution. Then

$$\begin{aligned} y(t) &= (At^2 + Bt + C)e^{-t} \\ y'(t) &= (2At + B)e^{-t} - (At^2 + Bt + C)e^{-t} = (-At^2 + (2A - B)t + (B - C))e^{-t} \\ y''(t) &= (-2At + 2A - B)e^{-t} - (-At^2 + (2A - B)t + (B - C))e^{-t} \\ &= (At^2 + (B - 4A)t + (2A - 2B + C))e^{-t}, \end{aligned}$$

so that

$$y''(t) - 9y(t) = (-8At^2 - (4A + 8B)t + (2A - 2B - 8C))e^{-t} = (t^2 + 1)e^{-t}.$$

Thus $A = -\frac{1}{8}$, $B = \frac{1}{16}$, and $C = -\frac{11}{64}$, so that the particular solution is

$$y_p(t) = -\frac{1}{64}(8t^2 - 4t + 11)e^{-t}.$$

D2.3.27 The characteristic polynomial is $r^2 - r - 2 = (r - 2)(r + 1)$, so that the general solution involves exponentials. Thus we can use $(At + B) \cos t + (Ct + D) \sin t$ as a trial solution. Then

$$\begin{aligned} y(t) &= (At + B) \cos t + (Ct + D) \sin t \\ y'(t) &= A \cos t - (At + B) \sin t + C \sin t + (Ct + D) \cos t \\ &= (Ct + A + D) \cos t + (-At - B + C) \sin t \\ y''(t) &= C \cos t - (Ct + A + D) \sin t - A \sin t + (-At - B + C) \cos t \\ &= (-At - B + 2C) \cos t + (-Ct - 2A - D) \sin t, \end{aligned}$$

so that

$$\begin{aligned} y''(t) - y'(t) - 2y(t) &= ((-3A - C)t + (-A - 3B + 2C - D)) \cos t + ((A - 3C)t + (-2A + B - C - 3D)) \sin t \\ &= t \cos t - t \sin t. \end{aligned}$$

This gives the system of equations

$$\begin{aligned} -3A & & - C & & = 1 \\ -A - 3B + 2C & - D & = 0 \\ A & & - 3C & & = -1 \\ -2A + B - C - 3D & = 0. \end{aligned}$$

Solving gives $A = -\frac{2}{5}$, $B = \frac{9}{50}$, $C = \frac{1}{5}$, and $D = \frac{13}{50}$, so that the particular solution is

$$y_p(t) = \frac{1}{50}((-20t + 9) \cos t + (10t + 13) \sin t).$$

D2.3.28 The characteristic polynomial is $r^2 + 2y - 4$, so the equation has exponential general solution. Thus we can use $(At^2 + Bt + C) \cos t + (Dt^2 + Et + F) \sin t$ as a trial solution. Then

$$\begin{aligned} y(t) &= (At^2 + Bt + C) \cos t + (Dt^2 + Et + F) \sin t \\ y'(t) &= (2At + B) \cos t - (At^2 + Bt + C) \sin t + (2Dt + E) \sin t + (Dt^2 + Et + F) \cos t \\ &= (Dt^2 + (2A + E)t + B + F) \cos t + (-At^2 + (-B + 2D)t + E - C) \sin t \\ y''(t) &= (2Dt + 2A + E) \cos t - (Dt^2 + (2A + E)t + B + F) \sin t \\ &\quad + (-2At - B + 2D) \sin t + (-At^2 + (-B + 2D)t + E - C) \cos t \\ &= (-At^2 + (-B + 4D)t + 2A + 2E - C) \cos t + (-Dt^2 + (-4A - E)t + 2D - 2B - F) \sin t. \end{aligned}$$

Thus

$$\begin{aligned} y''(t) + 2y'(t) - 4y(t) &= ((-5A + 2D)t^2 + (4A - 5B + 4D + 2E)t + (2A + 2B - 5C + 2E + 2F)) \cos t \\ &\quad + ((-2A - 5D)t^2 + (-4A - 2B + 4D - 5E)t + (-2B - 2C + 2D + 2E - 5F)) \sin t \\ &= t^2 \cos t. \end{aligned}$$

This gives the system of equations

$$\begin{aligned} -5A & & + 2D & & = 1 \\ 4A - 5B & & + 4D + 2E & & = 0 \\ 2A + 2B - 5C & & + 2E + 2F & = 0 \\ -2A & & - 5D & & = 0 \\ -4A - 2B & & + 4D - 5E & & = 0 \\ & - 2B - 2C + 2D + 2E - 5F & = 0. \end{aligned}$$

Solving gives the particular solution

$$y_p(t) = \frac{1}{24389}((-4205t^2 - 116t + 1054)\cos t + (1682t^2 + 4756t + 2200)\sin t).$$

D2.3.29 We recognize the general solution to $y'' - y = 0$ as $c_1e^t + c_2e^{-t}$, so that $3e^t$ is a solution to the homogeneous equation. Thus we must use Ate^t as our trial solution. Then

$$\begin{aligned} y''(t) - y(t) &= (Ate^t)'' - (Ate^t) = (Ae^t + Ate^t)' - Ate^t = Ae^t + Ae^t + Ate^t - Ate^t \\ &= 2Ae^t = 3e^t, \end{aligned}$$

so that $A = \frac{3}{2}$, and the particular solution is

$$y_p(t) = \frac{3}{2}te^t.$$

D2.3.30 We recognize the general solution to $y'' + y = 0$ as $c_1 \cos t + c_2 \sin t$, so that $3 \cos t$ is a solution to the homogeneous equation. Thus we must use $At \cos t + Bt \sin t$ as our trial solution. Then

$$\begin{aligned} y''(t) + y(t) &= (At \cos t + Bt \sin t)'' + At \cos t + Bt \sin t \\ &= (A \cos t - At \sin t + B \sin t + Bt \cos t)' + At \cos t + Bt \sin t \\ &= -A \sin t - A \sin t - At \cos t + B \cos t + B \cos t - Bt \sin t + At \cos t + Bt \sin t \\ &= -2A \sin t + 2B \cos t = 3 \cos t, \end{aligned}$$

so that $A = 0$ and $B = \frac{3}{2}$. Thus the particular solution is

$$y_p(t) = \frac{3}{2}t \sin t.$$

D2.3.31 The characteristic polynomial is $r^2 + r - 6 = (r + 3)(r - 2)$, so that $4e^{-3t}$ is a solution to the homogeneous equation. Thus we must use Ate^{-3t} as our trial solution. Then

$$\begin{aligned} y''(t) + y'(t) - 6y(t) &= (Ae^{-3t} - 3Ate^{-3t})' + (Ae^{-3t} - 3Ate^{-3t}) - 6Ate^{-3t} \\ &= -3Ae^{-3t} - 3Ae^{-3t} + 9Ate^{-3t} + Ae^{-3t} - 3Ate^{-3t} - 6Ate^{-3t} \\ &= -5Ae^{-3t} = 4e^{-3t}, \end{aligned}$$

so that $A = -\frac{4}{5}$, and the particular solution is

$$y_p(t) = -\frac{4}{5}te^{-3t}.$$

D2.3.32 We recognize the general solution to $y'' + 4y = 0$ as $c_1 \cos 2t + c_2 \sin 2t$, so that $\cos 2t$ is a solution to the homogeneous equation. Thus we must use $At \cos 2t + Bt \sin 2t$ as our trial solution. Then

$$\begin{aligned} y(t) &= At \cos 2t + Bt \sin 2t \\ y'(t) &= A \cos 2t - 2At \sin 2t + B \sin 2t + 2Bt \cos 2t = (A + 2Bt) \cos 2t + (B - 2At) \sin 2t \\ y''(t) &= 2B \cos 2t - 2(A + 2Bt) \sin 2t - 2A \sin 2t + 2(B - 2At) \cos 2t \\ &= (-4At + 4B) \cos 2t + (-4Bt - 4A) \sin 2t, \end{aligned}$$

and we get

$$y''(t) + 4y(t) = 4B \cos 2t - 4A \sin 2t = \cos 2t,$$

so that $B = \frac{1}{4}$ and $A = 0$. Hence the particular solution is

$$y_p(t) = \frac{1}{4}t \sin 2t.$$

D2.3.33 The characteristic polynomial is $r^2 + 5r + 6 = (r + 2)(r + 3)$, so that $2e^{-2t}$ is a solution of the homogeneous equation. Thus we must use Ate^{-2t} as the trial solution. Then

$$\begin{aligned} y''(t) + 5y'(t) + 6y(t) &= (Ae^{-2t} - 2Ate^{-2t})' + 5(Ae^{-2t} - 2Ate^{-2t}) + 6Ate^{-2t} \\ &= -2Ae^{-2t} - 2Ae^{-2t} + 4Ate^{-2t} + 5Ae^{-2t} - 10Ate^{-2t} + 6Ate^{-2t} \\ &= Ae^{-2t} = 2e^{-2t}, \end{aligned}$$

so that $A = 2$. Hence the particular solution is

$$y_p(t) = 2te^{-2t}.$$

D2.3.34 The characteristic polynomial is $r^2 + 2r + 1 = (r + 1)^2$, so that $4e^{-t}$ and te^{-t} are both solutions of the homogeneous equation. Thus we must use At^2e^{-t} as the trial solution. Then

$$\begin{aligned} y''(t) + 2y'(t) + y(t) &= (2Ate^{-t} - At^2e^{-t})' + 2(2Ate^{-t} - At^2e^{-t}) + At^2e^{-t} \\ &= 2Ae^{-t} - 2Ate^{-t} - 2Ate^{-t} + At^2e^{-t} + 4Ate^{-t} - 2At^2e^{-t} + At^2e^{-t} \\ &= 2Ae^{-t} = 4e^{-t}. \end{aligned}$$

Then $A = 2$, so that the particular solution is

$$y_p(t) = 2t^2e^{-t}.$$

D2.3.35

- False. You must use the most general polynomial of the highest degree occurring on the right-hand side; in this case $At^3 + Bt^2 + Ct + D$.
- False. You should use the trial solution $A \sin t + B \cos t$.
- True. The characteristic polynomial of the homogeneous equation is $r^2 + 10r + 25 = (r + 5)^2$, so that the linearly independent solutions are e^{-5t} and te^{-5t} . Because e^{5t} is not a solution of the homogeneous equation, we can use Ae^{5t} as a trial solution.

D2.3.36 The characteristic polynomial is $r^2 - 9 = (r + 3)(r - 3)$, so that the general solution to the homogeneous equation is $c_1e^{3t} + c_2e^{-3t}$. To find a particular solution of the nonhomogeneous equation, then, we use a trial solution of the form Ae^{-t} . Then

$$y''(t) - 9y(t) = Ae^{-t} - 9Ae^{-t} = -8Ae^{-t} = 2e^{-t},$$

so that $A = -\frac{1}{4}$ and the particular solution is $-\frac{1}{4}e^{-t}$. Thus the general solution to the nonhomogeneous equation is

$$y(t) = c_1e^{3t} + c_2e^{-3t} - \frac{1}{4}e^{-t}.$$

Then $y(0) = c_1 + c_2 - \frac{1}{4} = 0$, so that $c_1 + c_2 = \frac{1}{4}$. Also, $y'(t) = 3c_1e^{3t} - 3c_2e^{-3t} + \frac{1}{4}e^{-t}$, so that $y'(0) = 3c_1 - 3c_2 + \frac{1}{4} = 0$ and thus $c_1 - c_2 = -\frac{1}{12}$. Solving this pair of equations gives $c_1 = \frac{1}{12}$ and $c_2 = \frac{1}{6}$, so that the solution to the initial value problem is

$$y(t) = \frac{1}{12}(e^{3t} + 2e^{-3t} - 3e^{-t}).$$

D2.3.37 The characteristic polynomial is $r^2 + 1 = (r + i)(r - i)$, so that the general solution to the homogeneous equation is $e^0(c_1 \cos t + c_2 \sin t) = c_1 \cos t + c_2 \sin t$. Thus $4 \sin 2t$ is not a solution to the homogeneous equation, so we use $A \sin 2t + B \cos 2t$ as the trial solution for the nonhomogeneous problem. Then

$$y''(t) + y(t) = -4A \sin 2t - 4B \cos 2t + A \sin 2t + B \cos 2t = -3A \sin 2t - 3B \cos 2t = 4 \sin 2t.$$

Thus $A = -\frac{4}{3}$ and $B = 0$, so that the particular solution is $-\frac{4}{3}\sin 2t$. Thus the general solution to the nonhomogeneous equation is

$$y(t) = c_1 \cos t + c_2 \sin t - \frac{4}{3} \sin 2t.$$

Then $y(0) = c_1 = 1$. Further, $y'(t) = -c_1 \sin t + c_2 \cos t - \frac{8}{3} \cos 2t$, so that $y'(0) = c_2 - \frac{8}{3} = 0$, and thus $c_2 = \frac{8}{3}$. Then the general solution to the initial value problem is

$$y(t) = \cos t + \frac{8}{3} \sin t - \frac{4}{3} \sin 2t.$$

D2.3.38 The characteristic polynomial is $r^2 + 3r + 2 = (r + 2)(r + 1)$ with roots -1 and -2 , so that the general solution to the homogeneous equation is $c_1 e^{-t} + c_2 e^{-2t}$. Thus $2e^{-3t}$ is not a solution to the homogeneous equation, so we use Ae^{-3t} as the trial solution for the nonhomogeneous problem. Then

$$y''(t) + 3y'(t) + 2y(t) = 9Ae^{-3t} + 3 \cdot (-3Ae^{-3t}) + 2Ae^{-3t} = 2Ae^{-3t} = 2e^{-3t},$$

so that $A = 1$. Thus the particular solution is e^{-3t} . Thus the general solution to the nonhomogeneous equation is

$$y(t) = c_1 e^{-t} + c_2 e^{-2t} + e^{-3t}.$$

Then $y(0) = c_1 + c_2 + 1 = 0$, so that $c_1 + c_2 = -1$. Further, $y'(t) = -c_1 e^{-t} - 2c_2 e^{-2t} - 3e^{-3t}$, so that $y'(0) = -c_1 - 2c_2 - 3 = 1$, so that $c_1 + 2c_2 = -4$. Solving this pair of equations gives $c_1 = 2$ and $c_2 = -3$, so that the solution to the initial value problem is

$$y(t) = 2e^{-t} - 3e^{-2t} + e^{-3t}.$$

D2.3.39 The characteristic polynomial is $r^2 + 4r + 5$, which has complex roots $-2 \pm i$, so that the general solution to the homogeneous equation is $y(t) = e^{-2t}(c_1 \cos t + c_2 \sin t)$. Because 12 is not a solution to the homogeneous equation, we use A as a trial solution. Then $A'' + 4A' + 5A = 12$, so that $A = \frac{12}{5}$ is the particular solution. Thus the general solution to the nonhomogeneous equation is

$$y(t) = e^{-2t}(c_1 \cos t + c_2 \sin t) + \frac{12}{5}.$$

Now, $y(0) = c_1 + \frac{12}{5} = 1$, so that $c_1 = -\frac{7}{5}$. Also,

$$\begin{aligned} y'(t) &= -2e^{-2t}(c_1 \cos t + c_2 \sin t) + e^{-2t}(-c_1 \sin t + c_2 \cos t) \\ &= e^{-2t}((c_2 - 2c_1) \cos t + (-2c_2 - c_1) \sin t), \end{aligned}$$

so that $y'(0) = c_2 - 2c_1 = -1$, and then $c_2 = -\frac{19}{5}$. Thus the solution to the initial value problem is

$$y(t) = -\frac{1}{5}e^{-2t}(7 \cos t + 19 \sin t) + \frac{12}{5}.$$

D2.3.40 The homogeneous equation has general solution $c_1 e^t + c_2 e^{-t}$. Because $2e^{-t} \sin t$ is not a solution of the homogeneous equation, we use $Ae^{-t} \sin t + Be^{-t} \cos t$ as a trial solution. Then

$$\begin{aligned} y''(t) - y(t) &= (-Ae^{-t} \sin t + Ae^{-t} \cos t - Be^{-t} \cos t - Be^{-t} \sin t)' - Ae^{-t} \sin t - Be^{-t} \cos t \\ &= Ae^{-t} \sin t - Ae^{-t} \cos t - Ae^{-t} \cos t - Ae^{-t} \sin t + Be^{-t} \cos t + Be^{-t} \sin t \\ &\quad + Be^{-t} \sin t - Be^{-t} \cos t - Ae^{-t} \sin t - Be^{-t} \cos t \\ &= (2B - A)e^{-t} \sin t + (-2A - B)e^{-t} \cos t = 2e^{-t} \sin t. \end{aligned}$$

Thus $2B - A = 2$ and $-2A - B = 0$, so that $A = -\frac{2}{5}$ and $B = \frac{4}{5}$, so that the general solution to the nonhomogeneous problem is

$$y(t) = c_1 e^t + c_2 e^{-t} + \frac{1}{5}e^{-t}(-2 \sin t + 4 \cos t).$$

Then $y(0) = c_1 + c_2 + \frac{4}{5} = 4$, so that $c_1 + c_2 = \frac{16}{5}$. Further,

$$y'(t) = c_1 e^t - c_2 e^{-t} - \frac{1}{5} e^{-t} (-2 \sin t + 4 \cos t) + \frac{1}{5} e^{-t} (-2 \cos t - 4 \sin t),$$

so that $y'(0) = c_1 - c_2 - \frac{4}{5} - \frac{2}{5} = 0$ and thus $c_1 - c_2 = \frac{6}{5}$. Solving this pair of equations gives $c_1 = \frac{11}{5}$ and $c_2 = 1$, so that the solution to the initial value problem is

$$y(t) = \frac{1}{5} (11e^t + 5e^{-t} + e^{-t} (-2 \sin t + 4 \cos t)).$$

D2.3.41 The general solution to the homogeneous equation is $c_1 \cos 3t + c_2 \sin 3t$. Because $6 \cos 3t$ is a solution to the homogeneous equation, we must use $At \cos 3t + Bt \sin 3t$ as the trial solution. Then

$$\begin{aligned} y''(t) &= (A \cos 3t - 3At \sin 3t + B \sin 3t + 3Bt \cos 3t)' \\ &= -3A \sin 3t - 3A \sin 3t - 9At \cos 3t + 3B \cos 3t + 3B \cos 3t - 9Bt \sin 3t \\ &= -6A \sin 3t + 6B \cos 3t - 9At \cos 3t - 9Bt \sin 3t. \end{aligned}$$

Then $y''(t) + 9y(t) = 6B \cos 3t - 6A \sin 3t = 6 \cos 3t$, so that $A = 0$ and $B = 1$. Thus the general solution to the nonhomogeneous equation is

$$y(t) = c_1 \cos 3t + c_2 \sin 3t + t \sin 3t.$$

Then $y(0) = c_1 = 0$, and $y'(t) = -3c_1 \sin 3t + 3c_2 \cos 3t + \sin 3t + 3t \cos 3t$, so that $y'(0) = 3c_2 = 0$. Thus only $c_1 = c_2 = 0$ satisfies the initial conditions, so that the general solution to the nonhomogeneous problem is

$$y(t) = t \sin 3t.$$

D2.3.42 The characteristic polynomial is $r^2 + 2r + 2$, which has roots $\frac{-2 \pm \sqrt{4-8}}{2} = -1 \pm i$. Thus $5e^{-2t} \cos t$ is not a solution of the homogeneous equation, so we use $Ae^{-2t} \cos t + Be^{-2t} \sin t$ as a trial solution. Then

$$\begin{aligned} y(t) &= Ae^{-2t} \cos t + Be^{-2t} \sin t \\ y'(t) &= -2Ae^{-2t} \cos t - Ae^{-2t} \sin t - 2Be^{-2t} \sin t + Be^{-2t} \cos t \\ &= (B - 2A)e^{-2t} \cos t + (-A - 2B)e^{-2t} \sin t \\ y''(t) &= -2(B - 2A)e^{-2t} \cos t - (B - 2A)e^{-2t} \sin t + 2(A + 2B)e^{-2t} \sin t + (-A - 2B)e^{-2t} \cos t \\ &= (3A - 4B)e^{-2t} \cos t + (4A + 3B)e^{-2t} \sin t, \end{aligned}$$

Thus

$$y''(t) + 2y'(t) + 2y(t) = (A - 2B)e^{-2t} \cos t + (2A + B)e^{-2t} \sin t = 5e^{-2t} \cos t,$$

so that $A - 2B = 5$ and $2A + B = 0$. Solving this pair of equations gives $A = 1$ and $B = -2$. Then the particular solution is

$$y_p(t) = e^{-2t} \cos t - 2e^{-2t} \sin t.$$

D2.3.43 The characteristic polynomial is $r^2 + r - 1$, which has distinct positive roots $\frac{-1 \pm \sqrt{5}}{2}$. Thus $3t^4 - 3t^3 + t$ is not a solution of the homogeneous equation, so we use $At^4 + Bt^3 + Ct^2 + Dt + E$ as a trial solution.

$$\begin{aligned} y''(t) + y'(t) - y(t) &= 12At^2 + 6Bt + 2C + 4At^3 + 3Bt^2 + 2Ct + D - At^4 - Bt^3 - Ct^2 - Dt - E \\ &= -At^4 + (4A - B)t^3 + (12A + 3B - C)t^2 + (6B + 2C - D)t + (2C + D - E) \\ &= 3t^4 - 3t^3 + t. \end{aligned}$$

This gives the system of equations

$$\begin{aligned} -A &= 3 \\ 4A - B &= -3 \\ 12A + 3B - C &= 0 \\ 6B + 2C - D &= 1 \\ 2C + D - E &= 0. \end{aligned}$$

Solving gives $A = -3$, $B = -9$, $C = -63$, $D = -181$, and $E = -307$, so that the particular solution is

$$y_p(t) = -3t^4 - 9t^3 - 63t^2 - 181t - 307.$$

D2.3.44 The characteristic polynomial is $r^2 + 2r + 5$, which has complex roots $-1 \pm 2i$, so that the general solution to the homogeneous problem is $e^{-t}(c_1 \cos 2t + c_2 \sin 2t)$. Thus $8e^{-t} \cos 2t$ is a solution to the homogeneous problem, so that we must use $Ate^{-t} \cos 2t + Bte^{-t} \sin 2t$ as our trial solution. Then

$$\begin{aligned} y(t) &= Ate^{-t} \cos 2t + Bte^{-t} \sin 2t \\ y'(t) &= Ae^{-t} \cos 2t - Ate^{-t} \cos 2t - 2Ate^{-t} \sin 2t + Be^{-t} \sin 2t - Bte^{-t} \sin 2t + 2Bte^{-t} \cos 2t \\ &= Ae^{-t} \cos 2t + (2B - A)te^{-t} \cos 2t + Be^{-t} \sin 2t + (-2A - B)te^{-t} \sin 2t \\ y''(t) &= -Ae^{-t} \cos 2t - 2Ae^{-t} \sin 2t + (2B - A)e^{-t} \cos 2t - (2B - A)te^{-t} \cos 2t \\ &\quad - 2(2B - A)te^{-t} \sin 2t - Be^{-t} \sin 2t + Be^{-t} \cos 2t + (-2A - B)e^{-t} \sin 2t \\ &\quad + (2A + B)te^{-t} \sin 2t + 2(-2A - B)te^{-t} \cos 2t \\ &= (-2A + 4B)e^{-t} \cos 2t + (-3A - 4B)te^{-t} \cos 2t + (-4A - 2B)e^{-t} \sin 2t \\ &\quad + (4A - 3B)te^{-t} \sin 2t. \end{aligned}$$

Then

$$y''(t) + 2y'(t) + 5y(t) = 4Be^{-t} \cos 2t - 4Ae^{-t} \sin 2t = 8e^{-t} \cos 2t,$$

so that $B = 2$ and $A = 0$, and the particular solution is

$$y_p(t) = 2te^{-t} \sin 2t.$$

D2.3.45 The characteristic polynomial is $r^2 - 1$, with roots ± 1 , so that the general solution to the homogeneous problem is $c_1 e^t + c_2 e^{-t}$. Thus we can use as a trial solution $(At + B)e^{-t} \sin 3t + (Ct + D)e^{-t} \cos 3t = e^{-t}((At + B) \sin 3t + (Ct + D) \cos 3t)$. Then

$$\begin{aligned} y(t) &= e^{-t}((At + B) \sin 3t + (Ct + D) \cos 3t) \\ y'(t) &= -e^{-t}((At + B) \sin 3t + (Ct + D) \cos 3t) + e^{-t}(A \sin 3t + 3(At + B) \cos 3t \\ &\quad + C \cos 3t - 3(Ct + D) \sin 3t) \\ &= e^{-t}[((-A - 3C)t + A - B - 3D) \sin 3t + ((3A - C)t + 3B + C - D) \cos 3t] \\ y''(t) &= -e^{-t}(((A - 3C)t + A - B - 3D) \sin 3t + ((3A - C)t + 3B + C - D) \cos 3t) \\ &\quad - e^{-t}[(-A - 3C) \sin 3t + 3((-A - 3C)t + A - B - 3D) \cos 3t \\ &\quad + (3A - C) \cos 3t - 3((3A - C)t + 3B + C - D) \sin 3t] \\ &= -2e^{-t}(((4A - 3C)t + A + 4B + 3C - 3D) \sin 3t \\ &\quad + ((3A + 4C)t - 3A + 3B + C + 4D) \cos 3t) \end{aligned}$$

Then

$$\begin{aligned} y''(t) - y(t) &= e^{-t}[((-9A + 6C)t - 2A - 9B - 6C + 6D) \sin 3t \\ &\quad + ((-6A - 9C)t + 6A - 6B - 2C - 9D) \cos 3t] \\ &= 25te^{-t} \sin 3t. \end{aligned}$$

This gives the system of equations

$$\begin{aligned} -9A &+ 6C &= 25 \\ -2A - 9B - 6C + 6D &= 0 \\ -6A &- 9C &= 0 \\ 6A - 6B - 2C - 9D &= 0. \end{aligned}$$

Solving these equations gives for the particular solution

$$y(t) = \frac{1}{1521} e^{-t} (-2925t \sin 3t - 1550 \sin 3t + 1950t \cos 3t - 1350 \cos 3t).$$

D2.3.46 Because y itself does not appear in the homogeneous equation, in order to get an $8t^3$ on the right-hand side, our trial solution must involve a quartic polynomial. Now, the characteristic polynomial of the homogeneous equation is $r^3 - r = r(r+1)(r-1)$, so that the linearly independent solutions of the homogeneous equation are 1 , e^t , and e^{-t} . Thus no quartic is a solution for the homogeneous equation, so we use $At^4 + Bt^3 + Ct^2 + Dt + E$ as our trial solution. Then

$$\begin{aligned} y'''(t) - y'(t) &= 24At + 6B - (4At^3 + 3Bt^2 + 2Ct + D) \\ &= -4At^3 - 3Bt^2 + (24A - 2C)t + (6B - D) = 8t^3. \end{aligned}$$

Equating coefficients and solving gives $A = -2$, $B = 0$, $C = -24$, and $D = 0$, so that the particular solution is

$$y_p(t) = -2t^4 - 24t^2.$$

D2.3.47 The characteristic polynomial of the homogeneous equation is $r^3 - 8 = (r-2)(r^2 + 2r + 4)$, so that the linearly independent solutions of the homogeneous equation are e^{2t} , $e^{-t} \cos \sqrt{3}t$, and $e^{-t} \sin \sqrt{3}t$. Thus e^t is not a solution of the homogeneous equation, so we use Ae^t as a trial solution. Then

$$y'''(t) - 8y(t) = Ae^t - 8Ae^t = -7Ae^t = 7e^t,$$

so that $A = -1$. Thus the particular solution is

$$y_p(t) = -e^t.$$

D2.3.48 The characteristic polynomial of the homogeneous equation is $r^4 - 1 = (r-1)(r+1)(r-i)(r+i)$, so that the linearly independent solutions of the homogeneous equation are $e^{\pm t}$, $\sin t$ and $\cos t$. Thus $5 \sin 2t$ is not a solution of the homogeneous equation, so we use $A \sin 2t + B \cos 2t$ as a trial solution. Then

$$y^{\text{iv}}(t) - y(t) = 16A \sin 2t + 16B \cos 2t - A \sin 2t - B \cos 2t = 15A \sin 2t + 15B \cos 2t = 5 \sin 2t,$$

so that $A = \frac{1}{3}$ and $B = 0$. Then the particular solution is

$$y_p(t) = \frac{1}{3} \sin 2t.$$

D2.3.49 The characteristic polynomial of the homogeneous equation is $r^4 - 3r^2 + 2 = (r^2 - 1)(r^2 - 2)$, so that the linearly independent solutions are $e^{\pm t}$ and $e^{\pm t\sqrt{2}}$. Because $6te^{2t}$ is not a solution of the homogeneous equation, we use $(At + B)e^{2t}$ as a trial solution. Then

$$\begin{aligned} y(t) &= (At + B)e^{2t} \\ y'(t) &= Ae^{2t} + 2(At + B)e^{2t} = (2At + A + 2B)e^{2t} \\ y''(t) &= 2Ae^{2t} + 2(2At + A + 2B)e^{2t} = (4At + 4A + 4B)e^{2t} \\ y'''(t) &= 4Ae^{2t} + 2(4At + 4A + 4B)e^{2t} = (8At + 12A + 8B)e^{2t} \\ y^{\text{iv}}(t) &= 8Ae^{2t} + 2(8At + 12A + 8B)e^{2t} = (16At + 32A + 16B)e^{2t}. \end{aligned}$$

Thus

$$y^{\text{iv}}(t) - 3y''(t) + 2y(t) = (6At + 20A + 6B)e^{2t} = 6te^{2t}.$$

Equating coefficients and solving gives $A = 1$ and $B = -\frac{10}{3}$, so that the particular solution is

$$(t - \frac{10}{3})e^{2t}.$$

D2.3.50

- a. The characteristic polynomial of the homogeneous equation is $r^2 + \omega_0^2$, which has roots $\pm\omega_0 i$. Thus the general solution of the homogeneous equation is $c_1 \sin \omega_0 t + c_2 \cos \omega_0 t$. Because we are assuming that $\omega \neq \omega_0$, $a \cos \omega t$ is not a solution of the homogeneous equation, so we can use $A \sin \omega t + B \cos \omega t$ as a trial solution. Then

$$\begin{aligned} y''(t) + \omega_0^2 y(t) &= -\omega^2 A \sin \omega t - \omega^2 B \cos \omega t + A\omega_0^2 \sin \omega t + B\omega_0^2 \cos \omega t \\ &= A(\omega_0^2 - \omega^2) \sin \omega t + B(\omega_0^2 - \omega^2) \cos \omega t = a \cos \omega t. \end{aligned}$$

Solving gives

$$A = 0, \quad B = \frac{a}{\omega_0^2 - \omega^2},$$

so that the general solution to the nonhomogeneous equation is

$$y(t) = c_1 \sin \omega_0 t + c_2 \cos \omega_0 t + \frac{a}{\omega_0^2 - \omega^2} \cos \omega t.$$

- b. From part (a), $y(0) = c_2 + \frac{a}{\omega_0^2 - \omega^2} = 0$, so that

$$c_2 = -\frac{a}{\omega_0^2 - \omega^2}.$$

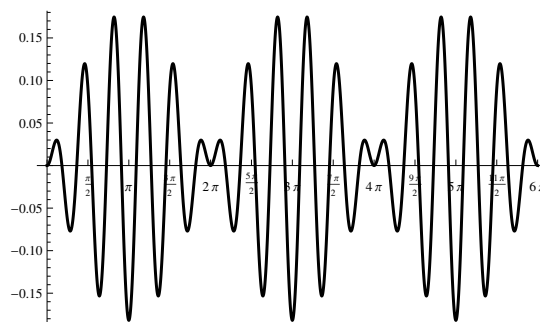
Further,

$$y'(t) = \omega_0 c_1 \cos \omega_0 t - \omega_0 c_2 \sin \omega_0 t - \omega \frac{a}{\omega_0^2 - \omega^2} \sin \omega t,$$

so that $y'(0) = \omega_0 c_1 = 0$, and thus $c_1 = 0$. Hence the solution to the initial value problem is

$$y(t) = -\frac{a}{\omega_0^2 - \omega^2} \cos \omega_0 t + \frac{a}{\omega_0^2 - \omega^2} \cos \omega t = \frac{a}{\omega_0^2 - \omega^2} (\cos \omega t - \cos \omega_0 t). \quad (17.3)$$

- c. A graph for $\omega = 5$ and $\omega_0 = 6$, with $a = 1$ is



The behavior is periodic with period 2π . Within each period, the curve oscillates with amplitude that is small at first, increases to a maximum, and then decreases again.

- d. Substituting the given identity into (17.3) gives

$$y = -\frac{a}{\omega_0^2 - \omega^2} \left(2 \sin \frac{(\omega_0 - \omega)t}{2} \sin \frac{(\omega_0 + \omega)t}{2} \right) = -\frac{2a}{\omega_0^2 - \omega^2} \sin \frac{(\omega_0 - \omega)t}{2} \sin \frac{(\omega_0 + \omega)t}{2}.$$

- e. The curve is a product of two sine functions, one with a frequency of $\frac{4\pi}{\omega_0 - \omega}$ and the other with a frequency of $\frac{4\pi}{\omega_0 + \omega}$. Thus as $\omega \rightarrow \omega_0$, the frequency of the first sine wave approaches ∞ , while the frequency of the second approaches $\frac{2\pi}{\omega_0}$. So the appearance of the curve is that of a sine wave of frequency $\frac{2\pi}{\omega_0}$ inside an envelope that is in fact a sine wave of a much longer frequency, approaching ∞ as $\omega \rightarrow \omega_0$.

D2.3.51 Suppose that y_{p1} is a solution of $y'' + py' + qy = f(t)$ and that y_{p2} is a solution of $y'' + py' + qy = g(t)$. Then

$$\begin{aligned}(y_{p1} + y_{p2})'' + p(y_{p1} + y_{p2})' + q(y_{p1} + y_{p2}) &= y_{p1}'' + y_{p2}'' + py_{p1}' + y_{p2}' + qy_{p1} + qy_{p2} \\ &= (y_{p1}'' + py_{p1}' + qy_{p1}) + (y_{p2}'' + py_{p2}' + qy_{p2}) \\ &= f(t) + g(t).\end{aligned}$$

Thus $y_{p1} + y_{p2}$ is a solution of $y'' + py' + qy = f(t) + g(t)$.

D2.3.52

- a. The homogeneous equation has characteristic polynomial $r^2 - 1 = (r + 1)(r - 1)$, so that the linearly independent solutions are e^t and e^{-t} . Thus the right-hand side of the nonhomogeneous equation, e^t , is a solution of the homogeneous equation.
- b. Assume that $y_p(t) = u_1(t)e^t + u_2(t)e^{-t}$. Assuming that $u_1'e^t + u_2'e^{-t} = 0$, we compute

$$y_p' = u_1'e^t + u_1e^t + u_2'e^{-t} - u_2e^{-t} = u_1e^t - u_2e^{-t}.$$

- c. Differentiating again gives

$$y_p'' = u_1'e^t + u_1e^t - u_2'e^{-t} + u_2e^{-t}.$$

Substituting into the differential equation gives

$$\begin{aligned}y_p'' - y_p &= e^t \\ u_1'e^t + u_1e^t - u_2'e^{-t} + u_2e^{-t} - u_1e^t - u_2e^{-t} &= e^t \\ u_1'e^t - u_2'e^{-t} &= e^t\end{aligned}$$

- d. We have a pair of equations for u_1' and u_2' , one from the assumed condition in part (b) and one from the last equation above:

$$\begin{aligned}e^t u_1' + e^{-t} u_2' &= 0 \\ e^t u_1' - e^{-t} u_2' &= e^t.\end{aligned}$$

Add the two equations together to get $2e^t u_1' = e^t$, so that $u_1' = \frac{1}{2}$.

- e. Integrate both sides to get $u_1 = \frac{1}{2}t + C_1$.
- f. Substitute $u_1' = \frac{1}{2}$ into the first of the simultaneous equations above and rearrange terms to get $e^{-t} u_2' = -\frac{1}{2}e^t$, so that $u_2' = -\frac{1}{2}e^{2t}$.
- g. Integrate both sides to get $u_2 = -\frac{1}{4}e^{2t} + C_2$.
- h. Plug the formulas for u_1 and u_2 into the formula for $y_p(t)$ to get

$$\begin{aligned}y_p(t) &= u_1e^t + u_2e^{-t} \\ &= \left(\frac{1}{2}t + C_1\right)e^t + \left(-\frac{1}{4}e^{2t} + C_2\right)e^{-t} \\ &= \frac{1}{2}te^t - \frac{1}{4}e^t + C_1e^t + C_2e^{-t}.\end{aligned}$$

The last two terms in the above are also solutions of the homogeneous equation, so can be ignored in the particular solution. Thus we get for a particular solution

$$y_p(t) = \frac{1}{2}te^t - \frac{1}{4}e^t.$$

(Note that $-\frac{1}{4}e^t$ is also a solution of the homogeneous equation, so that in fact it can be ignored as well if desired, to get a particular solution of $\frac{1}{2}te^t$).

Applications

D2.4.1 A system is *damped* if there is a force depending on velocity that affects the motion (for example, friction or resistance). It is *undamped* if there is no such force. A system is *forced* if there is an external force affecting the motion of the system (this results in a nonhomogeneous equation). A system is *free* if there is no such force, resulting in a homogeneous equation.

D2.4.2 In the free case, solutions for damped oscillations may take three distinct forms depending on the relative sizes of the damping and restoring forces. Such systems do not appear often in the real world since there is almost always some form of damping, like friction or electrical resistance. In the forced case, the solution depends on the form of the damping force, but in general the general solution to the corresponding free (homogeneous) system decays over time, so that as $t \rightarrow \infty$, the solution to the system approaches a particular solution of the forced system.

D2.4.3 Beats occur in the forced undamped case, when the natural frequency of the system and the frequency of the external force are close to each other.

D2.4.4 Resonance refers to the situation where the natural frequency of the system exactly matches the frequency of the external force. The result is that the amplitude of the solution increases without bound over time.

D2.4.5 For forced damped oscillations, the general form of the equation is $y'' + py' + qy = F_0(t)$ where $p > 0$. To solve this equation we first solve the corresponding homogeneous equation, then find a particular solution of the nonhomogeneous equation. The result is always a transient solution corresponding to the solution of the homogeneous equation, which goes to zero over time, plus a steady-state solution corresponding to the solution of the nonhomogeneous equation.

D2.4.6 The two situations are completely analogous, including the general form of the differential equations, except for terminology. The relationships between the terms used in each field are given in Table D2.3.

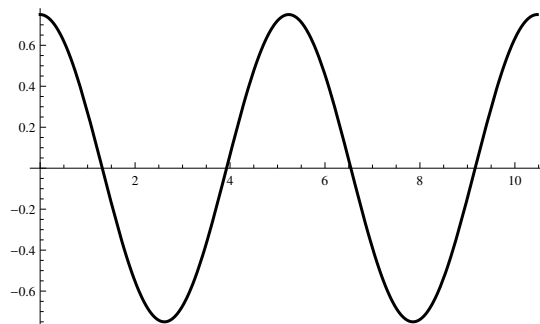
D2.4.7 This is a free undamped system; the equation has the form $y'' + \omega_0^2 y = 0$ where $\omega_0^2 = \frac{k}{m} = 1.44$. The general solution of this system is

$$y = c_1 \sin 1.2t + c_2 \cos 1.2t,$$

since $\sqrt{1.44} = 1.2$. Because $y(0) = c_2 = 0.75$, we get $c_2 = 0.75$. Also, $y'(t) = 1.2c_1 \cos 1.2t - 1.2c_2 \sin 1.2t$, so that $y'(0) = 1.2c_1 = 0$, so that $c_1 = 0$. Thus the solution to this system is

$$y(t) = 0.75 \cos 1.2t.$$

A graph of the solution over two periods is



The block oscillates with an amplitude of 0.75 and a period of $\frac{2\pi}{1.2} \approx 5.24$.

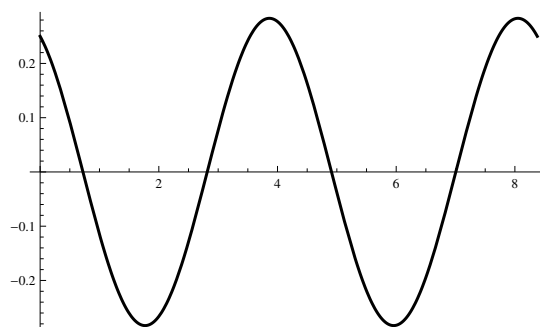
D2.4.8 This is a free undamped system; the equation has the form $y'' + \omega^2 y = 0$ where $\omega^2 = \frac{k}{m} = \frac{3.375}{1.5} = 2.25$, so that $\omega = 1.5$. The general solution of this system is

$$y(t) = c_1 \sin 1.5t + c_2 \cos 1.5t.$$

Now, $y(0) = c_2 = 0.25$, so that $c_2 = 0.25$. Also, $y'(t) = 1.5c_1 \cos 1.5t - 1.5c_2 \sin 1.5t$, so that $y'(0) = 1.5c_1 = -0.2$. Thus $c_1 \approx -0.133$, so the solution to this system is

$$y(t) = -0.133 \sin 1.5t + 0.25 \cos 1.5t.$$

A graph of the solution over two periods is



The block oscillates with period $\frac{2\pi}{1.5} \approx 4.189$, and the amplitude is $\sqrt{0.133^2 + 0.25^2} \approx 0.283$.

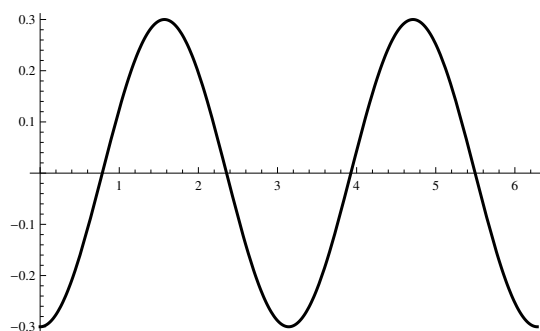
D2.4.9 The force exerted by the block when it is at rest is $mg = 2 \cdot 9.8 = 19.6$ N. By Hooke's law, this force is equal to ky , where y is the distance the spring is stretched when the block is attached. Thus $2.45k = 19.6$, so that $k = 8$. The block is then released at -0.3 m. This is a free undamped system, so the equation is $y'' + \omega_0^2 y = 0$, where $\omega_0^2 = \frac{k}{m} = 4$. Thus $\omega_0 = 2$. The general solution to this system is

$$y(t) = c_1 \sin 2t + c_2 \cos 2t.$$

Now, $y(0) = c_2 = -0.3$, so that $c_2 = -0.3$. Further, $y'(t) = 2c_1 \cos 2t - 2c_2 \sin 2t$, so that $y'(0) = 2c_1 = 0$ and thus $c_1 = 0$. Hence the solution to the system is

$$y(t) = -0.3 \cos 2t.$$

A graph of the solution over two periods is



The block oscillates with period $\frac{2\pi}{2} = \pi \approx 3.142$, and the amplitude is 0.3.

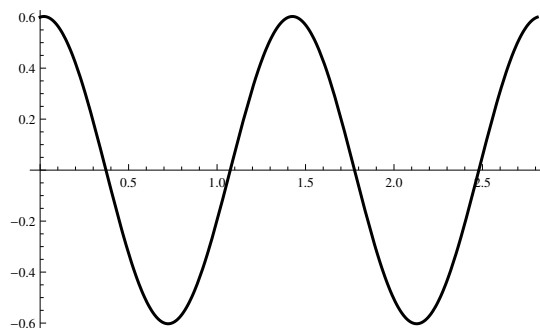
D2.4.10 The force exerted by the block when it is at rest is $mg = 0.5 \cdot 9.8 = 4.9$ N. By Hooke's law, this force is equal to ky , where y is the distance the spring is stretched when the block is attached. Thus $0.49k = 4.9$, so that $k = 10$. The block is then released at 0.6 m. This is a free undamped system, so the equation is $y'' + \omega_0^2 y = 0$, where $\omega_0^2 = \frac{k}{m} = 20$. Thus $\omega_0 = 2\sqrt{5}$. The general solution to this system is

$$y(t) = c_1 \sin 2\sqrt{5}t + c_2 \cos 2\sqrt{5}t.$$

Now, $y(0) = c_2 = 0.6$, so that $c_2 = 0.6$. Further, $y'(t) = 2\sqrt{5}c_1 \cos 2\sqrt{5}t - 2\sqrt{5}c_2 \sin 2\sqrt{5}t$, so that $y'(0) = 2\sqrt{5}c_1 = 0.25$ and thus $c_1 = \frac{1}{8\sqrt{5}} \approx 0.056$. Hence the solution to the system is

$$y(t) = 0.056 \sin 2\sqrt{5}t + 0.6 \cos 2\sqrt{5}t.$$

A graph of the solution over two periods is



The block oscillates with period $\frac{2\pi}{2\sqrt{5}} \approx 1.405$, and the amplitude is ≈ 0.603 .

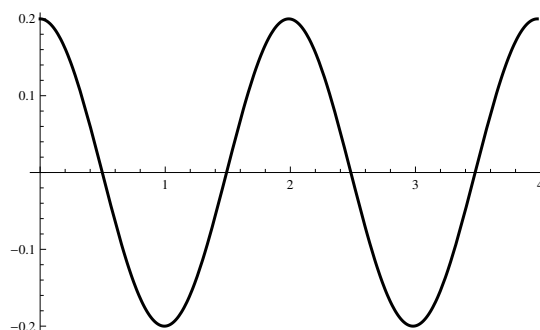
D2.4.11 To analyze this problem, we assume that “above” equilibrium means angles to the right of vertical, and “below” means angles to the left. The general solution to the equation $\theta'' + \omega_0^2 \theta = 0$, with $\omega^2 = \frac{g}{l} = \frac{9.8}{0.98} = 10$, is

$$y(t) = c_1 \sin \sqrt{10}t + c_2 \cos \sqrt{10}t.$$

Because the initial position of the pendulum is 0.2, we have $y(0) = c_2 = 0.2$, so that $c_2 = 0.2$. Further, $y'(t) = \sqrt{10}c_1 \cos \sqrt{10}t - \sqrt{10}c_2 \sin \sqrt{10}t$, so that $y'(0) = \sqrt{10}c_1 = 0$ and thus $c_1 = 0$. Hence the solution to the system is

$$y(t) = 0.2 \cos \sqrt{10}t.$$

A graph of the solution over two periods is



The pendulum oscillates with period $\frac{2\pi}{\sqrt{10}} \approx 1.987$, and the amplitude is 0.2.

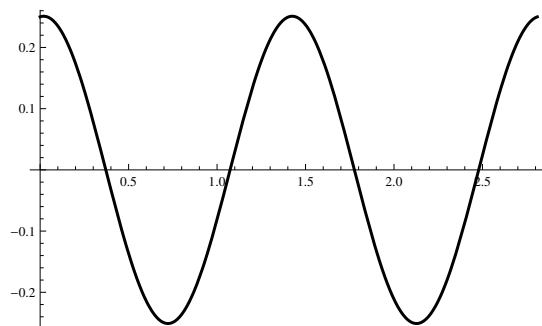
D2.4.12 To analyze this problem, we assume that “above” equilibrium means angles to the right of vertical, and “below” means angles to the left. The general solution to the equation $\theta'' + \omega_0^2 \theta = 0$, with $\omega^2 = \frac{g}{l} = \frac{9.8}{0.49} = 20$, is

$$y(t) = c_1 \sin 2\sqrt{5}t + c_2 \cos 2\sqrt{5}t.$$

Because the initial position of the pendulum is 0.25, we have $y(0) = c_2 = 0.25$, so that $c_2 = 0.25$. Further, $y'(t) = 2\sqrt{5}c_1 \cos 2\sqrt{5}t - 2\sqrt{5}c_2 \sin 2\sqrt{5}t$, so that $y'(0) = 2\sqrt{5}c_1 = 0.1$ and thus $c_1 = \frac{0.1}{2\sqrt{5}} \approx 0.0224$. Hence the solution to the system is

$$y(t) = 0.0224 \sin 2\sqrt{5}t + 0.25 \cos 2\sqrt{5}t.$$

A graph of the solution over two periods is



The pendulum oscillates with period $\frac{2\pi}{\sqrt{5}} \approx 1.405$, and the amplitude is ≈ 0.25 .

D2.4.13 This is a system with forced undamped oscillations and an external force in the form of a cosine curve, so the equation is

$$y'' + \omega_0^2 y = \frac{F_0}{m} \cos \omega t$$

where m is the mass, F_0 is the amplitude of the external force, ω_0 is the natural frequency of the system, and ω is the frequency of the external force. The general solution to this equation is

$$y(t) = c_1 \sin \omega_0 t + c_2 \cos \omega_0 t + \frac{F_0/m}{\omega_0^2 - \omega^2} \cos \omega t.$$

In our problem, $F_0 = 4$, $m = 2$ and $\omega_0^2 = \frac{k}{m} = \frac{5.12}{2} = 2.56$, so that $\omega_0 = 1.6$. Then by the analysis in the text of the general solution to this equation, we have

$$y(t) = c_1 \sin 1.6t + c_2 \cos 1.6t + \frac{2}{2.56 - \omega^2} \cos \omega t.$$

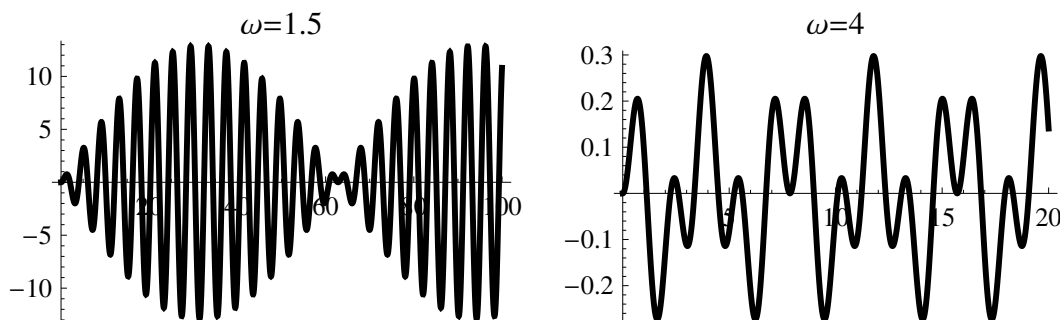
Because $y(0) = c_2 + \frac{2}{2.56 - \omega^2} = 0$, we get $c_2 = -\frac{2}{2.56 - \omega^2}$. Also,

$$y'(t) = 1.6c_1 \cos 1.6t - 1.6c_2 \sin 1.6t - \frac{2\omega}{2.56 - \omega^2} \sin \omega t,$$

so that $y'(0) = 1.6c_1 = 0$ and thus $c_1 = 0$. So the solution to the system is

$$y(t) = \frac{2}{2.56 - \omega^2} (\cos \omega t - \cos 1.6t).$$

Graphs for $\omega = 1.5$ (for $0 \leq t \leq 100$) and for $\omega = 4$ (for $0 \leq t \leq 20$) are:



This clearly shows beats for $\omega = 1.5$ and no beats for $\omega = 4$.

D2.4.14 As in the previous exercise, the general solution for this situation is

$$y(t) = c_1 \sin \omega_0 t + c_2 \cos \omega_0 t + \frac{F_0/m}{\omega_0^2 - \omega^2} \cos \omega t.$$

In this problem, $F_0 = 2$, $m = 0.5$ and $\omega_0^2 = \frac{k}{m} = \frac{4.5}{0.5} = 9$, so that $\omega_0 = 3$, and we have

$$y(t) = c_1 \sin 3t + c_2 \cos 3t + \frac{4}{9 - \omega^2} \cos \omega t.$$

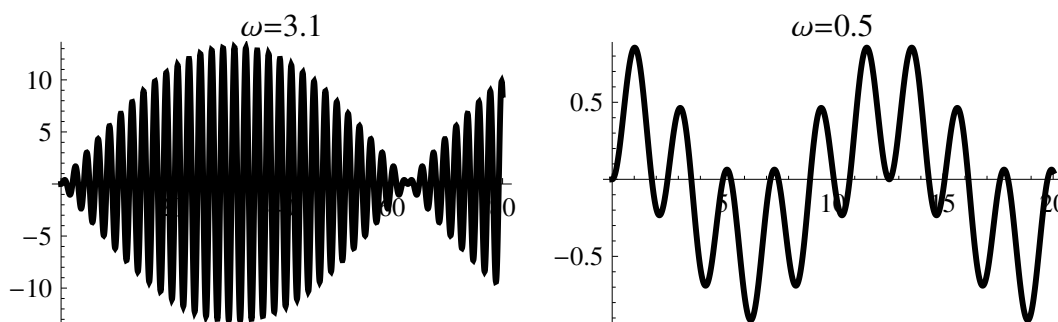
Because $y(0) = c_2 + \frac{4}{9 - \omega^2} = 0$, we get $c_2 = -\frac{4}{9 - \omega^2}$. Also,

$$y'(t) = 3c_1 \cos 3t - 3c_2 \sin 3t - \frac{4\omega}{9 - \omega^2} \sin \omega t,$$

so that $y'(0) = 3c_1 = 0$ and thus $c_1 = 0$. So the solution to the system is

$$y(t) = \frac{4}{9 - \omega^2} (\cos \omega t - \cos 3t).$$

Graphs for $\omega = 3.1$ (for $0 \leq t \leq 80$) and for $\omega = 0.5$ (for $0 \leq t \leq 20$) are:



This clearly shows beats for $\omega = 3.1$ and no beats for $\omega = 0.5$.

D2.4.15 The equation to be solved is

$$y'' + \omega_0^2 y = \frac{F_0}{m} \sin \omega t,$$

where $F_0 = 1$, $m = 0.25$, and $\omega_0^2 = \frac{k}{m} = \frac{4}{0.25} = 16$, so that $\omega_0 = 4$. Thus the equation is

$$y'' + 16y = 4 \sin \omega t.$$

The general solution to the homogeneous equation is $c_1 \sin 4t + c_2 \cos 4t$. In the case where $\omega \neq 4$, so that $4 \sin \omega t$ is not a solution of the homogeneous equation, we use a trial solution of $A \sin \omega t + B \cos \omega t$; then

$$y'' + 16y = (16A - A\omega^2) \sin \omega t + (16B - B\omega^2) \cos \omega t = 4 \sin \omega t,$$

so that $A = \frac{4}{16 - \omega^2}$ and $B = 0$. Thus the general solution is

$$y(t) = c_1 \sin 4t + c_2 \cos 4t + \frac{4}{16 - \omega^2} \sin \omega t.$$

Then $y(0) = c_2 = 0$, so that $c_2 = 0$. Also,

$$y'(t) = 4c_1 \cos 4t - 4c_2 \sin 4t + \frac{4\omega}{16 - \omega^2} \cos \omega t,$$

so that $y'(0) = 4c_1 + \frac{4\omega}{16 - \omega^2} = 0$, so that $c_1 = -\frac{\omega}{\omega^2 - 16}$, and then the solution to the system is (for $\omega \neq 4$)

$$y(t) = \frac{\omega}{\omega^2 - 16} \sin 4t + \frac{4}{16 - \omega^2} \sin \omega t = \frac{\omega \sin 4t - 4 \sin \omega t}{\omega^2 - 16}. \quad (17.4)$$

When $\omega = 4$, the trial solution must be $At \sin 4t + Bt \cos 4t$, and then

$$\begin{aligned} y'' + 16y &= 8A \cos 4t - 16At \sin 4t - 8B \sin 4t - 16Bt \cos 4t + 16At \sin 4t + 16Bt \cos 4t \\ &= 8A \cos 4t - 8B \sin 4t = 4 \sin 4t, \end{aligned}$$

so that $A = 0$ and $B = -\frac{1}{2}$. Then the general solution is

$$y(t) = c_1 \sin 4t + c_2 \cos 4t - \frac{1}{2}t \cos 4t.$$

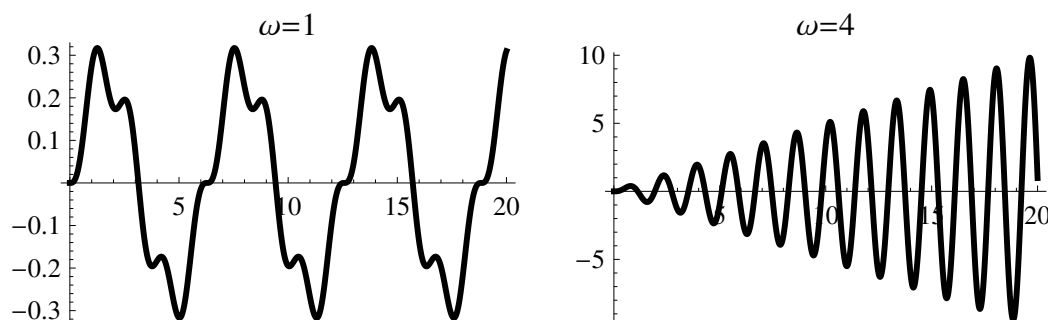
Then $y(0) = c_2 = 0$, so that $c_2 = 0$, and also

$$y'(t) = 4c_1 \cos 4t - 4c_2 \sin 4t - \frac{1}{2}(\cos 4t - 4t \sin 4t),$$

so that $y'(0) = 4c_1 - \frac{1}{2} = 0$ and thus $c_1 = \frac{1}{8}$. Thus the solution in the case $\omega = 4$ is

$$y(t) = \frac{1}{8} \sin 4t - \frac{1}{2}t \cos 4t. \quad (17.5)$$

Plots of (17.4) for $\omega = 1$, and of (17.5), on $[0, 20]$ are below:



The case $\omega = \omega_0 = 4$ exhibits resonance.

D2.4.16 From the work in the text, we know that the general solutions are

$$y(t) = c_1 \sin \omega_0 t + c_2 \cos \omega_0 t + \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos \omega t, \quad \omega \neq \omega_0,$$

$$y(t) = c_1 \sin \omega_0 t + c_2 \cos \omega_0 t + \frac{F_0}{2m\omega_0} t \sin \omega_0 t, \quad \omega = \omega_0.$$

For this problem, $F_0 = 9$, $m = \frac{4}{3}$, and $k = 12$, so that $\omega_0^2 = \frac{k}{m} = 9$ and $\omega_0 = 3$. The two solutions are thus

$$y(t) = c_1 \sin 3t + c_2 \cos 3t + \frac{27}{4(9 - \omega^2)} \cos \omega t, \quad \omega \neq 3,$$

$$y(t) = c_1 \sin 3t + c_2 \cos 3t + \frac{9}{8}t \sin 3t, \quad \omega = 3.$$

We also have

$$y'(t) = 3c_1 \cos 3t - 3c_2 \sin 3t - \frac{27\omega}{4(9 - \omega^2)} \sin \omega t, \quad \omega \neq 3,$$

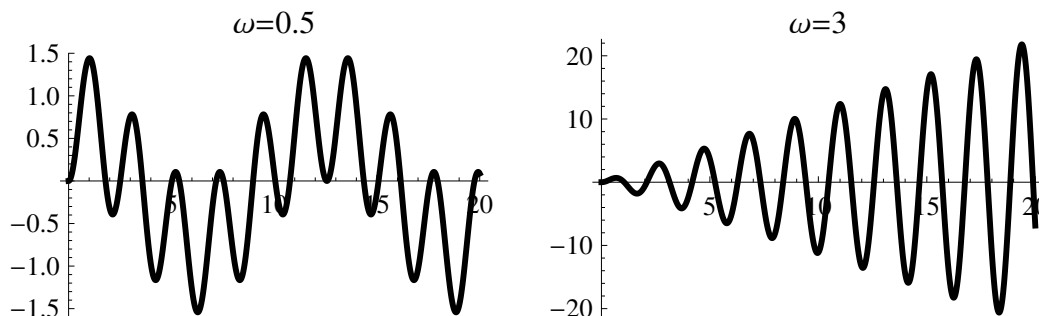
$$y'(t) = 3c_1 \cos 3t - 3c_2 \sin 3t + \frac{9}{8}(\sin 3t + t \cos 3t), \quad \omega = 3.$$

Then for $\omega \neq 3$, we have $y(0) = c_2 + \frac{27}{4(9-\omega^2)} = 0$, so that $c_2 = -\frac{27}{4(9-\omega^2)}$, and $y'(0) = 3c_1 = 0$. For $\omega = 3$, we have $y(0) = c_2 = 0$, so that $c_2 = 0$, and also $y'(0) = 3c_1 = 0$, so that $c_1 = 0$. Hence the two solutions to the system are

$$y(t) = \frac{27}{4(9-\omega^2)}(\cos \omega t - \cos 3t), \quad \omega \neq 3,$$

$$y(t) = \frac{9}{8}t \sin 3t, \quad \omega = 3.$$

A plot of the first of these for $\omega = 0.5$, and of the second, both on $[0, 20]$, is



The case $\omega = \omega_0 = 3$ exhibits resonance.

D2.4.17 From the text, the general differential equation for this situation is

$$y'' + \frac{c}{m}y' + \omega^2 y = 0,$$

where in our case $m = 0.3$ and $\omega^2 = \frac{k}{m} = \frac{30}{0.3} = 100$, so that $\omega = 10$. Thus the equation is

$$y'' + \frac{10c}{3}y' + 100y = 0.$$

The characteristic polynomial is $r^2 + \frac{10c}{3}r + 100$, whose roots are the same as the roots of $3r^2 + 10cr + 300$, which are

$$\frac{-10c \pm \sqrt{100c^2 - 3600}}{6} = -\frac{5c}{3} \pm \frac{5\sqrt{c^2 - 36}}{3}.$$

- a. When $c = 4.8$, we have $\sqrt{c^2 - 36} = \sqrt{23.04 - 36} = \sqrt{-12.96} = 3.6i$, so that the roots of the characteristic polynomial are the complex conjugates

$$-\frac{5 \cdot 4.8}{3} \pm \frac{18i}{3} = -8 \pm 6i,$$

and thus the general solution is

$$y(t) = e^{-8t}(c_1 \cos 6t + c_2 \sin 6t).$$

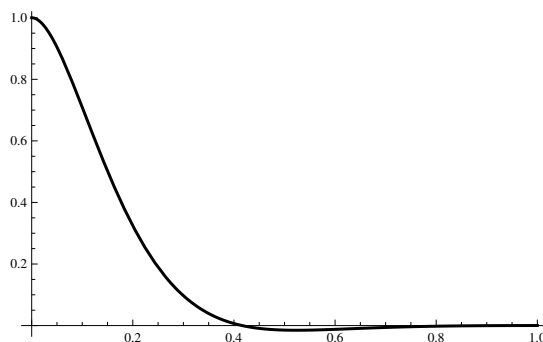
Then $y(0) = c_1 = 1$, so that $c_1 = 1$. Further,

$$y'(t) = (e^{-8t}(\cos 6t + c_2 \sin 6t))' = -8e^{-8t}(\cos 6t + c_2 \sin 6t) + e^{-8t}(-6 \sin 6t + 6c_2 \cos 6t),$$

so that $y'(0) = -8 + 6c_2 = 0$. Thus $c_2 = \frac{4}{3}$, and the solution to the system with $c = 4.8$ is

$$y(t) = e^{-8t}(\cos 6t + \frac{4}{3} \sin 6t)$$

with graph



Because the discriminant was negative, this is an underdamped system. There are visible oscillations in the solution, but they damp out pretty quickly because of the high multiple of t in the exponential.

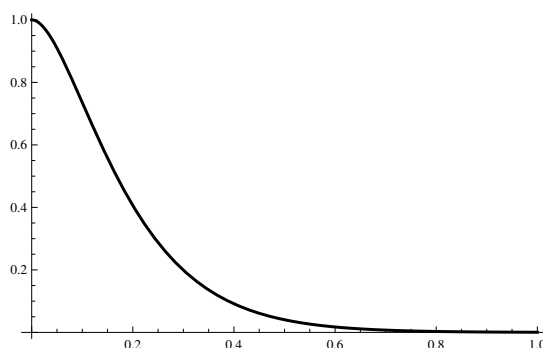
- b. With $c = 6$, we have $\sqrt{c^2 - 36} = 0$, so that the characteristic polynomial has the repeated root $r = -\frac{5 \cdot 6}{3} = -10$. In this case the general solution is

$$y(t) = c_1 e^{-10t} + c_2 t e^{-10t}.$$

Then $y(0) = c_1 = 1$, so that $c_1 = 1$. Further, $y'(t) = -10c_1 e^{-10t} + c_2 e^{-10t} - 10c_2 t e^{-10t}$, so that $y'(0) = -10c_1 + c_2 = 0$ and thus $c_2 = 10$. Hence the solution to the system when $c = 6$ is

$$y(t) = e^{-10t} + 10t e^{-10t}$$

with graph



Because the discriminant is zero, this system has critical damping.

- c. With $c = 7.5$, we have $\sqrt{56.25 - 36} = \sqrt{20.25} = 4.5$, so that the characteristic polynomial has the distinct roots

$$-\frac{5 \cdot 7.5}{3} \pm \frac{5 \cdot 4.5}{3} = -12.5 \pm 7.5 = \{-5, -20\}.$$

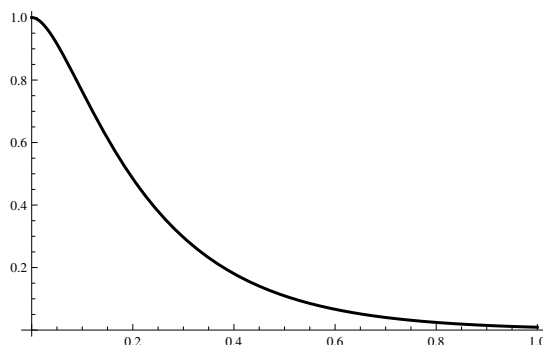
Thus the general solution in this case is

$$y(t) = c_1 e^{-5t} + c_2 e^{-20t}.$$

Then $y(0) = c_1 + c_2 = 1$. Also, $y'(t) = -5c_1 e^{-5t} - 20c_2 e^{-20t}$, so that $y'(0) = -5c_1 - 20c_2 = 0$. Solving these simultaneous equations gives $c_1 = \frac{4}{3}$ and $c_2 = -\frac{1}{3}$, so that the solution to the system for $c = 7.5$ is

$$y(t) = \frac{4}{3} e^{-5t} - \frac{1}{3} e^{-20t},$$

with graph



Because the discriminant is positive, this system is overdamped.

D2.4.18 From the text, the general differential equation for this situation is

$$y'' + \frac{c}{m}y' + \omega_0^2 y = 0,$$

where in our case $m = 10$, $c = 40$ and $\omega_0^2 = \frac{k}{10}$. Thus the equation is

$$y'' + 4y' + \frac{k}{10}y = 0.$$

The characteristic polynomial is $r^2 + 4r + \frac{k}{10}$, whose roots are the same as the roots of $10r^2 + 40r + k$, which are

$$\frac{-40 \pm \sqrt{1600 - 40k}}{20} = -2 \pm \frac{\sqrt{400 - 10k}}{10}.$$

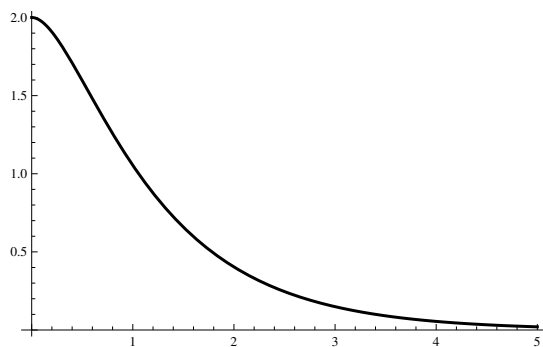
- a. When $k = 30$, the discriminant is $400 - 300 = 100$, so that the characteristic polynomial has the two real roots $-2 \pm 1 = \{-1, -3\}$. Then the general solution is

$$y(t) = c_1 e^{-t} + c_2 e^{-3t}, \quad y'(t) = -c_1 e^{-t} - 3c_2 e^{-3t}.$$

Thus $y(0) = c_1 + c_2 = 2$ and $y'(0) = -c_1 - 3c_2 = 0$. Solving these equations gives $c_1 = 3$ and $c_2 = -1$, so that the solution to the system for $k = 30$ is

$$y(t) = 3e^{-t} - e^{-3t},$$

with graph



Because the discriminant is positive, this system is overdamped.

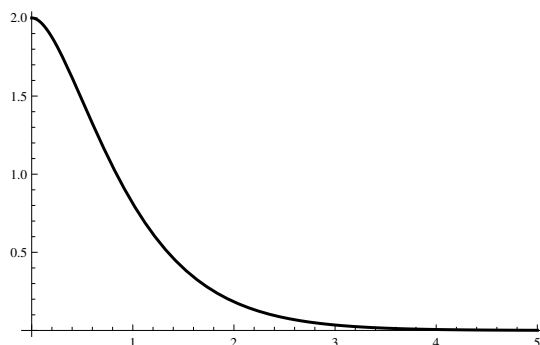
- b. When $k = 40$, the discriminant is zero, so the characteristic polynomial has the repeated root -2 , and the general solution is

$$y(t) = c_1 e^{-2t} + c_2 t e^{-2t}, \quad y'(t) = (-2c_1 + c_2) e^{-2t} - 2c_2 t e^{-2t}.$$

Thus $y(0) = c_1 = 2$ and $y'(0) = -2c_1 + c_2 = 0$. Solving these equations gives $c_1 = 2$ and $c_2 = 4$, so that the solution to the system for $k = 40$ is

$$y(t) = 2e^{-2t} + 4te^{-2t},$$

with graph



Because the discriminant is zero, this system is critically damped.

- c. When $k = 62.5$, the discriminant is $400 - 625 = -225$, so that the characteristic polynomial has the complex conjugate roots

$$-2 \pm \frac{\sqrt{-225}}{10} = -2 \pm \frac{3}{2}i,$$

so that the general solution is

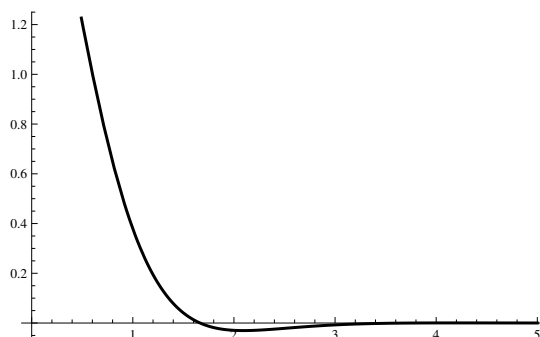
$$y(t) = e^{-2t} \left(c_1 \cos \frac{3}{2}t + c_2 \sin \frac{3}{2}t \right), \text{ with}$$

$$y'(t) = -2e^{-2t} \left(c_1 \cos \frac{3}{2}t + c_2 \sin \frac{3}{2}t \right) + \frac{3}{2}e^{-2t} \left(-c_1 \sin \frac{3}{2}t + c_2 \cos \frac{3}{2}t \right).$$

Then $y(0) = c_1 = 2$ and $y'(0) = -2c_1 + \frac{3}{2}c_2 = 0$, so that $c_2 = \frac{8}{3}$. Thus the solution to the system for $k = 62.5$ is

$$y(t) = e^{-2t} \left(2 \cos \frac{3}{2}t + \frac{8}{3} \sin \frac{3}{2}t \right),$$

with graph



Because the discriminant is negative, this system is underdamped. Although the damping occurs fairly quickly, by picking a graphing window judiciously, you can see the oscillation in the above graph.

D2.4.19

- a. The general equation for this free undamped oscillation is

$$y'' + \frac{c}{m}y' + \frac{k}{m}y = 0.$$

In this case we have $m = 250$ and $c = 500$, so that the equation is

$$y'' + 2y' + \frac{k}{250}y = 0.$$

The characteristic polynomial for this equation is $r^2 + 2r + \frac{k}{250}$. Critical damping occurs when this polynomial has discriminant zero; the discriminant is $b^2 - 4ac = 4 - 4\frac{k^2}{250^2}$, so that $k^2 = 250^2$ and $k = 250$. So $k = 250$ provides critical damping.

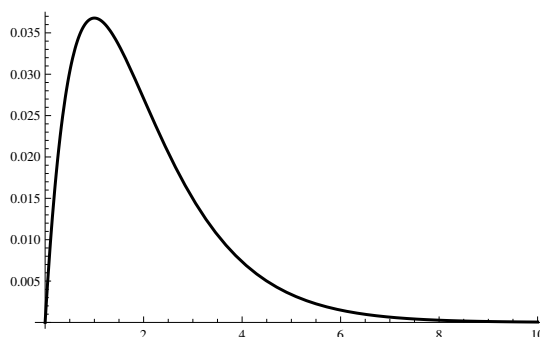
- b. With $k = 250$, the equation is $y'' + 2y' + y = 0$, which has the repeated root -1 , and thus the general solution is

$$y(t) = c_1e^{-t} + c_2te^{-t}, \quad y'(t) = (-c_1 + c_2)e^{-t} - c_2te^{-t}.$$

Then $y(0) = c_1 = 0$ and $y'(0) = -c_1 + c_2 = 0.1$, so that $c_2 = 0.1$ and the solution to the critically damped system is

$$y(t) = 0.1te^{-t}.$$

- c. $y(t) = 0.1te^{-t}$ looks like



The maximum displacement occurs when $y'(t) = 0$. Because

$$y'(t) = 0.1e^{-t} - 0.1te^{-t} = 0.1e^{-t}(1 - t),$$

we see that this happens at $t = 1$, when the displacement is $y(t) = 0.1e^{-1} \approx 0.0368$ m.

- d. Increasing k by 50% means $k = 375$, so that the equation is $y'' + 2y' + 1.5y = 0$, with complex roots $-1 \pm i\frac{\sqrt{2}}{2}$, so that the general solution is

$$y(t) = e^{-t}\left(c_1 \cos \frac{\sqrt{2}}{2}t + c_2 \sin \frac{\sqrt{2}}{2}t\right), \text{ with}$$

$$y'(t) = -e^{-t}\left(c_1 \cos \frac{\sqrt{2}}{2}t + c_2 \sin \frac{\sqrt{2}}{2}t\right) + \frac{\sqrt{2}}{2}e^{-t}\left(-c_1 \sin \frac{\sqrt{2}}{2}t + c_2 \cos \frac{\sqrt{2}}{2}t\right)$$

so that $y(0) = c_1 = 0$ and $y'(0) = -c_1 + \frac{\sqrt{2}}{2}c_2 = 0.1$. Thus $c_2 = \frac{0.2\sqrt{2}}{2} = 0.1\sqrt{2}$ and the solution to the system is

$$y(t) = e^{-t}\left(0.1\sqrt{2} \sin \frac{\sqrt{2}}{2}t\right).$$

The maximum displacement occurs when $y'(t) = 0$:

$$\begin{aligned} y'(t) &= -e^{-t}(0.1\sqrt{2} \sin \frac{\sqrt{2}}{2}t) + e^{-t}(0.1 \cos \frac{\sqrt{2}}{2}t) \\ &= e^{-t}(0.1 \cos \frac{\sqrt{2}}{2}t - 0.1\sqrt{2} \sin \frac{\sqrt{2}}{2}t), \end{aligned}$$

which is zero when $\cos \frac{\sqrt{2}}{2}t = \sqrt{2} \sin \frac{\sqrt{2}}{2}t$, or when $\tan \frac{\sqrt{2}}{2}t = \frac{\sqrt{2}}{2}$. We are concerned only with $t > 0$, and since this motion is damped, the maximum displacement will occur at the first positive solution of this equation, which is at $t \approx 0.870$. That displacement is $y(0.870) \approx 0.0342$ m.

When k is increased by 100%, we have $k = 500$, so that the equation is $y'' + 2y' + 2y = 0$ with characteristic polynomial $r^2 + 2r + 2$. The roots of this polynomial are the complex conjugate roots $-1 \pm i$, so that the general solution is

$$\begin{aligned} y(t) &= e^{-t}(c_1 \cos t + c_2 \sin t), \text{ with} \\ y'(t) &= -e^{-t}(c_1 \cos t + c_2 \sin t) + e^{-t}(-c_1 \sin t + c_2 \cos t) \\ &= e^{-t}((c_2 - c_1) \cos t - (c_1 + c_2) \sin t). \end{aligned}$$

Then $y(0) = c_1 = 0$ and $y'(0) = c_2 - c_1 = 0.1$, so that $c_2 = 0.1$. The solution to the system is

$$y(t) = 0.1e^{-t} \sin t, \text{ with } y'(t) = 0.1e^{-t}(\cos t - \sin t).$$

The maximum displacement occurs at the first positive solution of $y'(t) = 0$, which is at $t = \frac{\pi}{4} \approx 0.785$, and $y(0.785) \approx 0.0322$ m.

Thus as k increases, the maximum displacement of the car decreases, at least in this range. These systems are both underdamped, so although the maximum displacement decreases, the resulting oscillations create a feeling of a “bounce” in the car’s reaction to the bump.

- e. If we decrease k by 50%, then k becomes 125, and the equation is then $y'' + 2y' + \frac{1}{2}y = 0$, which has characteristic polynomial $r^2 + 2r + \frac{1}{2}$ with real solutions $-1 \pm \frac{\sqrt{2}}{2}$. Thus the general solution is

$$\begin{aligned} y(t) &= c_1 e^{(-1+\sqrt{2}/2)t} + c_2 e^{(-1-\sqrt{2}/2)t}, \text{ with} \\ y'(t) &= (-1 + \frac{\sqrt{2}}{2})c_1 e^{(-1+\sqrt{2}/2)t} - (1 + \frac{\sqrt{2}}{2})c_2 e^{(-1-\sqrt{2}/2)t}. \end{aligned}$$

Then $y(0) = c_1 + c_2 = 0$ and

$$y(0) = c_1 + c_2 = 0, \text{ and } y'(0) = (-1 + \frac{\sqrt{2}}{2})c_1 - (1 + \frac{\sqrt{2}}{2})c_2 = 0.1.$$

Solving gives $c_1 = 0.1 \frac{\sqrt{2}}{2}$ and $c_2 = -0.1 \frac{\sqrt{2}}{2}$, so that the solution to the system is

$$\begin{aligned} y(t) &= 0.1 \frac{\sqrt{2}}{2} (e^{(-1+\sqrt{2}/2)t} - e^{(-1-\sqrt{2}/2)t}), \text{ with} \\ y'(t) &= 0.1 \frac{\sqrt{2}}{4} ((\sqrt{2} + 2)e^{(-1-\sqrt{2}/2)t} + (\sqrt{2} - 2)e^{(-1+\sqrt{2}/2)t}). \end{aligned}$$

Then the maximum displacement occurs when $y' = 0$, which is when

$$\begin{aligned} (\sqrt{2} + 2)e^{(-1-\sqrt{2}/2)t} + (\sqrt{2} - 2)e^{(-1+\sqrt{2}/2)t} &= 0, \text{ or} \\ (2 + \sqrt{2})e^{(-1-\sqrt{2}/2)t} &= (2 - \sqrt{2})e^{(-1+\sqrt{2}/2)t}, \text{ or} \\ e^{t\sqrt{2}} &= \frac{2 + \sqrt{2}}{2 - \sqrt{2}}, \text{ or} \\ t &= \frac{1}{\sqrt{2}} \ln \frac{2 + \sqrt{2}}{2 - \sqrt{2}} \approx 1.246. \end{aligned}$$

Finally, $y(1.246) \approx 0.041$ m, so the maximum displacement increases in this overdamped case.

D2.4.20

- a. The general equation for this free undamped oscillation is

$$y'' + \frac{c}{m}y' + \frac{k}{m}y = 0.$$

In this case we have $m = 400$ and $k = 324$, so that the equation is

$$y'' + \frac{c}{400}y' + \frac{81}{100}y = 0.$$

The characteristic polynomial is $r^2 + \frac{c}{400}r + \frac{81}{100}$; this polynomial has the same roots as $400r^2 + cr + 324$; those roots are

$$\frac{-c \pm \sqrt{c^2 - 4 \cdot 400 \cdot 324}}{800} = -\frac{c}{800} \pm \frac{\sqrt{c^2 - 720^2}}{800}.$$

Thus critical damping occurs when $c = 720$.

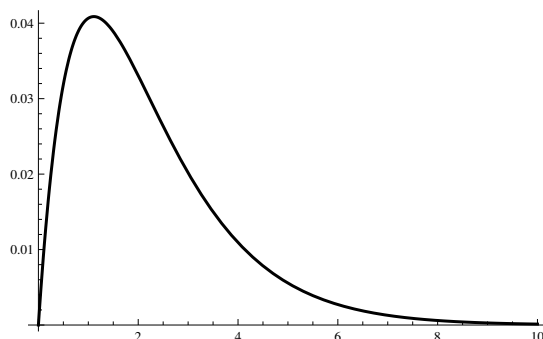
- b. When $c = 720$, the characteristic polynomial has the repeated root $-\frac{9}{10}$, so the system has the general solution

$$y(t) = c_1 e^{-9t/10} + c_2 t e^{-9t/10}, \text{ with } y'(t) = \left(-\frac{9}{10}c_1 + c_2\right)e^{-9t/10} - \frac{9}{10}c_2 t e^{-9t/10}.$$

Then $y(0) = c_1 = 0$ and $y'(0) = -\frac{9}{10}c_1 + c_2 = 0.1$, so that $c_1 = 0$ and $c_2 = 0.1$. Thus the solution to the system is

$$y(t) = 0.1t e^{-9t/10}, \text{ with } y'(t) = 0.1e^{-9t/10} - 0.1\frac{9}{10}t e^{-9t/10}.$$

- c. A graph of the solution above is



The maximum displacement occurs when $y'(t) = 0$. But

$$y'(t) = 0.1e^{-9t/10}\left(1 - \frac{9}{10}t\right),$$

so that $y'(t) = 0$ for $t = \frac{10}{9} \approx 1.111$. Because $y(1.111) \approx 0.0409$, the maximum displacement is about 0.0409 m.

- d. If the damping constant is increased to $c = 720\sqrt{2}$, then the roots of the characteristic polynomial are

$$-\frac{720\sqrt{2}}{800} \pm \frac{\sqrt{2 \cdot 720^2 - 720^2}}{800} = -\frac{9}{10}\sqrt{2} \pm \frac{9}{10} = \frac{9}{10}(\pm 1 - \sqrt{2}),$$

so that the system has general solution

$$y(t) = c_1 e^{\frac{9}{10}(1-\sqrt{2})t} + c_2 e^{\frac{9}{10}(-1-\sqrt{2})t}, \text{ with}$$

$$y'(t) = \frac{9}{10}(1-\sqrt{2})c_1 e^{\frac{9}{10}(1-\sqrt{2})t} - \frac{9}{10}(1+\sqrt{2})c_2 e^{\frac{9}{10}(-1-\sqrt{2})t}.$$

Then $y(0) = c_1 + c_2 = 0$ and $y'(0) = \frac{9}{10}(1 - \sqrt{2})c_1 - \frac{9}{10}(1 + \sqrt{2})c_2 = 0.1$. Solving gives $c_1 = \frac{1}{18}$ and $c_2 = -\frac{1}{18}$, so that the solution to the initial value problem is

$$y(t) = \frac{1}{18}(e^{\frac{9}{10}(1-\sqrt{2})t} - e^{\frac{9}{10}(-1-\sqrt{2})t}).$$

If the constant is decreased to $c = 360\sqrt{3}$, then the roots of the characteristic polynomial are

$$-\frac{360\sqrt{3}}{800} \pm \frac{\sqrt{3 \cdot 360^2 - 720^2}}{800} = -\frac{9\sqrt{3}}{20} \pm \frac{360}{800}i = \frac{9}{20}(-\sqrt{3} \pm i),$$

so that the system has general solution

$$\begin{aligned} y(t) &= e^{-9\sqrt{3}t/20}(c_1 \cos \frac{9t}{20} + c_2 \sin \frac{9t}{20}), \text{ with} \\ y'(t) &= \frac{9}{20}e^{-9\sqrt{3}t/20}((c_2 - c_1\sqrt{3}) \cos \frac{9t}{20} - (c_2\sqrt{3} + c_1) \sin \frac{9t}{20}). \end{aligned}$$

Then $y(0) = c_1 = 0$ and $y'(0) = \frac{9}{20}(c_2 - c_1\sqrt{3}) = 0.1$. Solving gives $c_1 = 0$, $c_2 = \frac{2}{9}$, so that the solution to the initial value problem is

$$y(t) = \frac{2}{9}e^{-9\sqrt{3}t/20} \sin \frac{9t}{20}.$$

D2.4.21

- a. The general equation is $y'' + by' + \omega_0^2 y = F_0 \cos \omega t$. The solution takes the form of a transient solution, which decays as $t \rightarrow \infty$, and a particular solution to the nonhomogeneous equation, which is a steady-state solution of the form $A \sin \omega t + B \cos \omega t$ (as long as $\omega \neq \omega_0$). The form of the transient solution is dictated by the values of b and ω_0 . For the problem at hand,

$$b = \frac{c}{m} = \frac{2}{1} = 2, \quad \omega_0^2 = \frac{k}{m} = \frac{5}{1} = 5.$$

Thus the equation is $y'' + 2y' + 5y = 4 \cos 2t$. The characteristic polynomial is $r^2 + 2r + 5$, with roots $-1 \pm 2i$, so that the general solution to the homogeneous equation is $y(t) = e^{-t}(c_1 \cos 2t + c_2 \sin 2t)$. Because $4 \cos 2t$ is not a solution to the homogeneous equation, we take $A \sin 2t + B \cos 2t$ as a trial solution, and compute

$$\begin{aligned} y'' + 2y' + 5y &= (A \sin 2t + B \cos 2t)'' + 2(A \sin 2t + B \cos 2t)' + 5(A \sin 2t + B \cos 2t) \\ &= -4A \sin 2t - 4B \cos 2t + 4A \cos 2t - 4B \sin 2t + 5A \sin 2t + 5B \cos 2t \\ &= (A - 4B) \sin 2t + (4A + B) \cos 2t = 4 \cos 2t. \end{aligned}$$

Thus $A - 4B = 0$ and $4A + B = 4$, so that $A = \frac{16}{17}$ and $B = \frac{4}{17}$, and the general solution to the nonhomogeneous equation, incorporating this particular solution, is

$$y(t) = e^{-t}(c_1 \cos 2t + c_2 \sin 2t) + \frac{1}{17}(16 \sin 2t + 4 \cos 2t).$$

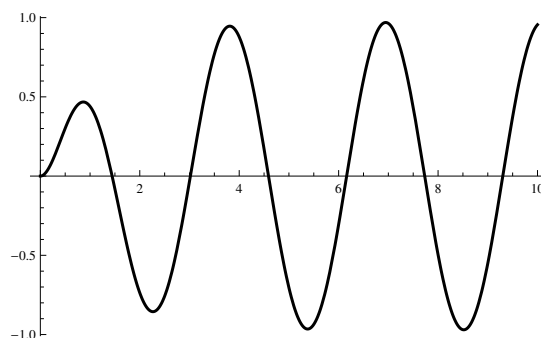
Then $y(0) = c_1 + \frac{4}{17} = 0$, so that $c_1 = -\frac{4}{17}$. Also,

$$y'(t) = -e^{-t}(c_1 \cos 2t + c_2 \sin 2t) + e^{-t}(-2c_1 \sin 2t + 2c_2 \cos 2t) + \frac{1}{17}(32 \cos 2t - 8 \sin 2t),$$

so that $y'(0) = -c_1 + 2c_2 + \frac{32}{17} = 0$, so that $c_2 = -\frac{18}{17}$. Thus the solution to the system is

$$y(t) = \frac{1}{17}[e^{-t}(-4 \cos 2t - 18 \sin 2t) + 16 \sin 2t + 4 \cos 2t].$$

A plot of this solution is

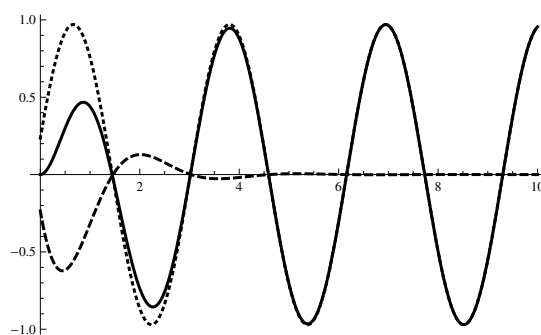


b. The transient and steady-state solutions are

$$y_{\text{transient}} = \frac{1}{17}e^{-t}(-4 \cos 2t - 18 \sin 2t)$$

$$y_{\text{ss}} = \frac{1}{17}(16 \sin 2t + 4 \cos 2t).$$

All three solutions (with $y(t)$ the solid line, $y_{\text{transient}}$ the dashed line, and y_{ss} the dotted line) look like



c. The transient solution becomes negligible after about 3 seconds, and the curves for $y(t)$ and $y_{\text{ss}}(t)$ are indistinguishable after about 4.5 seconds.

D2.4.22

a. The general equation is $y'' + by' + \omega_0^2 y = F_0 \sin \omega t$. The solution takes the form of a transient solution, which decays as $t \rightarrow \infty$, and a particular solution to the nonhomogeneous equation, which is a steady-state solution of the form $A \sin \omega t + B \cos \omega t$ (as long as $\omega \neq \omega_0$). The form of the transient solution is dictated by the values of b and ω_0 . For the problem at hand,

$$b = \frac{c}{m} = \frac{80}{20} = 4, \quad \omega_0^2 = \frac{k}{m} = \frac{180}{20} = 9.$$

Thus the equation is $y'' + 4y' + 9y = 3 \sin t$. The characteristic polynomial is $r^2 + 4r + 9$, with roots $-2 \pm i\sqrt{5}$, so that the general solution to the homogeneous equation is $y(t) = e^{-2t}(c_1 \cos \sqrt{5}t + c_2 \sin \sqrt{5}t)$. Because $3 \sin t$ is not a solution to the homogeneous equation, we take $A \sin t + B \cos t$ as a trial solution, and compute

$$\begin{aligned} y'' + 4y' + 9y &= (A \sin t + B \cos t)'' + 4(A \sin t + B \cos t)' + 9(A \sin t + B \cos t) \\ &= -A \sin t - B \cos t + 4A \cos t - 4B \sin t + 9A \sin t + 9B \cos t \\ &= (8A - 4B) \sin t + (4A + 8B) \cos t = 3 \sin t. \end{aligned}$$

Thus $8A - 4B = 3$ and $4A + 8B = 0$, so that $A = \frac{3}{10}$ and $B = -\frac{3}{20}$, and the general solution to the nonhomogeneous equation, incorporating this particular solution, is

$$y(t) = e^{-2t}(c_1 \cos \sqrt{5}t + c_2 \sin \sqrt{5}t) + \frac{3}{10} \sin t - \frac{3}{20} \cos t.$$

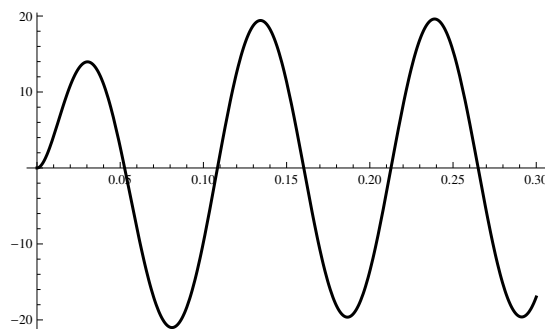
Thus $y(0) = c_1 - \frac{3}{20} = -1$, so that $c_1 = -\frac{17}{20}$. Also,

$$y'(t) = -2e^{-2t}(c_1 \cos \sqrt{5}t + c_2 \sin \sqrt{5}t) + e^{-2t}(-\sqrt{5}c_1 \sin \sqrt{5}t + \sqrt{5}c_2 \cos \sqrt{5}t) + \frac{3}{10} \cos t + \frac{3}{20} \sin t,$$

so that $y'(0) = -2c_1 + \sqrt{5}c_2 + \frac{3}{10} = 0$, and thus $c_2 = -\frac{2\sqrt{5}}{5}$. Hence the solution to the system is

$$y(t) = e^{-2t}\left(-\frac{17}{20} \cos \sqrt{5}t - \frac{2\sqrt{5}}{5} \sin \sqrt{5}t\right) + \frac{3}{10} \sin t - \frac{3}{20} \cos t.$$

A plot of this solution is

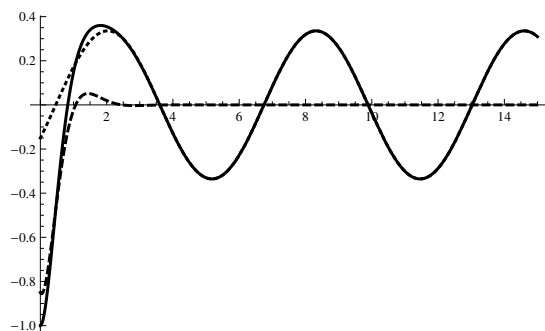


b. The transient and steady-state solutions are

$$y_{\text{transient}} = e^{-2t}\left(-\frac{17}{20} \cos \sqrt{5}t - \frac{2\sqrt{5}}{5} \sin \sqrt{5}t\right)$$

$$y_{\text{ss}} = \frac{3}{10} \sin t - \frac{3}{20} \cos t.$$

All three solutions (with $y(t)$ the solid line, $y_{\text{transient}}$ the dashed line, and y_{ss} the dotted line) look like



c. The transient solution becomes negligible after about 2.5 seconds.

D2.4.23

a. The characteristic polynomial of the homogeneous equation is $r^2 + r + \frac{5}{4}$, which has roots $-\frac{1}{2} \pm i$, so that the general solution to the homogeneous equation is $e^{-t/2}(c_1 \cos t + c_2 \sin t)$. Because $2 \cos \omega t$ is never a solution to the homogeneous equation, we can use $A \cos \omega t + B \sin \omega t$ as a trial solution; substituting gives

$$y'' + y' + \frac{5}{4}y = -A\omega^2 \cos \omega t - B\omega^2 \sin \omega t - A\omega \sin \omega t + B\omega \cos \omega t + \frac{5}{4}A \cos \omega t + \frac{5}{4}B \sin \omega t$$

$$= \left(\frac{5}{4}A + B\omega - A\omega^2\right) \cos \omega t + \left(\frac{5}{4}B - A\omega - B\omega^2\right) \sin \omega t = 2 \cos \omega t.$$

Solving for A and B gives

$$A = \frac{40 - 32\omega^2}{16\omega^4 - 24\omega^2 + 25}, \quad B = \frac{32\omega}{16\omega^4 - 24\omega^2 + 25}.$$

Thus the general solution to the nonhomogeneous equation is

$$y(t) = e^{-t/2}(c_1 \cos t + c_2 \sin t) + \frac{40 - 32\omega^2}{16\omega^4 - 24\omega^2 + 25} \cos \omega t + \frac{32\omega}{16\omega^4 - 24\omega^2 + 25} \sin \omega t.$$

b. Because

$$y'(t) = -\frac{1}{2}e^{-t/2}(c_1 \cos t + c_2 \sin t) + e^{-t/2}(-c_1 \sin t + c_2 \cos t) - \frac{40\omega - 32\omega^3}{16\omega^4 - 24\omega^2 + 25} \sin \omega t + \frac{32\omega^2}{16\omega^4 - 24\omega^2 + 25} \cos \omega t,$$

we have

$$y(0) = c_1 + \frac{40 - 32\omega^2}{16\omega^4 - 24\omega^2 + 25} = 0$$

$$y'(0) = -\frac{1}{2}c_1 + c_2 + \frac{32\omega^2}{16\omega^4 - 24\omega^2 + 25} = 0.$$

Solving for c_1 and c_2 gives for the solution of the initial value problem

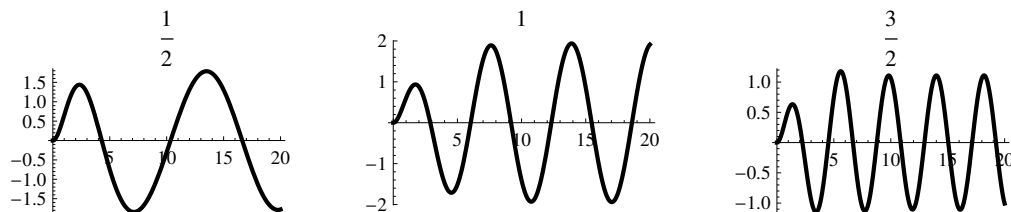
$$y(t) = e^{-t/2} \left(\frac{32\omega^2 - 40}{16\omega^4 - 24\omega^2 + 25} \cos t - \frac{16\omega^2 + 20}{16\omega^4 - 24\omega^2 + 25} \sin t \right) - \frac{32\omega^2 - 40}{16\omega^4 - 24\omega^2 + 25} \cos \omega t + \frac{32\omega}{16\omega^4 - 24\omega^2 + 25} \sin \omega t.$$

c. The transient and steady-state components are

$$y_{\text{transient}}(t) = e^{-t/2} \left(\frac{32\omega^2 - 40}{16\omega^4 - 24\omega^2 + 25} \cos t - \frac{16\omega^2 + 20}{16\omega^4 - 24\omega^2 + 25} \sin t \right)$$

$$y_{\text{ss}}(t) = -\frac{32\omega^2 - 40}{16\omega^4 - 24\omega^2 + 25} \cos \omega t + \frac{32\omega}{16\omega^4 - 24\omega^2 + 25} \sin \omega t.$$

d. Plots for the three given values of ω are



The steady-state oscillations have the greatest magnitude when $\omega = 1$.

D2.4.24

a. The characteristic polynomial of the homogeneous equation is $r^2 + 2r + (\omega_0^2 + 1)$, which has roots $-1 \pm \omega_0 i$. Thus the general solution of the homogeneous equation is

$$y(t) = e^{-t}(c_1 \cos \omega_0 t + c_2 \sin \omega_0 t).$$

Because $2 \cos t$ is never a solution of the homogeneous equation, we use $A \cos t + B \sin t$ as a trial solution for the nonhomogeneous equation, and compute

$$\begin{aligned} y'' + 2y' + (\omega_0^2 + 1)y &= -A \cos t - B \sin t - 2A \sin t + 2B \cos t \\ &\quad + (\omega_0^2 + 1)A \cos t + (\omega_0^2 + 1)B \sin t \\ &= (\omega_0^2 A + 2B) \cos t + (\omega_0^2 B - 2A) \sin t = 2 \cos t. \end{aligned}$$

Solving for A and B gives for a general solution to the nonhomogeneous equation

$$y(t) = e^{-t}(c_1 \cos \omega_0 t + c_2 \sin \omega_0 t) + \frac{2\omega_0^2}{4 + \omega_0^4} \cos t + \frac{4}{4 + \omega_0^4} \sin t.$$

b. Because

$$\begin{aligned} y'(t) &= -e^{-t}(c_1 \cos \omega_0 t + c_2 \sin \omega_0 t) + e^{-t}(-c_1 \omega_0 \sin \omega_0 t + c_2 \omega_0 \cos \omega_0 t) \\ &\quad - \frac{2\omega_0^2}{4 + \omega_0^4} \sin t + \frac{4}{4 + \omega_0^4} \cos t, \end{aligned}$$

we have

$$y(0) = c_1 + \frac{2\omega_0^2}{4 + \omega_0^4} = 0, \quad y'(0) = -c_1 + c_2 \omega_0 + \frac{4}{4 + \omega_0^4} = 0.$$

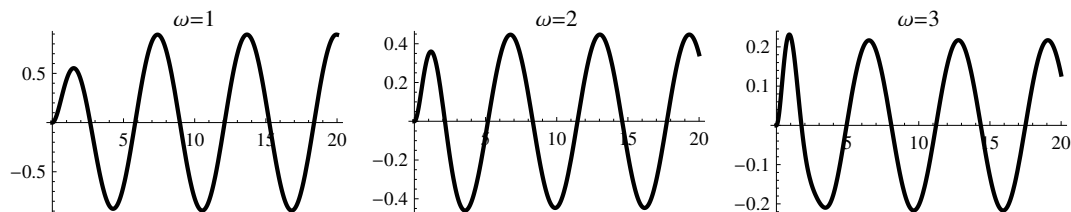
Solving gives for the solution to the initial value problem

$$y(t) = e^{-t}\left(-\frac{2\omega_0^2}{4 + \omega_0^4} \cos \omega_0 t - \frac{4 + 2\omega_0^2}{\omega_0(4 + \omega_0^4)} \sin \omega_0 t\right) + \frac{2\omega_0^2}{4 + \omega_0^4} \cos t + \frac{4}{4 + \omega_0^4} \sin t.$$

c. The transient and steady-state components of the solution are

$$\begin{aligned} y_{\text{transient}}(t) &= e^{-t}\left(-\frac{2\omega_0^2}{4 + \omega_0^4} \cos \omega_0 t - \frac{4 + 2\omega_0^2}{\omega_0(4 + \omega_0^4)} \sin \omega_0 t\right) \\ y_{\text{ss}}(t) &= \frac{2\omega_0^2}{4 + \omega_0^4} \cos t + \frac{4}{4 + \omega_0^4} \sin t. \end{aligned}$$

d. Graphs for $\omega_0 = 1, 2$, and 3 are



$\omega_0 = 1$ produces the largest steady-state oscillation.

D2.4.25 Start with equation (2) in the discussion of electrical circuits, which is $LQ'' + RQ' + \frac{1}{C}Q = E(t)$. If there is no inductor, then $L = 0$, and the equation reduces to $RQ' + \frac{1}{C}Q = E(t)$. For the case given, we have $R = 50$, $C = 0.001$, and $E(t) = 50$, so that the equation becomes $50Q' + 1000Q = 50$, or $Q' + 20Q = 1$. The solution to the homogeneous equation is $c_1 e^{-20t}$. It is easy to see that a particular solution to the nonhomogeneous equation is $Q_p(t) = \frac{1}{20}$ (to see this analytically, use a trial solution of $Q = A$; then $Q' + 20Q = 0 + 20A = 1$ so that $A = \frac{1}{20}$). Thus the general solution is

$$Q(t) = c_1 e^{-20t} + \frac{1}{20}.$$

Because the initial charge on the capacitor is zero, we have $Q(0) = c_1 + \frac{1}{20} = 0$, so that the solution to the system is

$$Q(t) = -\frac{1}{20}e^{-20t} + \frac{1}{20}.$$

The charge on the capacitor approaches a steady-state value of $\frac{1}{20}$ farads.

D2.4.26 Start with equation (1) in the discussion of electrical circuits, and recall that a capacitor of C farads produces a voltage drop of $\frac{Q}{C}$ volts. If there is no capacitor, then there is obviously no voltage drop, so that this term in (1) is zero. That leaves the equation $LI' + RI = E(t)$. For the case given, we have $R = 200$ and $L = 2$, with $E(t) = 100$, so the equation is $2I' + 200I = 100$, or $I' + 100I = 50$. The solution to this first-order equation is $I(t) = c_1e^{-100t} + \frac{1}{2}$; with $I(0) = 0$, we get $c_1 = -\frac{1}{2}$ so that the solution to the initial value problem is

$$I(t) = \frac{1}{2}(1 - e^{-100t}).$$

The current in the circuit approaches a steady-state value of 50 amps.

D2.4.27

- a. The general equation for an LCR circuit is

$$I'' + \frac{R}{L}I' + \frac{1}{LC}I = \frac{1}{L}E'(t).$$

In the case given, we have $R = 10$, $L = 0.1$, $C = \frac{1}{240}$, and $E(t) = 100$, so that $E'(t) = 0$ and the equation becomes

$$I'' + 100I' + 2400I = 0.$$

The characteristic polynomial for this equation is $r^2 + 100r + 2400$, with roots $-50 \pm 10 = \{-40, -60\}$. Thus the general solution for the homogeneous equation is $I(t) = c_1e^{-40t} + c_2e^{-60t}$. Then $I(0) = c_1 + c_2 = 0$. To determine the value for $I'(0) = -40c_1 - 60c_2$, start with the fact that $Q(0) = 0$, and evaluate equation (1) from the discussion at $t = 0$:

$$LI'(0) + RI(0) + \frac{1}{C}Q(0) = 100 \quad \Rightarrow \quad 0.1I'(0) = 100,$$

so that $I'(0) = 1000$. Solving $c_1 + c_2 = 0$ and $-40c_1 - 60c_2 = 1000$ gives $c_1 = 50$ and $c_2 = -50$, so that the solution to the initial value problem is

$$I(t) = 50e^{-40t} - 50e^{-60t},$$

- b. The transient current is $I(t)$, and the steady-state current is zero. Thus the current in the circuit approaches a steady-state value of zero amps.

D2.4.28

- a. Because the external voltage is no longer constant, with $E'(t) = 15000 \cos 150t$, the differential equation for the circuit is now

$$I'' + 100I' + 2400I = \frac{1}{L}15000 \cos 150t = 150000 \cos 150t.$$

The general solution to the homogeneous problem is still $c_1e^{-40t} + c_2e^{-60t}$. Start by using $A \cos 150t + B \sin 150t$ as a trial solution for the nonhomogeneous problem:

$$\begin{aligned} I'' + 100I' + 2400I &= -22500A \cos 150t - 22500B \sin 150t - 15000A \sin 150t + 15000B \cos 150t \\ &\quad + 2400A \cos 150t + 2400B \sin 150t \\ &= (15000B - 20100A) \cos 150t - (20100B + 15000A) \sin 150t \\ &= 150000 \cos 150t. \end{aligned}$$

Thus (dividing through by 100), we get $150B - 201A = 1500$ and $201B + 150A = 0$. Solving gives for the general solution

$$I(t) = c_1 e^{-40t} + c_2 e^{-60t} - \frac{33500}{6989} \cos 150t + \frac{25000}{6989} \sin 150t.$$

We know that $I(0) = 0$. To determine $I'(0)$, evaluate equation (1) from the discussion at $t = 0$:

$$0.1I'(0) + 10I(0) + 240Q(0) = E(0) \quad \Rightarrow \quad 0.1I'(0) = 0,$$

so that $I'(0) = 0$. Then

$$I(0) = c_1 + c_2 - \frac{33500}{6989} = 0, \quad I'(0) = -40c_1 - 60c_2 + \frac{25000 \cdot 150}{6989} = 0.$$

Solving gives for a solution to the initial value problem

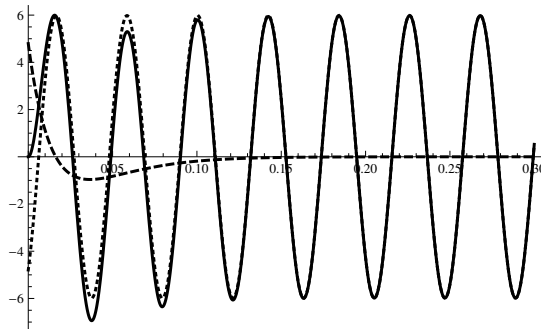
$$I(t) = -\frac{3000}{241} e^{-40t} + \frac{500}{29} e^{-60t} - \frac{33500}{6989} \cos 150t + \frac{25000}{6989} \sin 150t.$$

b. The transient and steady-state currents are

$$I_{\text{transient}} = -\frac{3000}{241} e^{-40t} + \frac{500}{29} e^{-60t},$$

$$I_{\text{ss}} = -\frac{33500}{6989} \cos 150t + \frac{25000}{6989} \sin 150t.$$

c. All three solutions (with $I(t)$ the solid line, $I_{\text{transient}}$ the dashed line, and I_{ss} the dotted line) look like



D2.4.29

a. With $R = 80$, $L = 0.5$, $C = \frac{1}{3230}$, and $E(t) = 200$ (so that $E'(t) = 0$), the equation for the circuit is

$$I'' + \frac{80}{0.5} I' + \frac{3230}{0.5} I = 0, \text{ or } I'' + 160I' + 6460I = 0.$$

The characteristic polynomial is $r^2 + 160r + 6460$, with roots $-80 \pm 2i\sqrt{15}$, so that the general solution is

$$I(t) = e^{-80t} (c_1 \cos 2\sqrt{15}t + c_2 \sin 2\sqrt{15}t), \text{ with}$$

$$I'(t) = e^{-80t} ((-80c_1 + 2\sqrt{15}c_2) \cos 2\sqrt{15}t - (2\sqrt{15}c_1 + 80c_2) \sin 2\sqrt{15}t).$$

We know that $I(0) = 0$. To determine $I'(0)$, evaluate equation (1) from the discussion at $t = 0$:

$$0.5I'(0) + 80I(0) + 3230Q(0) = E(0) \quad \Rightarrow \quad 0.5I'(0) = 200,$$

so that $I'(0) = 400$. Then

$$I(0) = c_1 = 0, \quad I'(0) = -80c_1 + 2\sqrt{15}c_2 = 400.$$

Solving gives $c_1 = 0$ and $c_2 = \frac{40\sqrt{15}}{3}$, so that the solution to the initial value problem is

$$I(t) = \frac{40\sqrt{15}}{3}e^{-80t} \sin 2\sqrt{15}t.$$

b. The entire current is transient; since this is a homogeneous equation, the steady-state current is zero.

D2.4.30

a. With $R = 80$, $L = 0.5$, $C = \frac{1}{3232}$, and $E(t) = 90.5 \sin 64t$ (so that $E'(t) = 5792 \cos 64t$), the equation for the circuit is

$$I'' + \frac{80}{0.5}I' + \frac{3232}{0.5}I = \frac{5792}{0.5} \cos 64t, \text{ or } I'' + 160I' + 6464I = 11584 \cos 64t.$$

The characteristic polynomial for the homogeneous problem is $r^2 + 160r + 6464$, with roots $-80 \pm 8i$. Thus the general solution for the homogeneous equation is

$$I(t) = e^{-80t}(c_1 \cos 8t + c_2 \sin 8t).$$

We use $A \cos 64t + B \sin 64t$ as a trial solution for the nonhomogeneous problem:

$$\begin{aligned} I'' + 160I' + 6464I &= -4096A \cos 64t - 4096B \sin 64t - 10240A \sin 64t + 10240B \cos 64t \\ &\quad + 6464A \cos 64t + 6464B \sin 64t \\ &= (2368A + 10240B) \cos 64t + (2368B - 10240A) \sin 64t = 11584 \cos 64t. \end{aligned}$$

Solving gives $A = \frac{37}{149}$ and $B = \frac{160}{149}$, so that the general solution to the nonhomogeneous equation is

$$\begin{aligned} I(t) &= e^{-80t}(c_1 \cos 8t + c_2 \sin 8t) + \frac{37}{149} \cos 64t + \frac{160}{149} \sin 64t, \text{ with} \\ I'(t) &= e^{-80t}((-80c_1 + 8c_2) \cos 8t + (-8c_1 - 80c_2) \sin 8t) + \frac{10240}{149} \cos 64t - \frac{2368}{149} \sin 64t \end{aligned}$$

We know that $I(0) = Q(0) = 0$. To determine $I'(0)$, evaluate equation (1) from the discussion at $t = 0$:

$$0.5I'(0) + 80I(0) + 3232Q(0) = E(0) \quad \Rightarrow \quad 0.5I'(0) = 90.5,$$

so that $I'(0) = 181$. Then

$$I(0) = c_1 + \frac{37}{149} = 0, \quad I'(0) = -80c_1 + 8c_2 + \frac{10240}{149} = 181.$$

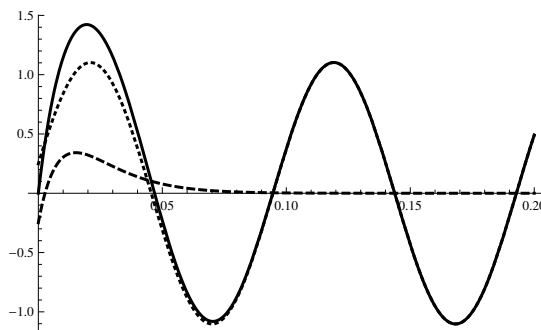
Solving gives $c_1 = -\frac{37}{149}$ and $c_2 = \frac{13769}{1192}$, so that the solution to the initial value problem is

$$I(t) = e^{-80t}\left(-\frac{37}{149} \cos 8t + \frac{13769}{1192} \sin 8t\right) + \frac{37}{149} \cos 64t + \frac{160}{149} \sin 64t.$$

b. The transient and steady-state currents are

$$\begin{aligned} I_{\text{transient}} &= e^{-80t}\left(-\frac{37}{149} \cos 8t + \frac{13769}{1192} \sin 8t\right) \\ I_{\text{ss}} &= \frac{37}{149} \cos 64t + \frac{160}{149} \sin 64t. \end{aligned}$$

c. All three solutions (with $I(t)$ the solid line, $I_{\text{transient}}$ the dashed line, and I_{ss} the dotted line) look like



D2.4.31 Because $I(t) = Q'(t)$, we start by finding the charge on the capacitor, $Q(t)$. The differential equation to be solved is

$$LQ'' + RQ' + \frac{1}{C}Q = E(t),$$

which with $R = 16$, $L = 2$, and $C = \frac{1}{50}$, together with $E = 120$, gives

$$2Q'' + 16Q' + 50Q = 120, \quad \text{or} \quad Q'' + 8Q' + 25Q = 60.$$

The homogeneous equation has characteristic polynomial $r^2 + 8r + 25$, with complex conjugate roots $-4 \pm 3i$. Also, it is clear that $\frac{120}{50} = \frac{12}{5}$ is a solution to the nonhomogeneous equation. Thus

$$Q(t) = e^{-4t}(c_1 \cos 3t + c_2 \sin 3t) + \frac{12}{5}$$

is the general solution to the nonhomogeneous equation. Differentiating $Q(t)$ gives

$$Q'(t) = e^{-4t}((-4c_1 + 3c_2) \cos 3t + (-3c_1 - 4c_2) \sin 3t),$$

so that $Q(0) = c_1 + \frac{12}{5} = 0$ and $Q'(0) = -4c_1 + 3c_2 = 0$. Thus $c_1 = -\frac{12}{5}$ and $c_2 = -\frac{16}{5}$, and the solution to the initial value problem is

$$Q(t) = \frac{1}{5}(e^{-4t}(-12 \cos 3t - 16 \sin 3t) + 12)$$

It follows that

$$\begin{aligned} I(t) = Q'(t) &= \frac{1}{5}(-4e^{-4t}(-12 \cos 3t - 16 \sin 3t) + e^{-4t}(36 \sin 3t - 48 \cos 3t)) \\ &= \frac{1}{5}e^{-4t}(100 \sin 3t) = 20e^{-4t} \sin 3t. \end{aligned}$$

D2.4.32 Because $I(t) = Q'(t)$, we start by finding the charge on the capacitor, $Q(t)$. The differential equation to be solved is

$$LQ'' + RQ' + \frac{1}{C}Q = E(t),$$

which with $R = 4$, $L = \frac{1}{2}$, and $C = \frac{2}{25}$, together with $E(t) = 30 \sin t$, gives

$$\frac{1}{2}Q'' + 4Q' + \frac{25}{2}Q = 30 \sin t, \quad \text{or} \quad Q'' + 8Q' + 25Q = 60 \sin t.$$

The characteristic polynomial for the homogeneous equation is $r^2 + 8r + 25$, which has roots $-4 \pm \frac{\sqrt{64-100}}{2} = -4 \pm 3i$. Thus the general solution to the homogeneous equation is

$$Q(t) = e^{-4t}(c_1 \cos 3t + c_2 \sin 3t).$$

Using $A \cos t + B \sin t$ as a trial solution for the nonhomogeneous equation, we get

$$\begin{aligned} Q'' + 8Q' + 25Q &= -A \cos t - B \sin t - 8A \sin t + 8B \cos t + 25A \cos t + 25B \sin t \\ &= (24A + 8B) \cos t + (24B - 8A) \sin t = 60 \sin t. \end{aligned}$$

Solving for A and B gives $A = -\frac{3}{4}$ and $B = \frac{9}{4}$, so that the general solution to the nonhomogeneous equation is

$$Q(t) = e^{-4t}(c_1 \cos 3t + c_2 \sin 3t) - \frac{3}{4} \cos t + \frac{9}{4} \sin t, \text{ with}$$

$$Q'(t) = e^{-4t}((-4c_1 + 3c_2) \cos 3t + (-3c_1 - 4c_2) \sin 3t) + \frac{3}{4} \sin t + \frac{9}{4} \cos t.$$

Because $Q(0) = c_1 - \frac{3}{4} = 0$ and $Q'(0) = I(0) = -4c_1 + 3c_2 + \frac{9}{4} = 0$, so that $c_1 = \frac{3}{4}$ and $c_2 = \frac{1}{4}$, and the solution to the system is

$$Q(t) = e^{-4t}\left(\frac{3}{4} \cos 3t + \frac{1}{4} \sin 3t\right) - \frac{3}{4} \cos t + \frac{9}{4} \sin t.$$

Finally,

$$I(t) = Q'(t) = e^{-4t}\left(-\frac{9}{4} \cos 3t - \frac{13}{4} \sin 3t\right) + \frac{3}{4} \sin t + \frac{9}{4} \cos t.$$

D2.4.33

- True. When $b \neq 0$, the real part of the roots of the characteristic polynomial are nonzero, so that there is always an exponential piece to the general solution. For an example, see any of Exercises 17–24.
- True. Resonance occurs when the frequency of the external force is the same as the natural frequency of the system; clearly this cannot happen if there is no external force.
- True. Because the solution of the homogeneous equation includes an exponential piece, the right-hand side (which in the text is $A \cos \omega t$ or $A \sin \omega t$) cannot be a solution of the homogeneous equation, so that a particular solution of the nonhomogeneous piece will have the form $C \cos \omega t + D \sin \omega t$, which has fixed amplitude and thus does not exhibit resonance.
- False. When $R = 0$, the equation is $LQ'' + \frac{1}{C}Q = E(t)$ or $LI'' + \frac{1}{C}I = E'(t)$, both of which are undamped since there is no linear term, which is the damping term.
- False. Beats occur when both the homogeneous and the particular solution are oscillatory and have frequencies close to each other. In the damped case, solutions always involve an exponential term.
- True, as long as $b > 0$. In this case, the characteristic polynomial is $r^2 + br + \omega_0^2$, with roots

$$\frac{-b \pm \sqrt{b^2 - 4\omega_0^2}}{2}.$$

If the discriminant is negative, then both roots have real part $-\frac{b}{2} < 0$, so that the solution includes a multiple of $e^{-bt/2}$, which decays to zero over time. If the discriminant is positive, then surely it is less than b^2 , so that its square root is less than b and both roots are negative. So in this case the solution is a sum of negative exponentials, which again decay over time. Thus this transient component decays over time, while the solution to the nonhomogeneous piece is an oscillatory constant-amplitude steady-state piece.

D2.4.34

- The characteristic polynomial for this equation is $r^2 + 2r + 5$, with roots $-1 \pm 2i$. Thus the general solution for the homogeneous equation is $e^{-t}(c_1 \cos 2t + c_2 \sin 2t)$. Because $10 \cos t$ is not a solution of the homogeneous equation, we use $A \cos t + B \sin t$ as a trial solution:

$$\begin{aligned} y'' + 2y' + 5y &= -A \cos t - B \sin t - 2A \sin t + 2B \cos t + 5A \cos t + 5B \sin t \\ &= (4A + 2B) \cos t + (4B - 2A) \sin t = 10 \cos t. \end{aligned}$$

Thus $4A + 2B = 10$ and $-2A + 4B = 0$, so that $A = 2$ and $B = 1$. Thus the general solution to the nonhomogeneous problem is

$$y(t) = e^{-t}(c_1 \cos 2t + c_2 \sin 2t) + 2 \cos t + \sin t, \text{ with}$$

$$y'(t) = e^{-t}((-c_1 + 2c_2) \cos 2t + (-2c_1 - c_2) \sin 2t) - 2 \sin t + \cos t.$$

Applying the initial conditions gives $y(0) = c_1 + 2 = 1$ and $y'(0) = -c_1 + 2c_2 + 1 = 0$, so that $c_1 = c_2 = -1$. Thus the solution to the initial value problem is

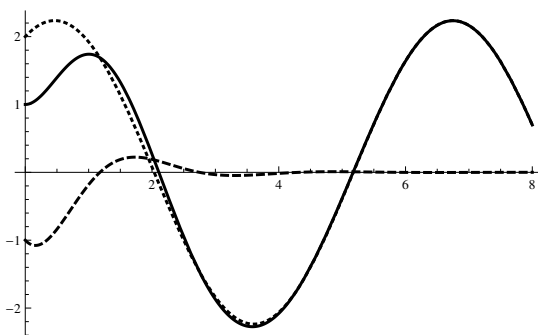
$$y(t) = -e^{-t}(\cos 2t + \sin 2t) + 2 \cos t + \sin t.$$

The transient and steady-state portions are

$$y_{\text{transient}}(t) = -e^{-t}(\cos 2t + \sin 2t)$$

$$y_{\text{ss}}(t) = 2 \cos t + \sin t.$$

- b. Graphs of the full (solid black), transient (dashed), and steady-state (dotted) solutions superimposed are



- c. The transient solution becomes almost zero after about three seconds. As $t \rightarrow \infty$, we have

$$\lim_{t \rightarrow \infty} |y_{\text{transient}}(t)| = \lim_{t \rightarrow \infty} (e^{-t} \sqrt{\cos^2 2t + \sin^2 2t}) = \lim_{t \rightarrow \infty} (e^{-t} \cdot 1) = 0.$$

so that the transient solution approaches zero, and thus the solution of the initial value problem approaches the steady-state solution.

D2.4.35

- a. The characteristic polynomial for this equation is $r^2 + \frac{1}{2}r + \frac{17}{16}$, with roots $-\frac{1}{4} \pm i$. Thus the general solution for the homogeneous equation is $e^{-t/4}(c_1 \cos t + c_2 \sin t)$. Because $65 \sin t$ is not a solution of the homogeneous equation, we use $A \cos t + B \sin t$ as a trial solution:

$$y'' + \frac{1}{2}y' + \frac{17}{16}y = -A \cos t - B \sin t - \frac{1}{2}A \sin t + \frac{1}{2}B \cos t + \frac{17}{16}A \cos t + \frac{17}{16}B \sin t$$

$$= \left(\frac{A}{16} + \frac{B}{2}\right) \cos t + \left(-\frac{A}{2} + \frac{B}{16}\right) \sin t = 65 \sin t.$$

Thus

$$\frac{A}{16} + \frac{B}{2} = 0, \quad -\frac{A}{2} + \frac{B}{16} = 65,$$

so that $A = -128$ and $B = 16$, and the general solution to the nonhomogeneous problem is

$$y(t) = e^{-t/4}(c_1 \cos t + c_2 \sin t) - 128 \cos t + 16 \sin t, \text{ with}$$

$$y'(t) = e^{-t/4}\left(-\frac{1}{4}c_1 + c_2\right) \cos t + \left(-c_1 - \frac{1}{4}c_2\right) \sin t + 128 \sin t + 16 \cos t.$$

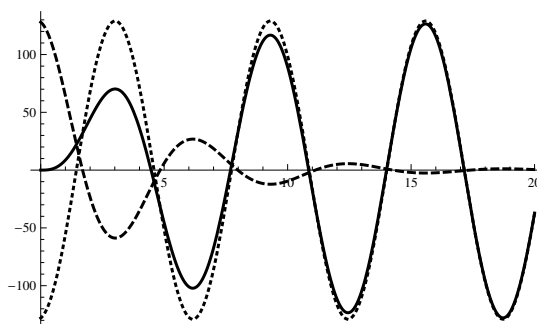
Applying the initial conditions gives $y(0) = c_1 - 128 =$ and $y'(0) = -\frac{1}{4}c_1 + c_2 + 16 = 0$, so that $c_1 = 128$ and $c_2 = 16$. Thus the solution to the initial value problem is

$$y(t) = e^{-t/4}(128 \cos t + 16 \sin t) - 128 \cos t + 16 \sin t.$$

The transient and steady-state portions are

$$\begin{aligned} y_{\text{transient}}(t) &= e^{-t/4}(128 \cos t + 16 \sin t) \\ y_{\text{ss}}(t) &= -128 \cos t + 16 \sin t. \end{aligned}$$

- b. Graphs of the full (solid black), transient (dashed), and steady-state (dotted) solutions superimposed are



- c. The transient solution becomes almost zero after about 15 seconds. As $t \rightarrow \infty$, we have

$$\lim_{t \rightarrow \infty} |y_{\text{transient}}(t)| = \lim_{t \rightarrow \infty} (e^{-t/4} \sqrt{128^2 \cos^2 2t + 16^2 \sin^2 2t}) \leq \lim_{t \rightarrow \infty} (e^{-t/4} \cdot 128) = 0.$$

so that the transient solution approaches zero, and thus the solution of the initial value problem approaches the steady-state solution.

D2.4.36

- a. The characteristic polynomial of the homogeneous equation is $r^2 + 2r + 2$, with roots $-1 \pm i$. Thus the general solution to the homogeneous equation is $y(t) = e^{-t}(c_1 \cos t + c_2 \sin t)$. Using $A \cos 4t + B \sin 4t$ as a trial solution, we compute

$$\begin{aligned} y'' + 2y' + 2y &= -16A \cos 4t - 16B \sin 4t - 8A \sin 4t + 8B \cos 4t + 2A \cos 4t + 2B \sin 4t \\ &= (8B - 14A) \cos 4t + (-8A - 14B) \sin 4t = 130 \sin 4t. \end{aligned}$$

Thus $-14A + 8B = 0$ and $-8A - 14B = 130$. Solving gives $A = -4$ and $B = -7$, so that the general solution to the nonhomogeneous problem is

$$\begin{aligned} y(t) &= e^{-t}(c_1 \cos t + c_2 \sin t) - 4 \cos 4t - 7 \sin 4t, \text{ with} \\ y'(t) &= e^{-t}((-c_1 + c_2) \cos t + (-c_1 - c_2) \sin t) - 28 \cos 4t + 16 \sin 4t. \end{aligned}$$

Then $y(0) = c_1 - 4 = 0$ and $y'(0) = -c_1 + c_2 - 28 = 0$, so that $c_1 = 4$ and $c_2 = 32$, and the solution to the system is

$$y(t) = e^{-t}(4 \cos t + 32 \sin t) - 4 \cos 4t - 7 \sin 4t,$$

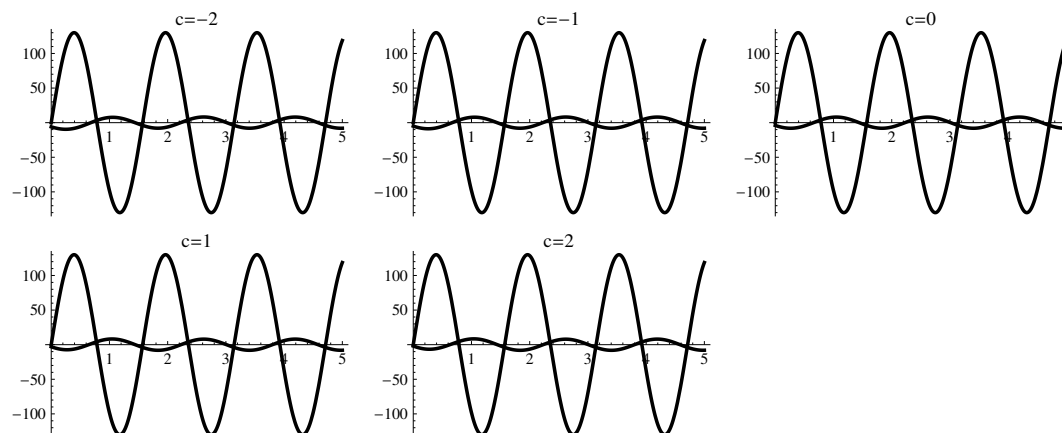
with transient and steady-state parts

$$\begin{aligned} y_{\text{transient}}(t) &= e^{-t}(4 \cos t + 32 \sin t) \\ y_{\text{ss}}(t) &= -4 \cos 4t - 7 \sin 4t. \end{aligned}$$

b. Setting $c_2 = 0$ in the general solution gives

$$y(t) = c_1 e^{-t} \cos t - 4 \cos 4t - 7 \sin 4t.$$

Graphs of this equation for $c_1 \in \{-2, -1, 0, 1, 2\}$, plotted along with the external forcing function $130 \sin 4t$, are:



c. In amplitude-phase form, the steady-state solution is

$$A \sin(4t + \varphi) = -4 \cos 4t - 7 \sin 4t, \text{ where } A = \sqrt{4^2 + 7^2} = \sqrt{65}, \tan \varphi = \frac{-4}{-7}.$$

Thus $\varphi = \arctan \frac{-4}{-7}$ in the third quadrant, so that the steady-state solution is $\approx 8.06 \sin(4t + 3.66) = 8.06 \sin(4(t + 0.915))$. Thus the amplitude of the steady-state solution is much smaller than the amplitude of the forcing function, and is shifted 0.915 radians to the left.

D2.4.37

a. The characteristic polynomial of the homogeneous equation is $r^2 + 3r + 2$, with roots $\frac{-3 \pm 1}{2} = \{-2, -1\}$. Thus the general solution to the homogeneous equation is $y(t) = c_1 e^{-2t} + c_2 e^{-t}$. Using $A \cos 4t + B \sin 4t$ as a trial solution, we compute

$$\begin{aligned} y'' + 3y' + 2y &= -16A \cos 4t - 16B \sin 4t - 12A \sin 4t + 12B \cos 4t + 2A \cos 4t + 2B \sin 4t \\ &= (12B - 14A) \cos 4t + (-12A - 14B) \sin 4t = 170 \cos 4t. \end{aligned}$$

Thus $12B - 14A = 170$ and $-12A - 14B = 0$, so that $A = -7$ and $B = 6$, and the general solution to the nonhomogeneous equation is

$$\begin{aligned} y(t) &= c_1 e^{-2t} + c_2 e^{-t} - 7 \cos 4t + 6 \sin 4t, \text{ with} \\ y'(t) &= -2c_1 e^{-2t} - c_2 e^{-t} + 24 \cos 4t + 28 \sin 4t. \end{aligned}$$

Then $y(0) = c_1 + c_2 - 7 = 0$ and $y'(0) = -2c_1 - c_2 + 24 = 0$. Solving gives $c_1 = 17$ and $c_2 = -10$, so that the solution to the initial value problem is

$$y(t) = 17e^{-2t} - 10e^{-t} - 7 \cos 4t + 6 \sin 4t,$$

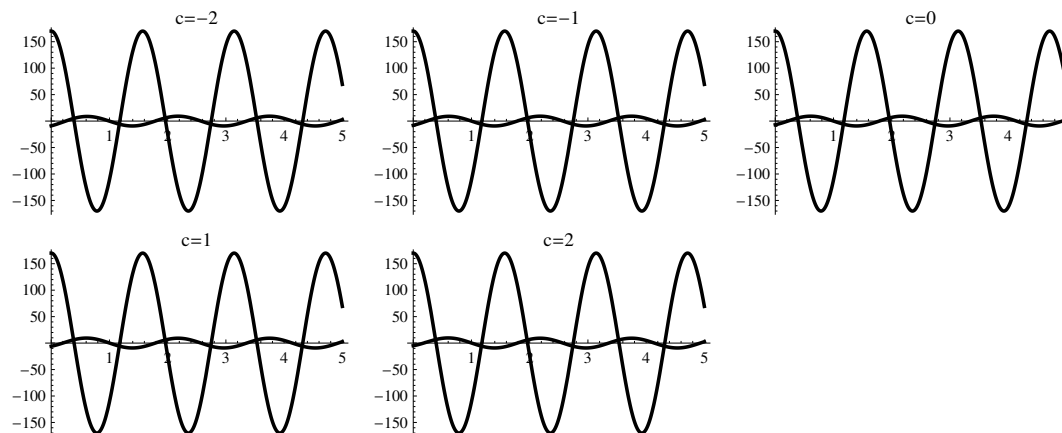
with transient and steady-state parts

$$\begin{aligned} y_{\text{transient}} &= 17e^{-2t} - 10e^{-t}, \\ y_{\text{ss}} &= -7 \cos 4t + 6 \sin 4t. \end{aligned}$$

b. Setting $c_2 = 0$ in the general solution gives

$$y(t) = c_1 e^{-2t} - 7 \cos 4t + 6 \sin 4t.$$

Graphs of this equation for $c_1 \in \{-2, -1, 0, 1, 2\}$, plotted along with the external forcing function $170 \cos 4t$, are:



c. In amplitude-phase form, the steady-state solution is

$$A \sin(4t + \varphi) = -7 \cos 4t + 6 \sin 4t, \text{ where } A = \sqrt{7^2 + 6^2} = \sqrt{85}, \tan \varphi = \frac{-7}{6}.$$

Thus $\varphi = \arctan \frac{-7}{6}$, so that the steady-state solution is $\approx 9.22 \sin(4t - 0.862) = 9.22 \sin(4(t - 0.216))$. Thus the amplitude of the steady-state solution is much smaller than the amplitude of the forcing function, and is shifted 0.216 radians to the right.

D2.4.38 Differentiating gives

$$y_p'' = \frac{F_0}{m(\omega_0^2 - \omega^2)} \cdot (-\omega^2) \cos \omega t,$$

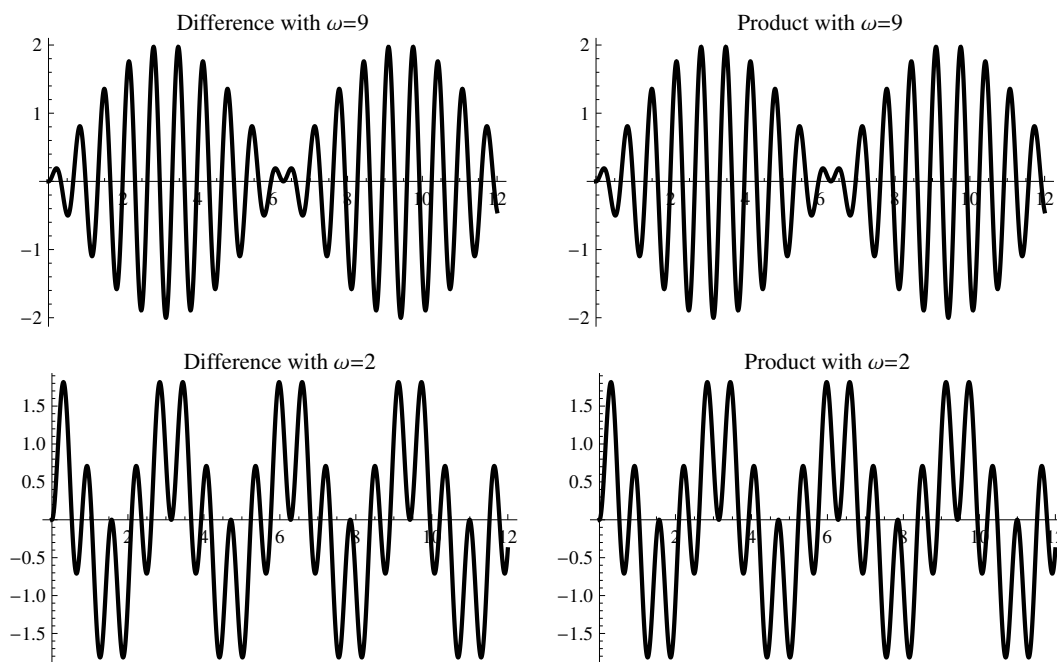
so that

$$\begin{aligned} y_p'' + \omega_0^2 y_p &= \frac{F_0}{m(\omega_0^2 - \omega^2)} \cdot (-\omega^2) \cos \omega t + \omega_0^2 \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos \omega t \\ &= \frac{F_0}{m(\omega_0^2 - \omega^2)} \cdot (\omega_0^2 - \omega^2) \cos \omega t \\ &= \frac{F_0}{m} \cos \omega t. \end{aligned}$$

D2.4.39

a. The identity follows trivially on substituting $A = \omega t$ and $B = \omega_0 t$ into the given identity.

b. See the plots below:



Beats are present in the first case, when $\omega = 9$, since ω and ω_0 are relatively close.

D2.4.40

- a. Divide through by m to get the equation in standard form

$$y'' + \frac{c}{m}y' + \frac{k}{m}y = \frac{F_0}{m}\cos\omega t.$$

Write $b = \frac{c}{m}$ and (since $\frac{k}{m} > 0$) $\omega_0^2 = \frac{k}{m}$. Then the characteristic polynomial of the homogeneous equation is $r^2 + br + \omega_0^2$ with $b > 0$. Its roots are then

$$r_1, r_2 = \frac{-b \pm \sqrt{b^2 - 4\omega_0^2}}{2}.$$

There are three possibilities:

- b. $b^2 - 4\omega_0^2 > 0$. Then surely $0 < b^2 - \omega_0^2 < b^2$, so that $\sqrt{b^2 - 4\omega_0^2} < b$. It follows that both $-b + \sqrt{b^2 - 4\omega_0^2}$ and $-b - \sqrt{b^2 - 4\omega_0^2}$ are negative. Thus the distinct real roots r_1 and r_2 are both negative, so that the general solution of the homogeneous equation is $c_1e^{r_1t} + c_2e^{r_2t}$; since $r_1, r_2 < 0$, this solution decays in time to zero.
- c. $b^2 - 4\omega_0^2 = 0$. Then the roots are $r_1 = r_2 = -\frac{b}{2}$, and the general solution to the homogeneous equation is $c_1e^{-bt/2} + c_2te^{-bt/2}$. This solution also decays in time: the first terms because it is exponential decay, and the second because $e^{-bt/2}$ decays faster than t grows (use L'Hôpital's Rule, for example).
- d. $b^2 - 4\omega_0^2 < 0$. Then the roots are complex conjugates, so that the general solution is

$$e^{-bt/2}(c_1 \cos \beta t + c_2 \sin \beta t), \text{ where } \beta = \frac{\sqrt{4\omega_0^2 - b^2}}{2}.$$

This solution also decays in time, since the exponential term decays while the magnitude of the second term is bounded by $\sqrt{c_1^2 + c_2^2}$.

Thus assuming all the parameters are positive, the homogeneous solution will always decay to zero over time.

- e. First note that $A \sin \omega t + B \cos \omega t$ is never a solution of the homogeneous equation, since it does not decay to zero over time. Using that as a trial solution gives

$$\begin{aligned} my'' + cy' + ky &= m(-A\omega^2 \sin \omega t - B\omega^2 \cos \omega t) + c(A\omega \cos \omega t - B\omega \sin \omega t) \\ &\quad + k(A \sin \omega t + B \cos \omega t) \\ &= ((k - m\omega^2)A - c\omega B) \sin \omega t + (c\omega A + (k - m\omega^2)B) \cos \omega t. \end{aligned}$$

Equating this to the forcing term $F_0 \cos \omega t$ gives the pair of simultaneous equations

$$\begin{aligned} (k - m\omega^2)A - c\omega B &= 0 \\ c\omega A + (k - m\omega^2)B &= F_0, \end{aligned}$$

which has solutions

$$A = \frac{c\omega F_0}{(c\omega)^2 + (k - m\omega^2)^2}, \quad B = \frac{(k - m\omega^2)F_0}{(c\omega)^2 + (k - m\omega^2)^2}.$$

- f. Start with

$$C^* \sin(\omega t + \varphi) = C^*(\sin \omega t \cos \varphi + \cos \omega t \sin \varphi) = (C^* \cos \varphi) \sin \omega t + (C^* \sin \varphi) \cos \omega t.$$

Setting this equal to $A \sin \omega t + B \cos \omega t$ gives $A = C^* \cos \varphi$ and $B = C^* \sin \varphi$. Squaring and adding gives $A^2 + B^2 = C^{*2} \cos^2 \varphi + C^{*2} \sin^2 \varphi = C^{*2}$ so that $C^* = \sqrt{A^2 + B^2}$. Dividing the equation for B by that for A gives

$$\frac{B}{A} = \frac{C^* \sin \varphi}{C^* \cos \varphi} = \tan \varphi$$

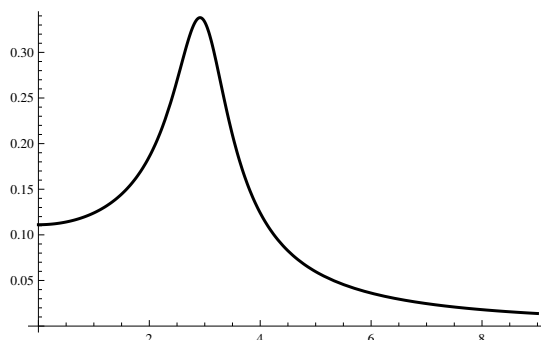
- g. We have

$$\begin{aligned} C^* = \sqrt{A^2 + B^2} &= \sqrt{\left(\frac{c\omega F_0}{(c\omega)^2 + (k - m\omega^2)^2}\right)^2 + \left(\frac{(k - m\omega^2)F_0}{(c\omega)^2 + (k - m\omega^2)^2}\right)^2} \\ &= \frac{\sqrt{(c\omega F_0)^2 + ((k - m\omega^2)F_0)^2}}{(c\omega)^2 + (k - m\omega^2)^2} \\ &= \frac{F_0 \sqrt{(c\omega)^2 + (k - m\omega^2)^2}}{(c\omega)^2 + (k - m\omega^2)^2} \\ &= \frac{F_0}{\sqrt{(c\omega)^2 + (k - m\omega^2)^2}}. \end{aligned}$$

Now, if $\omega_0^2 = \frac{k}{m}$, then $k = m\omega_0^2$. Making that substitution gives

$$C^* = \frac{F_0}{\sqrt{(c\omega)^2 + (k - m\omega^2)^2}} = \frac{F_0}{\sqrt{(c\omega)^2 + (m\omega_0^2 - m\omega^2)^2}} = \frac{F_0}{\sqrt{(c\omega)^2 + m^2(\omega_0^2 - \omega^2)^2}}.$$

- h. Because the numerator of C^* is constant, the largest amplitude results from the smallest possible denominator. The denominator has its smallest possible value when $\omega_0^2 - \omega^2 = 0$, so when $\omega = \omega_0$ (since we are assuming $\omega > 0$). In forced undamped motion, resonance (which is an unbounded increase in amplitude) occurs when $\omega = \omega_0$. Although amplitude cannot increase without bound in this case due to the damping, we still get the largest possible amplitude when $\omega = \omega_0$.
- i. With $m = c = F_0 = 1$ and $\omega_0 = 3$, a plot of C^* as a function of ω is



The amplitude increases rapidly from its initial value (in this case $\frac{1}{9}$) to a maximum at $\omega = \omega_0 = 3$, and then decreases rapidly, approaching zero asymptotically as the denominator gets increasingly large.

D2.4.41 Starting with

$$C^* = \frac{F_0}{\sqrt{c^2\omega^2 + m^2(\omega_0^2 - \omega^2)^2}},$$

substitute L for m , R for c , $\frac{1}{CL}$ for ω_0^2 , and ωE_0 for F_0 , to get

$$\begin{aligned} C^* &= \frac{\omega E_0}{\sqrt{R^2\omega^2 + L^2(\frac{1}{CL} - \omega^2)^2}} = \frac{\omega E_0}{\sqrt{R^2\omega^2 + (\frac{1}{C} - L\omega^2)^2}} = \frac{\omega E_0}{\sqrt{R^2\omega^2 + \omega^2(\frac{1}{C\omega} - L\omega)^2}} \\ &= \frac{E_0}{\sqrt{R^2 + (\frac{1}{C\omega} - L\omega)^2}} = \frac{E_0}{\sqrt{R^2 + (L\omega - \frac{1}{C\omega})^2}}. \end{aligned}$$

Because both terms in the denominator are squares, the denominator is minimized when the second term is zero, which happens when $\omega L = \frac{1}{C\omega}$. Solving for ω gives $\omega^2 = \frac{1}{CL}$.

D2.4.42

- When the spring is at equilibrium, the sum of the upward force and the downward force must be zero. The downward force is the force due to gravity, which is mg ; the restorative force due to the spring is $-k\Delta y$, where Δy is the signed amount by which the spring is stretched (positive) or compressed (negative) from its neutral (unstretched) position. Because the two forces must balance, we must have $mg + (-k\Delta y) = 0$, or $mg = k\Delta y$.
- The net force on the block, which is my'' (mass times acceleration) is the difference of the downward force, which is mg , and the upward force, which is $-k$ times the displacement, or $-k(y + \Delta y)$. Thus $my'' = -k(y + \Delta y) + mg$.
- Because $mg = k\Delta y$, we have

$$my'' = -k(y + \Delta y) + mg = -ky - k\Delta y + mg = -ky.$$

D2.4.43

- With $m = 2$, $c = 6$, $k = 8$, and no external force, the equation is

$$2x'' + 6x' + 8x = 0, \quad \text{or} \quad x'' + 3x' + 4x = 0.$$

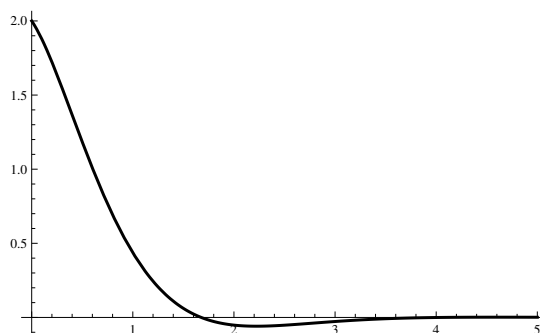
The characteristic polynomial is $r^2 + 3r + 4$, with roots $\frac{-3 \pm i\sqrt{7}}{2}$, so that the general solution is

$$\begin{aligned} x(t) &= e^{-3t/2} \left(c_1 \cos \frac{\sqrt{7}}{2}t + c_2 \sin \frac{\sqrt{7}}{2}t \right), \text{ with} \\ x'(t) &= e^{-3t/2} \left(\frac{-3c_1 + \sqrt{7}c_2}{2} \cos \frac{\sqrt{7}}{2}t + \frac{-\sqrt{7}c_1 - 3c_2}{2} \sin \frac{\sqrt{7}}{2}t \right). \end{aligned}$$

The initial conditions give $x(0) = c_1 = 2$ and $x'(0) = \frac{-3c_1 + \sqrt{7}c_2}{2} = -1$. Solving gives $c_1 = 2$ and $c_2 = \frac{4\sqrt{7}}{7}$, so that the solution to the system is

$$x(t) = e^{-3t/2} \left(2 \cos \frac{\sqrt{7}}{2} t + \frac{4\sqrt{7}}{7} \sin \frac{\sqrt{7}}{2} t \right).$$

b. The graph of $x(t)$ is



c. The motion is underdamped. It decreases to very close to zero in the first two seconds and then has very low-amplitude damped oscillations around zero, asymptotically approaching rest at position zero.

D2.4.44

a. With $m = 4$, $c = 8$, $k = 140$, and no external force, the equation is

$$4x'' + 8x' + 140x = 0, \quad \text{or} \quad x'' + 2x' + 35x = 0.$$

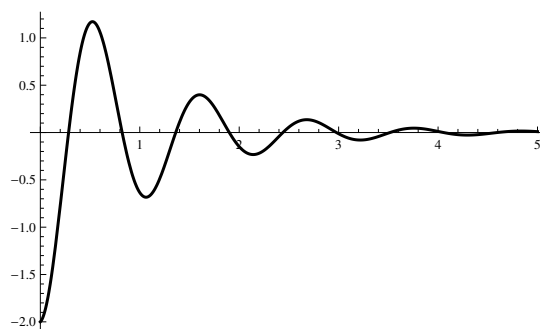
The characteristic polynomial is $r^2 + 2r + 35$, with roots $\frac{-2 \pm i\sqrt{136}}{2} = -1 \pm i\sqrt{34}$, so that the general solution is

$$\begin{aligned} x(t) &= e^{-t} (c_1 \cos \sqrt{34} t + c_2 \sin \sqrt{34} t), \quad \text{with} \\ x'(t) &= e^{-t} ((-c_1 + \sqrt{34} c_2) \cos \sqrt{34} t + (-\sqrt{34} c_1 - c_2) \sin \sqrt{34} t). \end{aligned}$$

The initial conditions give $x(0) = c_1 = -2$ and $x'(0) = -c_1 + \sqrt{34} c_2 = 1$. Solving gives $c_1 = -2$ and $c_2 = -\frac{\sqrt{34}}{34}$, so that the solution to the system is

$$x(t) = -e^{-t} \left(2 \cos \sqrt{34} t + \frac{\sqrt{34}}{34} \sin \sqrt{34} t \right).$$

b. The graph of $x(t)$ is



c. The motion is underdamped. It starts with an amplitude of -2 , and is damped by a little less than a factor of 2 at each extremum. It settles down to close to zero by about $t = 4$.

D2.4.45

- a. With $m = 4$, $c = 4$, $k = 17$, and $F_{\text{ext}} = 148 \sin t$, the equation is

$$4x'' + 4x' + 17x = 148 \sin t, \quad \text{or} \quad x'' + x' + \frac{17}{4}x = 37 \sin t.$$

The characteristic polynomial of the homogeneous equation is $4r^2 + 4r + 17$, with roots $\frac{-1}{2} \pm 2i$, so that the general solution of the homogeneous equation is

$$x(t) = e^{-t/2}(c_1 \cos 2t + c_2 \sin 2t).$$

Using $A \cos t + B \sin t$ as a trial solution, we get

$$\begin{aligned} x'' + x' + \frac{17}{4}x &= -A \cos t - B \sin t - A \sin t + B \cos t + \frac{17}{4}A \cos t + \frac{17}{4}B \sin t \\ &= \left(\frac{13}{4}A + B\right) \cos t + \left(-A + \frac{13}{4}B\right) \sin t = 37 \sin t. \end{aligned}$$

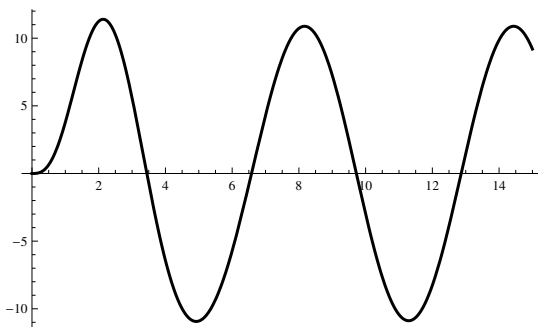
Solving the two resulting equations for A and B gives $A = -\frac{16}{5}$ and $B = \frac{52}{5}$. Thus the general solution of the nonhomogeneous equation is

$$\begin{aligned} x(t) &= e^{-t/2}(c_1 \cos 2t + c_2 \sin 2t) - \frac{16}{5} \cos t + \frac{52}{5} \sin t, \quad \text{with} \\ x'(t) &= e^{-t/2}\left(-\frac{1}{2}c_1 + 2c_2\right) \cos 2t + \left(-2c_1 - \frac{1}{2}c_2\right) \sin 2t + \frac{52}{5} \cos t + \frac{16}{5} \sin t. \end{aligned}$$

The initial conditions give $x(0) = c_1 - \frac{16}{5} = 0$ and $x'(0) = -\frac{1}{2}c_1 + 2c_2 + \frac{52}{5} = 0$. Solving gives $c_1 = \frac{16}{5}$ and $c_2 = -\frac{22}{5}$, so that the solution to the system is

$$x(t) = e^{-t/2}\left(\frac{16}{5} \cos 2t - \frac{22}{5} \sin 2t\right) - \frac{16}{5} \cos t + \frac{52}{5} \sin t.$$

- b. The graph of $x(t)$ is



- c. The transient motion is almost invisible (you can see it very close to $t = 0$, where the graph is flatter than it should be), so that essentially all of the motion is steady-state motion, with an amplitude of about 11 (the amplitude is $\frac{\sqrt{16^2 + 52^2}}{5} \approx 10.881$).

D2.4.46

- a. With $m = 8$, $c = 16$, $k = 136$, and $F_{\text{ext}} = 130 \cos t$, the equation is

$$8x'' + 16x' + 136x = 130 \cos t, \quad \text{or} \quad x'' + 2x' + 17x = \frac{65}{4} \cos t.$$

The characteristic polynomial of the homogeneous equation is $r^2 + 2r + 17$, with roots $-1 \pm 4i$, so that the general solution of the homogeneous equation is

$$x(t) = e^{-t}(c_1 \cos 4t + c_2 \sin 4t).$$

Using $A \cos t + B \sin t$ as a trial solution, we get

$$\begin{aligned} x'' + 2x' + 17x &= -A \cos t - B \sin t - 2A \sin t + 2B \cos t + 17A \cos t + 17B \sin t \\ &= (16A + 2B) \cos t + (-2A + 16B) \sin t = \frac{65}{4} \cos t. \end{aligned}$$

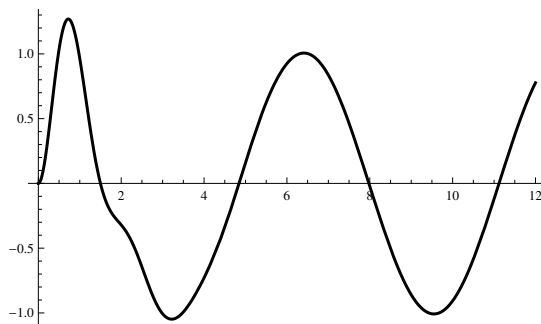
Solving the two resulting equations for A and B gives $A = 1$ and $B = \frac{1}{8}$. Thus the general solution of the nonhomogeneous equation is

$$\begin{aligned} x(t) &= e^{-t}(c_1 \cos 4t + c_2 \sin 4t) + \cos t + \frac{1}{8} \sin t, \text{ with} \\ x'(t) &= e^{-t}((-c_1 + 4c_2) \cos 4t + (-4c_1 - c_2) \sin 4t) + \frac{1}{8} \cos t - \sin t. \end{aligned}$$

The initial conditions give $x(0) = c_1 + 1 = 0$ and $x'(0) = -c_1 + 4c_2 + \frac{1}{8} = 0$. Solving gives $c_1 = -1$ and $c_2 = -\frac{9}{32}$, so that the solution to the system is

$$x(t) = -e^{-t}(\cos 4t + \frac{9}{32} \sin 4t) + \cos t + \frac{1}{8} \sin t.$$

b. The graph of $x(t)$ is



c. This is a system that decays to the steady state solution, which has amplitude $\sqrt{1 + \frac{1}{64}} = \frac{\sqrt{65}}{8} \approx 1.01$.

D2.4.47

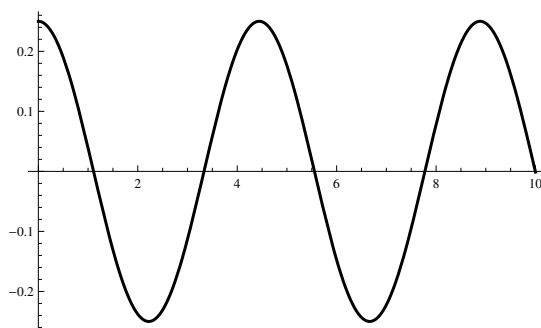
- To compute the component of the gravitational force in the direction of motion, note that if you extend the pendulum line past the end of the pendulum, it forms an angle of θ with the vertical line from the end of the pendulum; it is then clear that the component of the gravitational force in the direction of motion (perpendicular to the pendulum) is $mg \sin \theta$. By Newton's second law, that component of gravitational force is equal to the inertial force, which is given by $-ml \sin \theta''(t)$. Thus these two are equal, so that $ml \sin \theta''(t) = -mg \sin \theta(t)$.
- Rewriting gives $ml \sin \theta''(t) + mg \sin \theta(t) = 0$; divide through by ml to get the equation $\sin \theta''(t) + \frac{g}{l} \sin \theta(t) = 0$. With $\omega_0^2 = \frac{g}{l}$, this becomes $\sin \theta''(t) + \omega_0^2 \sin \theta(t) = 0$.
- Assuming that θ is small enough so that $\sin \theta \approx \theta$ is a good approximation, making that substitution gives $\sin \theta''(t) + \omega_0^2 \theta(t) = 0$.
- The solution to this equation is $\theta(t) = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t$, so that the frequency is $\omega_0 = \sqrt{\frac{g}{l}}$. Thus the period is $T = \frac{2\pi}{\omega_0} = 2\pi \sqrt{\frac{l}{g}}$, and doubling the length will increase the period by a factor of $\sqrt{2}$.

D2.4.48

- a. Here $\omega_0^2 = \frac{g}{l} = \frac{9.8}{4.9} = 2$ and $m = 1$, so that the pendulum equation is $\theta''(t) + 2\theta(t) = 0$; the general solution is $\theta(t) = c_1 \cos \sqrt{2}t + c_2 \sin \sqrt{2}t$. The initial condition $\theta(0) = 0.25$ gives $c_1 = 0.25$; the initial condition $\theta'(0) = \sqrt{2}c_2 = 0$ gives $c_2 = 0$. Thus the solution to the initial value problem is

$$\theta(t) = 0.25 \cos \sqrt{2}t.$$

A graph of this solution over time shows the expected sinusoidal steady-state solution:



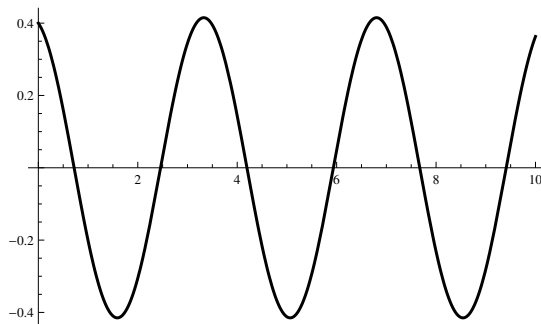
- b. The solution is already in phase-amplitude form; it has amplitude 0.25.
 c. The period of the pendulum is $\frac{2\pi}{\omega_0} = \frac{2\pi}{\sqrt{2}} = \pi\sqrt{2}$.

D2.4.49

- a. With $\omega_0^2 = \frac{g}{l} = \frac{9.8}{3} = \frac{49}{15}$, we get the equation $\theta'' + \frac{49}{15}\theta = 0$, with general solution $\theta(t) = c_1 \cos \frac{7\sqrt{15}}{15}t + c_2 \sin \frac{7\sqrt{15}}{15}t$. The initial condition $\theta(0) = 0.4$ gives $c_1 = 0.4$; the initial condition $\theta'(0) = -0.2$ gives $\frac{7\sqrt{15}}{15}c_2 = -0.2$, or $c_2 = -\frac{3}{7\sqrt{15}} = -\frac{\sqrt{15}}{35}$. Thus the solution to the initial value problem is

$$\theta(t) = \frac{2}{5} \cos \frac{7\sqrt{15}}{15}t - \frac{\sqrt{15}}{35} \sin \frac{7\sqrt{15}}{15}t$$

A plot of this solution is



- b. The amplitude of the solution is

$$\sqrt{\frac{4}{25} + \frac{15}{1225}} = \sqrt{\frac{211}{1225}} = \frac{\sqrt{211}}{35} \approx 0.415,$$

and the phase is given by

$$\tan \phi = \frac{2/5}{-\sqrt{15}/35} = -\frac{14\sqrt{15}}{15},$$

so that $\phi \approx -1.301$. Thus the phase-amplitude form of the solution is

$$\theta(t) \approx 0.415 \sin(1.807(t - 0.720)),$$

so that θ has an amplitude of 0.415 and is shifted right by 0.720 radians. (This can also be expressed in terms of a left shift by adding π to the phase).

- c. The period of the pendulum is $\frac{2\pi}{\omega_0} = \frac{2\pi}{7\sqrt{15}/15} = \frac{2\sqrt{15}\pi}{7} \approx 3.476$.

D2.4.50

- a. At rest, the upward buoyant force on the cylinder balances the gravitational force, so that the upward buoyant force is $mg = \rho ALg$ (because AL is the volume of the cylinder, and ρ its density). The volume of water displaced by the cylinder is $\frac{y^*}{L} \cdot AL = Ay^*$, since $\frac{y^*}{L}$ is the fraction of the cylinder that is submerged. The weight of that much water is $Ay^*\rho_w g$. Archimedes' principle says that the buoyant force and the weight of displaced water are equal, so that $\rho ALg = Ay^*\rho_w g$. Canceling Ag and solving gives

$$\frac{y^*}{L} = \frac{\rho}{\rho_w}.$$

- b. The force on the cylinder due to gravity does not change as the depth of the cylinder changes. However, as the cylinder's depth changes, the weight of water displaced changes, and thus the buoyant force changes. The additional weight of water displaced when the cylinder is at depth $y(t)$ (where $y = 0$ corresponds to equilibrium) is $y(t)A\rho_w g$, so that the additional buoyant force is $-y(t)A\rho_w g = -\rho_w A y g$.
- c. Because from the previous part we have $my'' = F_{\text{ext}} = -\rho_w A y g$ and $m = \rho AL$ is the mass of the cylinder, substituting and dividing through by ρAL gives $y'' = -\frac{\rho_w g}{\rho L} y$, or $y'' = -\omega_0^2 y$, where $\omega_0^2 = \frac{\rho_w g}{\rho L}$. This equation represents undamped oscillation with no external forcing function.
- d. The period of the oscillation is

$$T = \frac{2\pi}{\omega_0} = 2\pi \sqrt{\frac{\rho L}{\rho_w g}}$$

(note the similarity with the result of Exercise 47(d)). The period varies as the square root of the length of the cylinder, and is larger when $\rho \approx \rho_w$ than when $\rho \ll \rho_w$, from the form of the equation for the period.

- e. When $\rho = \rho_w$, the period is $T = 2\pi \sqrt{\frac{L}{g}}$. Because $\rho = \rho_w$, the result of part (a) shows that at equilibrium, the entire cylinder is submerged - the buoyant force when the entire cylinder is submerged exactly balances the weight of water displaced. Any displacement in the depth of the cylinder will result in an oscillatory motion around this equilibrium.

D2.4.51

- a. With the values given, we have

$$\omega_0^2 = \frac{\rho_w g}{\rho L} = \frac{\rho_w}{\rho} \frac{g}{L} = 2 \cdot \frac{9.8}{4.9} = 4,$$

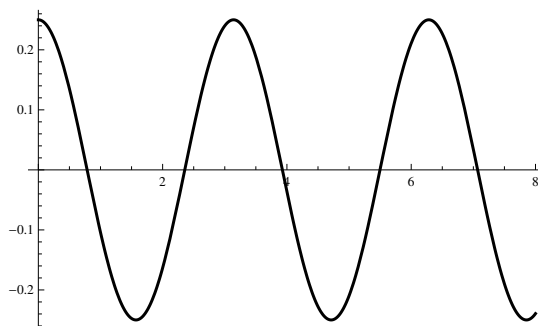
so that the initial value problem is

$$y'' + 4y = 0, \quad y(0) = 0.25, \quad y'(0) = 0.$$

The general solution is $y(t) = c_1 \cos 2t + c_2 \sin 2t$. Setting $y(0) = 0.25$ gives $c_1 = 0.25$; setting $y'(0) = 0$ gives $c_2 = 0$. Thus the solution to the initial value problem is

$$y(t) = 0.25 \cos 2t,$$

with graph



b. The solution is already in phase-amplitude form.

c. The period of the oscillation is $\frac{2\pi}{\omega_0} = \pi$.

D2.4.52

a. With the values given, we have

$$\omega_0^2 = \frac{\rho_w g}{\rho L} = \frac{\rho_w}{\rho} \frac{g}{L} = \frac{10}{7} \cdot \frac{9.8}{2} = 7,$$

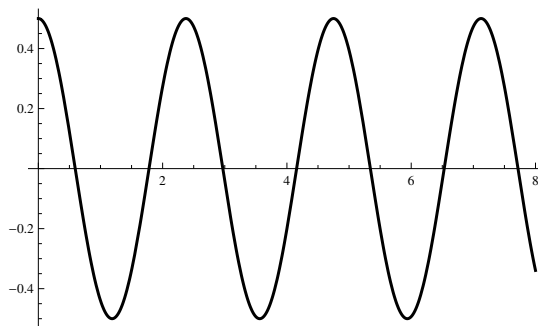
so that the initial value problem is

$$y'' + 7y = 0, \quad y(0) = 0.5, \quad y'(0) = 0.$$

The general solution is $y(t) = c_1 \cos \sqrt{7}t + c_2 \sin \sqrt{7}t$. Setting $y(0) = 0.5$ gives $c_1 = 0.5$; setting $y'(0) = 0$ gives $c_2 = 0$. Thus the solution to the initial value problem is

$$y(t) = 0.5 \cos \sqrt{7}t,$$

with graph



b. The solution is already in phase-amplitude form.

c. The period of the oscillation is $\frac{2\pi}{\omega_0} = \frac{2\pi\sqrt{7}}{7}$.

D2.4.53

a. Differentiating the first equation with respect to t gives

$$x_1'' = -k_1 x_1' + k_2 x_2' + f'(t).$$

Substituting for x_2' from the second equation gives

$$x_1'' = -k_1 x_1' + k_2(k_1 x_1 - k_2 x_2 - k_3 x_2) + f'(t) = -k_1 x_1' + k_1 k_2 x_1 - (k_2 + k_3)k_2 x_2 + f'(t).$$

Now substitute for k_2x_2 using the first equation to get

$$\begin{aligned}x_1'' &= -k_1x_1' + k_1k_2x_1 - (k_2 + k_3)k_2x_2 + f'(t) \\&= -k_1x_1' + k_1k_2x_1 - (k_2 + k_3)(x_1' + k_1x_1 - f(t)) + f'(t) \\&= (-k_1 - k_2 - k_3)x_1' - k_1k_3x_1 + (k_2 + k_3)f(t) + f'(t).\end{aligned}$$

Replacing x_1 by x gives the desired equation:

$$x'' = -(k_1 + k_2 + k_3)x' - k_1k_3x + (k_2 + k_3)f(t) + f'(t).$$

- b. The initial condition $x_1(0) = A$ is the same as $x(0) = A$ since x is just x_1 renamed. Evaluating the first of the original equations at $t = 0$ and setting it equal to zero for the second initial condition gives

$$x'(0) = x_1'(0) = -k_1x_1(0) + k_2x_2(0) + f(0) = -k_1A + f(0)$$

since $x_1(0) = A$ and $x_2(0) = 0$. Thus the resulting initial conditions are $x(0) = A$, $x'(0) = f(0) - k_1A$.

D2.4.54

- a. From the previous exercise, the initial value problem for the mass of drug in compartment 1 is

$$x'' = -(0.1 + 0.02 + 0.05)x' - 0.005x + 0.07, \quad x(0) = x_1(0) = 0, \quad x'(0) = f(0) - k_1x(0) = 1,$$

or

$$x'' = -0.17x' - 0.005x + 0.07, \quad x(0) = 0, \quad x'(0) = 1.$$

Reorganizing gives $x'' + 0.17x' + 0.005x = 0.07$. The characteristic polynomial for the homogeneous equation is thus $r^2 + 0.17r + 0.005$, which has roots $r \approx -0.132$, -0.038 , so that the general solution to the homogeneous equation is

$$x(t) = c_1e^{-0.132t} + c_2e^{-0.038t}.$$

Further, $\frac{0.07}{0.005} = 14$ is obviously a particular solution to the nonhomogeneous equation, so that the general solution to the nonhomogeneous equation is

$$x(t) = c_1e^{-0.132t} + c_2e^{-0.038t} + 14.$$

Then $x(0) = c_1 + c_2 + 14 = 0$ and $x'(0) = -0.132c_1 - 0.038c_2 = 1$. Solving this pair of equations gives $c_1 = -4.979$ and $c_2 = -9.021$, so that the solution to the initial value problem for $x = x_1$ is

$$x(t) = -4.979e^{-0.132t} - 9.021e^{-0.038t} + 14.$$

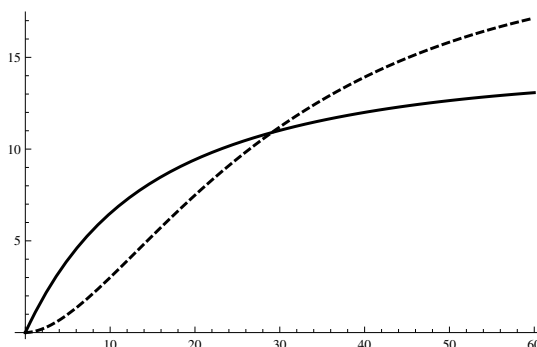
- b. Substituting for $x_1(t)$ and $x_1'(t)$ in the equation

$$x_1' = -k_1x_1 + k_2x_2 + f(t) = -0.1x_1 + 0.02x_2 + 1$$

and simplifying gives

$$x_2(t) = 7.966e^{-0.132t} - 27.965e^{-0.038t} + 20.$$

- c. A graph of $x_1(t)$ (solid black) and $x_2(t)$ (dashed) is:



In this situation, a constant infusion of one unit per unit time is given, represented by $f(t)$, and the initial concentration of the drug is zero. Over time, the concentrations increase; it appears that the concentration of the drug in the bloodstream will level off at 14 (and indeed $\lim_{t \rightarrow \infty} x_1(t) = 14$). Similarly, the level in the kidneys will asymptotically approach 20.

D2.4.55

- a. From Exercise 53, the initial value problem for the mass of drug in compartment 1 is

$$\begin{aligned}x'' &= -(0.1 + 0.02 + 0.05)x' - 0.005x, & x(0) &= x_1(0) = 10, \\x'(0) &= f(0) - k_1x(0) = 0 - 0.1 \cdot 10 = -1, & \text{or} \\x'' &= -0.17x' - 0.005x, & x(0) &= 10, \quad x'(0) = -1.\end{aligned}$$

Reorganizing gives $x'' + 0.17x' + 0.005x = 0$. The characteristic polynomial for the homogeneous equation is thus $r^2 + 0.17r + 0.005$, which has roots $r \approx -0.132, -0.038$, so that the general solution to the equation is

$$x(t) = c_1e^{-0.132t} + c_2e^{-0.038t}.$$

Then $x(0) = c_1 + c_2 = 10$ and $x'(0) = -0.132c_1 - 0.038c_2 = -1$. Solving this pair of equations gives $c_1 = 6.596$ and $c_2 = 3.404$, so that the solution to the initial value problem for $x = x_1$ is

$$x_1(t) = 6.596e^{-0.132t} + 3.404e^{-0.038t}.$$

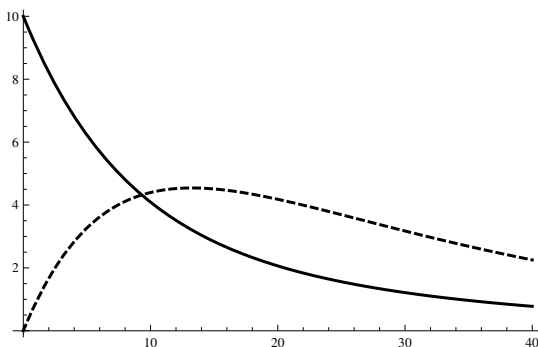
- b. Substituting for $x_1(t)$ and $x'_1(t)$ in the equation

$$x'_1 = -k_1x_1 + k_2x_2 + f(t) = -0.1x_1 + 0.02x_2$$

and simplifying gives

$$x_2(t) = -10.554e^{-0.132t} + 10.552e^{-0.038t}.$$

- c. A graph of $x_1(t)$ (solid black) and $x_2(t)$ (dashed) is:



Starting with the initial amount of 10 in the blood, the drug enters the kidney over time, reaching its maximum concentration at about $t = 12$. The concentration then declines in the kidney as well, and over time the concentration in both the bloodstream and the kidneys asymptotically approaches zero, although it does so more slowly in the kidneys.

Complex Forcing Functions

D2.5.1 The transfer function $H(\omega) = \frac{1}{-\omega^2 + ib\omega + \omega_0^2}$ is written in terms of the gain and phase lag functions as

$$H(\omega) = g(\omega)e^{i\gamma(\omega)},$$

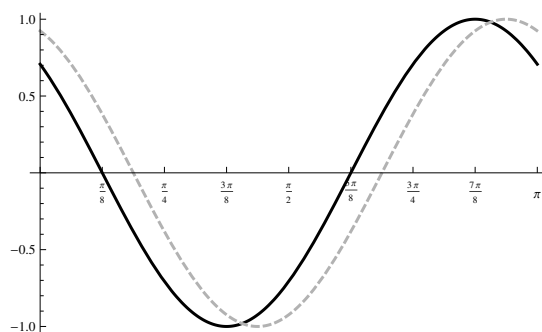
where $g(\omega)$ is the gain function and $\gamma(\omega)$ is the phase lag function.

D2.5.2 The gain function represents the change in amplitude between the input and the output. The amplitude of the output is $g(\omega)F_0$, where F_0 is the input amplitude.

D2.5.3 The phase lag function gives the amount by which the output lags the input: the output is

$$g(\omega)F_0 \cos(\omega t + \alpha + \gamma(\omega)).$$

D2.5.4 With $\omega = 2$, $\alpha = \frac{\pi}{4}$, and $\gamma(\omega) = -\frac{\pi}{8}$, a plot of the input $\cos(\omega t + \alpha)$ and the output $\cos(\omega t + \alpha + \gamma(\omega))$ is below, with the input in black and the output the dashed line in gray:



Note that the output curve is to the right of (lags) the input curve. The amount of the lag is $\frac{\pi}{16}$, which is $\frac{\gamma(\omega)}{\omega}$. This is what you would expect, since the output can also be written as

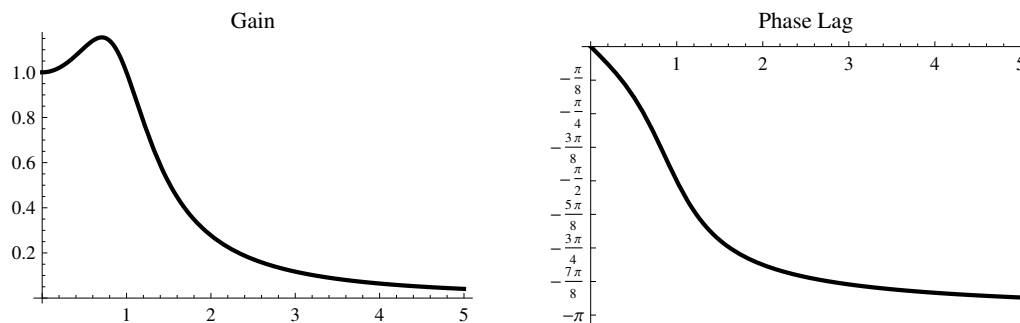
$$\cos\left(\omega\left(t + \frac{\gamma}{\omega}\right) + \alpha\right).$$

D2.5.5

a. With $b = 1$ and $\omega_0 = 1$, the gain and phase lag functions are

$$g(\omega) = \frac{1}{\sqrt{(\omega_0^2 - \omega^2)^2 + b^2\omega^2}} = \frac{1}{\sqrt{\omega^4 - \omega^2 + 1}}, \quad \tan \gamma(\omega) = -\frac{b\omega}{\omega_0^2 - \omega^2} = \frac{\omega}{\omega^2 - 1},$$

and then we compute $\gamma(\omega)$ using $(-\pi, 0]$ as the range of $\tan^{-1}$. Plots of $g(\omega)$ and $\gamma(\omega)$ are shown below:



b. Because $b = 1 < \sqrt{2}\omega_0$, this system has weak damping, so that the gain function has its maximum at

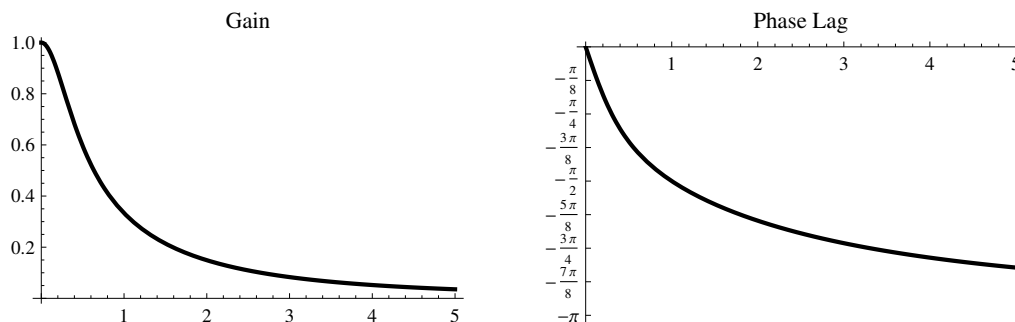
$$\omega = \sqrt{\omega_0^2 - \frac{b^2}{2}} = \sqrt{\frac{1}{2}} = \frac{\sqrt{2}}{2}.$$

D2.5.6

- a. With $b = 3$ and $\omega_0 = 1$, the gain and phase lag functions are

$$g(\omega) = \frac{1}{\sqrt{(\omega_0^2 - \omega^2)^2 + b^2\omega^2}} = \frac{1}{\sqrt{\omega^4 + 7\omega^2 + 1}}, \quad \tan \gamma(\omega) = -\frac{b\omega}{\omega_0^2 - \omega^2} = \frac{3\omega}{\omega^2 - 1},$$

and then we compute $\gamma(\omega)$ using $(-\pi, 0]$ as the range of $\tan^{-1}$. Plots of these two functions are shown below:



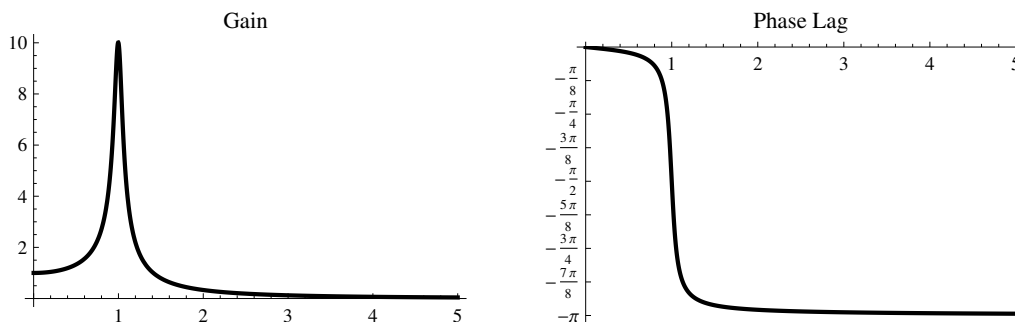
- b. Because $b = 3 > \sqrt{2}\omega_0$, this system has strong damping, so that the gain function has its maximum at $\omega = 0$.

D2.5.7

- a. With $b = 0.1$ and $\omega_0 = 1$, the gain and phase lag functions are

$$g(\omega) = \frac{1}{\sqrt{(\omega_0^2 - \omega^2)^2 + b^2\omega^2}} = \frac{1}{\sqrt{\omega^4 - 1.99\omega^2 + 1}}, \quad \tan \gamma(\omega) = -\frac{b\omega}{\omega_0^2 - \omega^2} = \frac{0.1\omega}{\omega^2 - 1},$$

and then we compute $\gamma(\omega)$ using $(-\pi, 0]$ as the range of $\tan^{-1}$. Plots of these two functions are shown below:



- b. Because $b = 0.1 < \sqrt{2}\omega_0$, this system has weak damping, so that the gain function has its maximum at

$$\omega = \sqrt{\omega_0^2 - \frac{b^2}{2}} = \sqrt{0.995} \approx 0.9975.$$

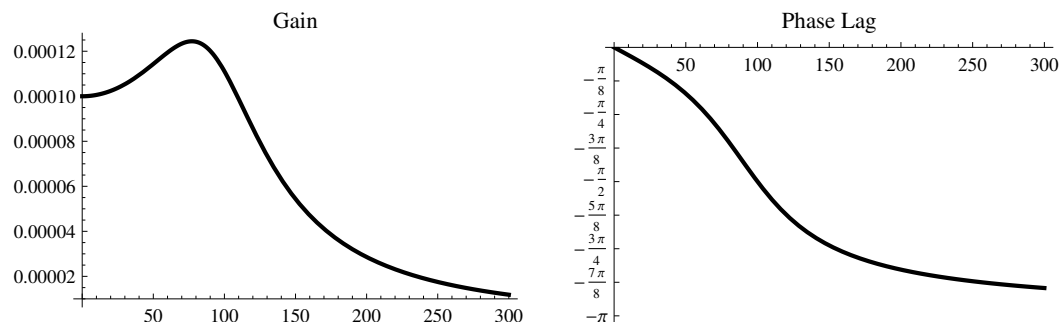
D2.5.8

- a. With $b = 90$ and $\omega_0 = 100$, the gain and phase lag functions are

$$g(\omega) = \frac{1}{\sqrt{(\omega_0^2 - \omega^2)^2 + b^2\omega^2}} = \frac{1}{\sqrt{\omega^4 - 11900\omega^2 + 10^8}},$$

$$\tan \gamma(\omega) = -\frac{b\omega}{\omega_0^2 - \omega^2} = \frac{90\omega}{\omega^2 - 10000},$$

and then we compute $\gamma(\omega)$ using $(-\pi, 0]$ as the range of $\tan^{-1}$. Plots of these two functions are shown below:



b. Because $b = 90 < \sqrt{2}\omega_0$, this system has weak damping, so that the gain function has its maximum at

$$\omega = \sqrt{\omega_0^2 - \frac{b^2}{2}} = \sqrt{10000 - \frac{8100}{2}} \approx 77.1362.$$

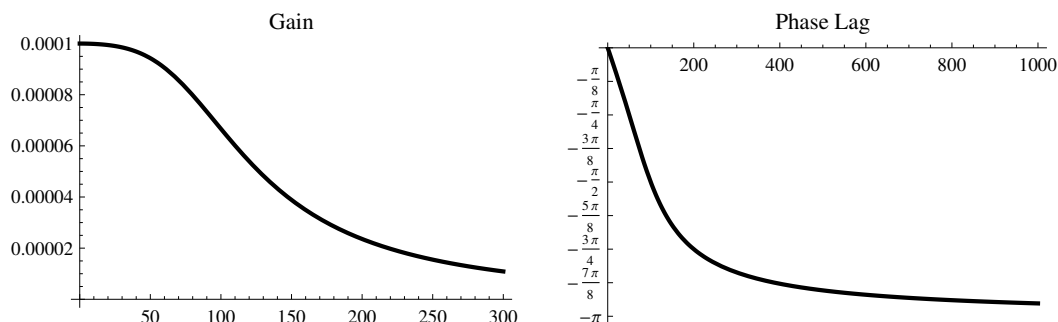
D2.5.9

a. With $b = 150$ and $\omega_0 = 100$, the gain and phase lag functions are

$$g(\omega) = \frac{1}{\sqrt{(\omega_0^2 - \omega^2)^2 + b^2\omega^2}} = \frac{1}{\sqrt{\omega^4 + 2500\omega^2 + 10^8}},$$

$$\tan \gamma(\omega) = -\frac{b\omega}{\omega_0^2 - \omega^2} = \frac{150\omega}{\omega^2 - 10000},$$

and then we compute $\gamma(\omega)$ using $(-\pi, 0]$ as the range of $\tan^{-1}$. Plots of these two functions are shown below:



b. Because $b = 150 > \sqrt{2}\omega_0 \approx 141.4$, this system has strong damping, so that the gain function has its maximum at $\omega = 0$.

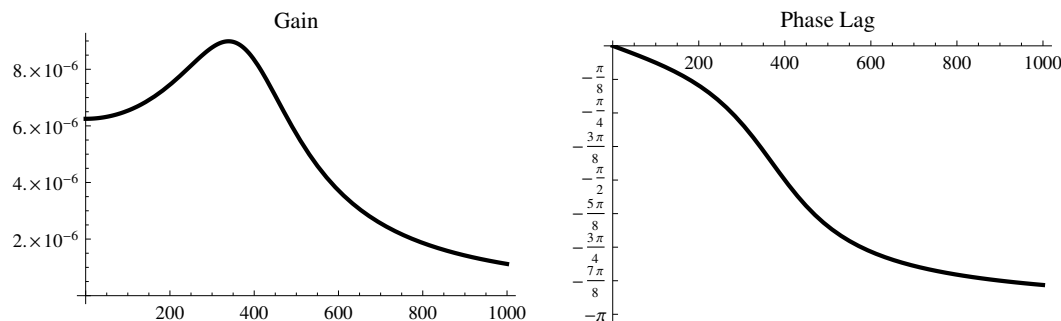
D2.5.10

a. With $b = 300$ and $\omega_0 = 400$, the gain and phase lag functions are

$$g(\omega) = \frac{1}{\sqrt{(\omega_0^2 - \omega^2)^2 + b^2\omega^2}} = \frac{1}{\sqrt{\omega^4 - 230000\omega^2 + 160000^2}},$$

$$\tan \gamma(\omega) = -\frac{b\omega}{\omega_0^2 - \omega^2} = \frac{300\omega}{\omega^2 - 160000},$$

and then we compute $\gamma(\omega)$ using $(-\pi, 0]$ as the range of $\tan^{-1}$. Plots of these two functions are shown below:



- b. Because $b = 300 < \sqrt{2}\omega_0$, this system has weak damping, so that the gain function has its maximum at

$$\omega = \sqrt{\omega_0^2 - \frac{b^2}{2}} = \sqrt{160000 - \frac{90000}{2}} \approx 339.116.$$

D2.5.11

- a. From Exercise 5, and using $F_0 = 2$ and $\omega = 2$, the gain and phase lag functions are

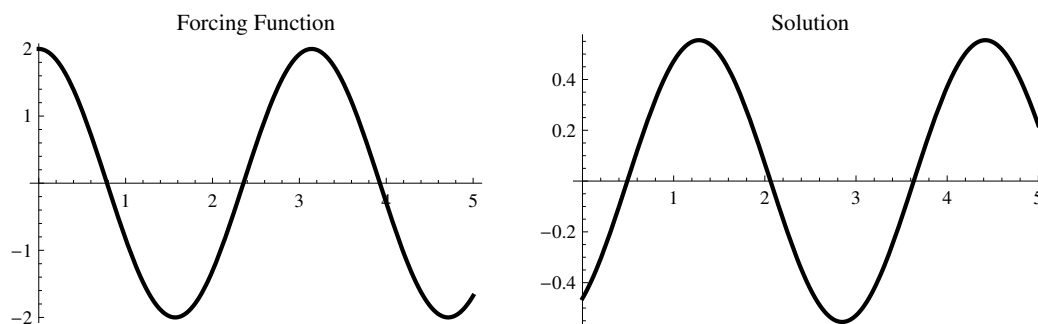
$$g(\omega) = \frac{1}{\sqrt{\omega^4 - \omega^2 + 1}} = \frac{1}{\sqrt{16 - 4 + 1}} = \frac{\sqrt{13}}{13}$$

$$\tan \gamma(\omega) = \frac{\omega}{\omega^2 - 1} = \frac{2}{3}.$$

- b. Using the appropriate range for $\tan^{-1}$, we see that $\gamma(\omega)$ is the third-quadrant angle with tangent $\frac{2}{3}$; this angle is about -2.554 radians. Thus the real part of the solution is

$$\begin{aligned} \operatorname{Re}\{y_p(t)\} &= g(\omega)F_0 \cos(\omega t + \alpha + \gamma(\omega)) \\ &= \frac{1}{\sqrt{16 - 4 + 1}} \cdot 2 \cos(2t - 2.554) \\ &= \frac{2\sqrt{13}}{13} \cos(2t - 2.554). \end{aligned}$$

- c. Graphs of the forcing function $2 \cos 2t$ and the solution above are:



Note that the amplitude of the solution is roughly 0.5, which is 0.25 times the amplitude of the input. This matches $g(\omega) = \frac{\sqrt{13}}{13} \approx 0.277$. Also, the peak of the output curve is shifted right by about 1.2, which is roughly half of the phase lag of 2.554. (We expect it to be half since $\cos(2t - 2.554) = \cos(2(t - 2.554/2))$).

D2.5.12

a. From Exercise 6, and using $F_0 = 10$ and $\omega = 1$, the gain and phase lag functions are

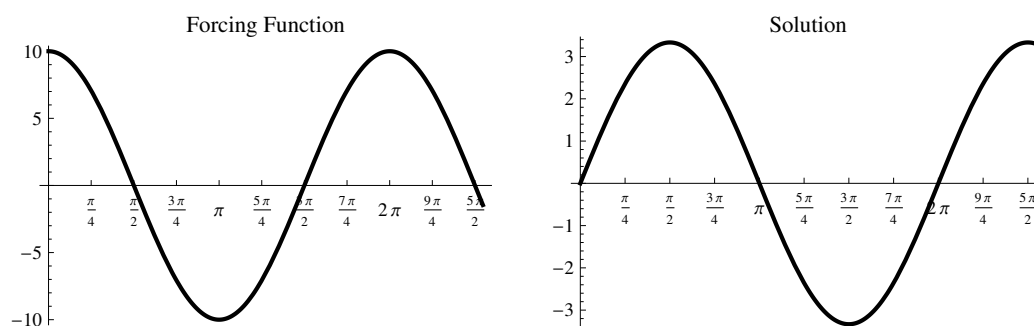
$$g(\omega) = \frac{1}{\sqrt{\omega^4 + 7\omega^2 + 1}} = \frac{1}{\sqrt{1 + 7 + 1}} = \frac{1}{3}$$

$$\tan \gamma(\omega) = \frac{3\omega}{\omega^2 - 1} = \infty.$$

b. The above gives $\gamma(\omega) = -\frac{\pi}{2}$. Thus the real part of the solution is

$$\begin{aligned} \operatorname{Re}\{y_p(t)\} &= g(\omega)F_0 \cos(\omega t + \alpha + \gamma(\omega)) \\ &= \frac{1}{\sqrt{1 + 7 + 1}} \cdot 10 \cdot \cos\left(t - \frac{\pi}{2}\right) \\ &= \frac{10}{3} \cos\left(t - \frac{\pi}{2}\right). \end{aligned}$$

c. Graphs of the forcing function $10 \cos t$ and the solution above are:



Note that the amplitude of the solution is a little more than 3, which is a little more than 0.3 times the amplitude of the input. This matches $g(\omega) = \frac{1}{\sqrt{9}} \approx 0.333$. Also, the initial peak of the output curve is shifted right by exactly $\frac{\pi}{2}$ from the peak of the input curve at 0, which matches the phase lag.

D2.5.13

- a. From Exercise 7, and using $F_0 = 5$ and $\omega = 3$, the gain and phase lag functions are

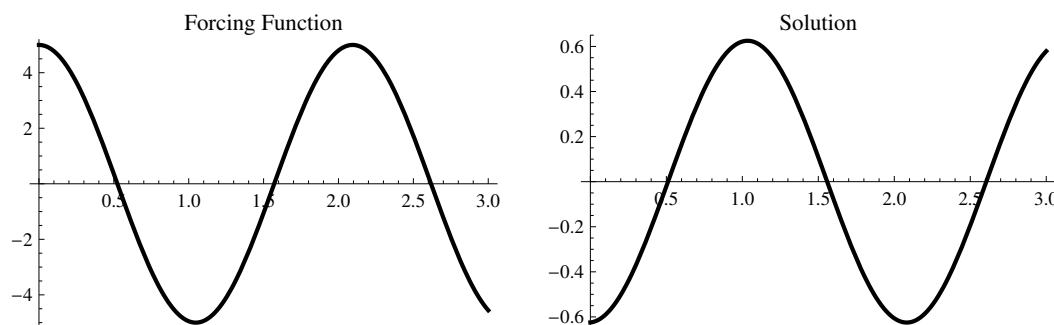
$$g(\omega) = \frac{1}{\sqrt{\omega^4 - 1.99\omega^2 + 1}} = \frac{1}{\sqrt{81 - 9 \cdot 1.99 + 1}} \approx 0.125$$

$$\tan \gamma(\omega) = \frac{0.1\omega}{\omega^2 - 1} = \frac{0.3}{8} = \frac{3}{80}.$$

- b. The above gives for $\gamma(\omega)$ the third-quadrant angle whose tangent is $\frac{3}{80}$; this angle is about -3.104 radians. Thus the real part of the solution is

$$\begin{aligned} \operatorname{Re}\{y_p(t)\} &= g(\omega)F_0 \cos(\omega t + \alpha + \gamma(\omega)) \\ &\approx 0.125 \cdot 5 \cos(3t - 3.104) = 0.625 \cos(3t - 3.104). \end{aligned}$$

- c. Graphs of the forcing function $5 \cos 3t$ and the solution above are:



From the graphs, you can see that the output is shifted right by about one unit from the input, which is what we would expect (the shift should be about $\frac{3.104}{3} \approx 1$). Also, the ratio of the amplitude of the output to that of the input is about $\frac{0.6}{5} = \frac{3}{25} = 0.12$, which is pretty close to 0.125.

D2.5.14

- a. From Exercise 8, and using $F_0 = 50$ and $\omega = 50$, the gain and phase lag functions are

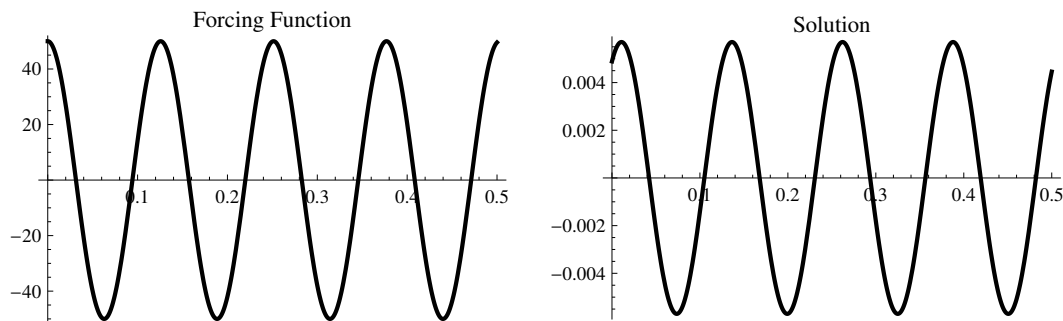
$$g(\omega) = \frac{1}{\sqrt{\omega^4 - 11900\omega^2 + 10^8}} = \frac{1}{\sqrt{50^4 - 11900 \cdot 50^2 + 10^8}} \approx 0.000114$$

$$\tan \gamma(\omega) = \frac{90\omega}{\omega^2 - 10000} = -0.6$$

- b. The above gives for $\gamma(\omega)$ the fourth-quadrant angle whose tangent is -0.6 ; this angle is about -0.540 radians. Thus the real part of the solution is

$$\begin{aligned} \operatorname{Re}\{y_p(t)\} &= g(\omega)F_0 \cos(\omega t + \alpha + \gamma(\omega)) \\ &\approx 0.000114 \cdot 50 \cos(50t - 0.540) = 0.0057 \cos(50t - 0.540). \end{aligned}$$

- c. Graphs of the forcing function $50 \cos 50t$ and the solution above are:



The ratio of the two amplitudes is about $\frac{0.0055}{50} \approx 0.00011$, which is pretty close to the actual 0.000114. The initial peak of the input curve is shifted right to between $t = 0.01$ and $t = 0.02$. This is roughly what we would expect, since the actual shift should be $\approx \frac{0.540}{50} \approx 0.011$; this is within the range of our ability to read the graph.

D2.5.15

- a. From Exercise 9, and using $F_0 = 80$ and $\omega = 200$, the gain and phase lag functions are

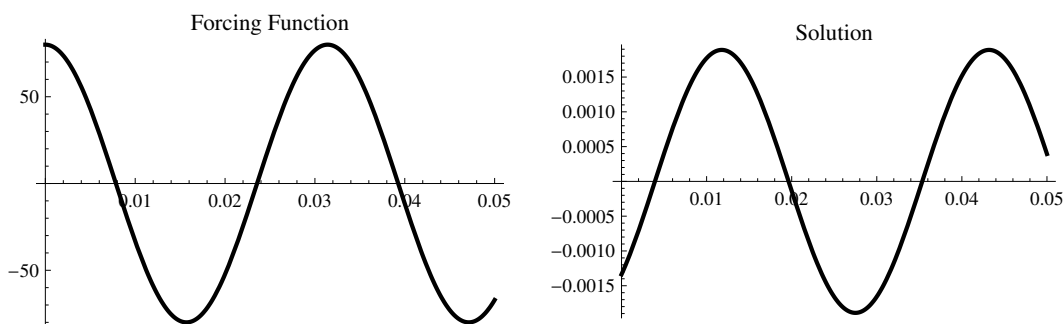
$$g(\omega) = \frac{1}{\sqrt{\omega^4 + 2500\omega^2 + 10^8}} = \frac{1}{\sqrt{200^4 + 2500 \cdot 200^2 + 10^8}} \approx 2.357 \times 10^{-5}$$

$$\tan \gamma(\omega) = \frac{150\omega}{\omega^2 - 10000} = \frac{150 \cdot 200}{200^2 - 10000} = 1.$$

- b. The above gives for $\gamma(\omega)$ the third-quadrant angle whose tangent is 1; this angle is $-\frac{3\pi}{4}$. Thus the real part of the solution is

$$\begin{aligned} \operatorname{Re}\{y_p(t)\} &= g(\omega)F_0 \cos(\omega t + \alpha + \gamma(\omega)) \\ &\approx 2.357 \times 10^{-5} \cdot 80 \cos(200t - \frac{3\pi}{4}) \approx 0.00189 \cos(200t - \frac{3\pi}{4}). \end{aligned}$$

- c. Graphs of the forcing function $80 \cos 200t$ and the solution above are:



The ratio of the two amplitudes is about $\frac{0.0019}{80} \approx 0.00002375$, which is pretty close to the actual value. The initial peak of the input curve is shifted right to about 0.012. This is roughly what we would expect, since the actual shift should be $\approx \frac{3\pi/4}{200} \approx 0.0118$.

D2.5.16

- a. From Exercise 10, and using $F_0 = 150$ and $\omega = 200$, the gain and phase lag functions are

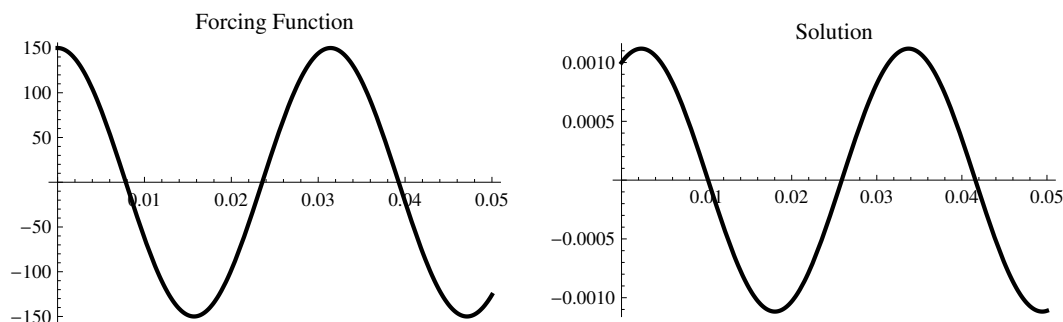
$$g(\omega) = \frac{1}{\sqrt{\omega^4 - 230000\omega^2 + 160000^2}} = \frac{1}{\sqrt{200^4 - 230000 \cdot 200^2 + 160000^2}} \approx 7.454 \times 10^{-6}$$

$$\tan \gamma(\omega) = \frac{300\omega}{\omega^2 - 160000} = \frac{60000}{200^2 - 160000} = -\frac{1}{2}.$$

- b. The above gives for $\gamma(\omega)$ the fourth-quadrant angle whose tangent is $-\frac{1}{2}$; this angle is ≈ -0.464 radians. Thus the real part of the solution is

$$\begin{aligned}\operatorname{Re}\{y_p(t)\} &= g(\omega)F_0 \cos(\omega t + \alpha + \gamma(\omega)) \\ &\approx 7.454 \times 10^{-6} \cdot 150 \cos(200t - 0.464) \approx 0.00112 \cos(200t - 0.464).\end{aligned}$$

- c. Graphs of the forcing function $150 \cos 200t$ and the solution above are:



From the graphs, the ratio of the two amplitudes is about $\frac{0.0011}{150} \approx 7.33 \times 10^{-6}$, which is pretty close to the actual value. The initial peak of the input curve is shifted right by about 0.003. This is roughly what we would expect, since the actual shift should be $\approx \frac{0.464}{200} \approx 0.0023$.

D2.5.17

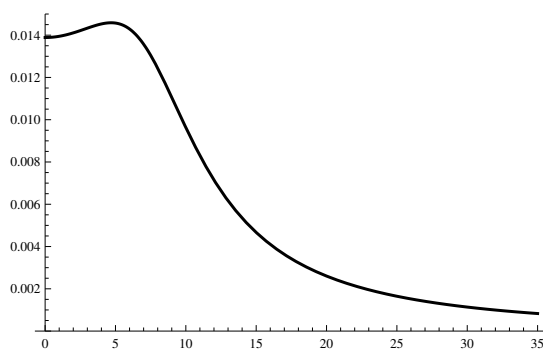
- a. With the values given, the equation is

$$v''_{\text{out}} + 10v'_{\text{out}} + 72v_{\text{out}} = 72v_{\text{in}}.$$

The gain function is thus

$$g(\omega) = \frac{1}{\sqrt{(72 - \omega^2)^2 + 100\omega^2}} = \frac{1}{\sqrt{\omega^4 - 44\omega^2 + 5184}}.$$

- b. The gain function looks like



The maximum of $g(\omega)$ for $\omega \geq 0$ occurs when the denominator is minimized, i.e. when $f(\omega) = \omega^4 - 44\omega^2 + 5184$ is minimized. Then

$$f'(\omega) = 4\omega^3 - 88\omega = 4\omega(\omega - \sqrt{22})(\omega + \sqrt{22}),$$

so that the local extrema for $\omega \geq 0$ are at $\omega = 0$ and $\omega = \sqrt{22}$. Because $f''(\omega) = 12\omega^2 - 88$ is negative at $\omega = 0$ and positive at $\omega = \sqrt{22}$, we see that $\sqrt{22}$ is a local minimum for f , so is a local maximum for g . Hence $g(\omega)$ has its maximum at $\omega = \sqrt{22} \approx 4.69$.

- c. From the graph and the above analysis, it is clear that the gain function is monotonically decreasing for $t \in (\sqrt{22}, \infty)$.
- d. From the general solution in the text, we know that

$$v_{\text{out}} = H(\omega) \frac{1}{LC} v_{\text{in}},$$

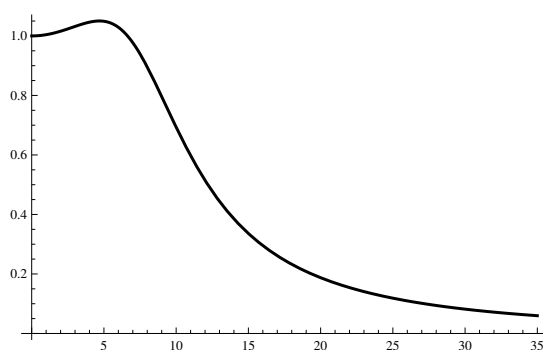
so that

$$\left| \frac{v_{\text{out}}}{v_{\text{in}}} \right| = \left| H(\omega) \frac{1}{LC} v_{\text{in}} \right| = \frac{g(\omega)}{LC},$$

since $g(\omega) = |H(\omega)|$. Substituting values, we get

$$\left| \frac{v_{\text{out}}}{v_{\text{in}}} \right| = \frac{72}{\sqrt{\omega^4 - 44\omega^2 + 5184}},$$

so that a plot of the ratio is



D2.5.18

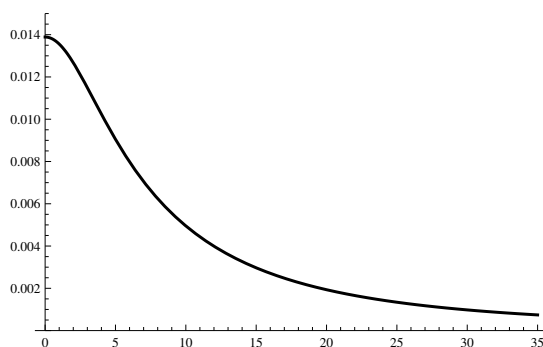
- a. With the values given, the equation is

$$v_{\text{out}}'' + 20v_{\text{out}}' + 72v_{\text{out}} = 72v_{\text{in}}.$$

The gain function is thus

$$g(\omega) = \frac{1}{\sqrt{(72 - \omega^2)^2 + 400\omega^2}} = \frac{1}{\sqrt{\omega^4 + 256\omega^2 + 5184}}.$$

- b. The gain function looks like



The maximum of $g(\omega)$ for $\omega \geq 0$ occurs when the denominator is minimized, i.e. when $f(\omega) = \omega^4 + 256\omega^2 + 5184$ is minimized. Then

$$f'(\omega) = 4\omega^3 + 512\omega = \omega(\omega - \sqrt{128})(\omega + \sqrt{128}),$$

so that the only local extremum is at $\omega = 0$. Because $f''(\omega) = 12\omega^2 + 512$ is positive at $\omega = 0$, we see that 0 is a local minimum for f and thus a local maximum for g . Hence $g(\omega)$ has its maximum at $\omega = 0$.

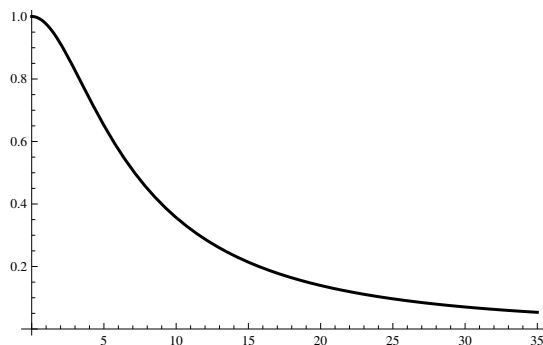
- c. From the graph and the above analysis, it is clear that the gain function is monotonically decreasing for all t .
- d. From either the text or the preceding exercise, we have

$$\left| \frac{v_{\text{out}}}{v_{\text{in}}} \right| = \frac{g(\omega)}{LC}.$$

Substituting values, we get

$$\left| \frac{v_{\text{out}}}{v_{\text{in}}} \right| = \frac{72}{\sqrt{\omega^4 + 256\omega^2 + 5184}},$$

so that a plot of the ratio is



D2.5.19

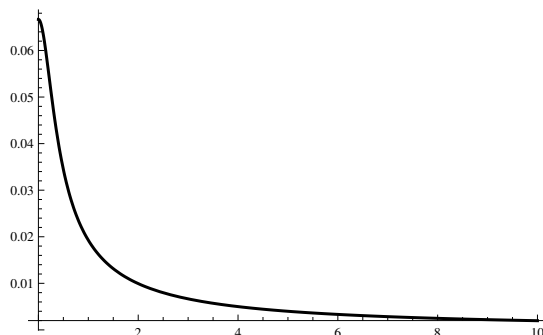
- a. With the values given, the equation is

$$v''_{\text{out}} + 50v'_{\text{out}} + 15v_{\text{out}} = 15v_{\text{in}}.$$

The gain function is thus

$$g(\omega) = \frac{1}{\sqrt{(15 - \omega^2)^2 + 2500\omega^2}} = \frac{1}{\sqrt{\omega^4 + 2470\omega^2 + 225}}.$$

- b. The gain function looks like



The maximum of $g(\omega)$ for $\omega \geq 0$ occurs when the denominator is minimized, i.e. when $f(\omega) = \omega^4 + 256\omega^2 + 5184$ is minimized. Then

$$f'(\omega) = 4\omega^3 + 512\omega = \omega(\omega - i\sqrt{128})(\omega + i\sqrt{128}),$$

so that the only local extremum is at $\omega = 0$. Because $f''(\omega) = 12\omega^2 + 512$ is positive at $\omega = 0$, we see that 0 is a local minimum for f and thus a local maximum for g . Hence $g(\omega)$ has its maximum at $\omega = 0$.

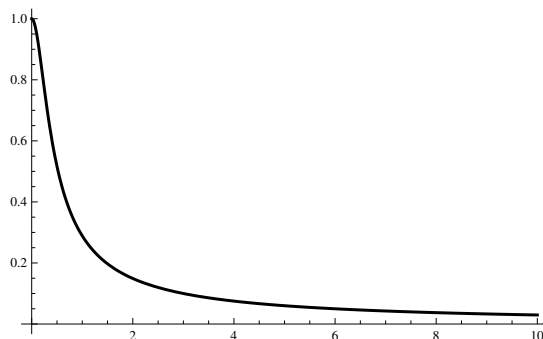
- c. From the graph and the above analysis, it is clear that the gain function is monotonically decreasing for all $t \geq 0$.
- d. From either the text or the preceding exercise, we have

$$\left| \frac{v_{\text{out}}}{v_{\text{in}}} \right| = \frac{g(\omega)}{LC}.$$

Substituting values, we get

$$\left| \frac{v_{\text{out}}}{v_{\text{in}}} \right| = \frac{72}{\sqrt{\omega^4 + 256\omega^2 + 5184}},$$

so that a plot of the ratio is



D2.5.20

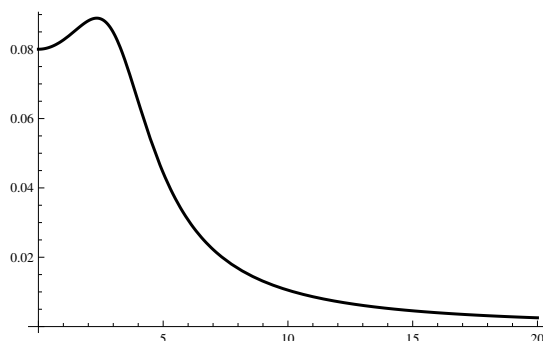
- a. With the values given, the equation is

$$v''_{\text{out}} + \frac{15}{4}v'_{\text{out}} + \frac{25}{2}v_{\text{out}} = \frac{25}{2}v_{\text{in}}.$$

The gain function is thus

$$g(\omega) = \frac{1}{\sqrt{(\frac{25}{2} - \omega^2)^2 + \frac{225}{16}\omega^2}} = \frac{4}{\sqrt{(50 - 4\omega^2)^2 + 225\omega^2}}.$$

- b. The gain function looks like



The maximum of $g(\omega)$ for $\omega \geq 0$ occurs when the denominator is minimized, i.e. when $f(\omega) = (50 - 4\omega^2)^2 + 225\omega^2 = 16\omega^4 - 175\omega^2 + 2500$ is minimized. Then

$$f'(\omega) = 64\omega^3 - 350\omega = 64\omega(\omega - \frac{5\sqrt{14}}{8})(\omega + \frac{5\sqrt{14}}{8}).$$

Because $f''(\omega) = 192\omega^2 - 350$ is negative at $\omega = 0$, we see that 0 is a local maximum for f and thus a local minimum for g . Rejecting the negative root of f' , we are left with the positive root, and $f''(\omega)$ is positive there, so that $\frac{5\sqrt{14}}{8}$ is a local maximum for g .

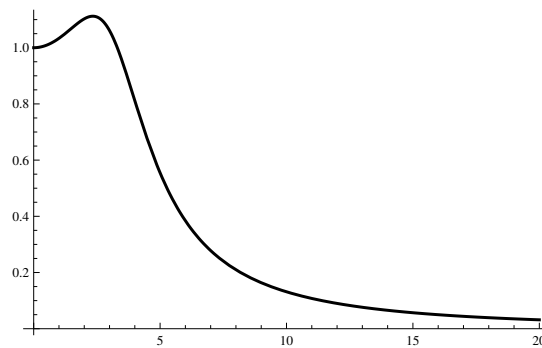
- c. From the graph and the above analysis, it is clear that the gain function is monotonically decreasing for $t \geq \frac{5\sqrt{14}}{8} \approx 2.339$.
- d. From either the text or the preceding exercise, we have

$$\left| \frac{v_{\text{out}}}{v_{\text{in}}} \right| = \frac{g(\omega)}{LC}.$$

Substituting values, we get

$$\left| \frac{v_{\text{out}}}{v_{\text{in}}} \right| = \frac{50}{\sqrt{(50 - 4\omega^2)^2 + 225\omega^2}},$$

so that a plot of the ratio is



D2.5.21

- a. True. The parameters that determine the transfer function are b and ω_0 , neither of which is related to the forcing function.
- b. True. If the oscillation is undamped, then $b = 0$, and then the transfer function is

$$H(\omega) = \frac{1}{\sqrt{(\omega_0^2 - \omega^2)^2 + 0^2\omega^2}} = \frac{1}{|\omega_0^2 - \omega^2|},$$

which is real-valued.

D2.5.22 Start with

$$H(\omega) = \frac{1}{-\omega^2 + ib\omega + \omega_0^2} = \frac{1}{(\omega_0^2 - \omega^2) + ib\omega}.$$

Multiply numerator and denominator by the conjugate of the denominator, $(\omega_0^2 - \omega^2) - ib\omega$, to make the denominator a real number:

$$\begin{aligned} H(\omega) &= \frac{(\omega_0^2 - \omega^2) - ib\omega}{((\omega_0^2 - \omega^2) + ib\omega)((\omega_0^2 - \omega^2) - ib\omega)} \\ &= \frac{(\omega_0^2 - \omega^2) - ib\omega}{(\omega_0^2 - \omega^2)^2 + b^2\omega^2} \\ &= \frac{\omega_0^2 - \omega^2}{(\omega_0^2 - \omega^2)^2 + b^2\omega^2} - i \frac{b\omega}{(\omega_0^2 - \omega^2)^2 + b^2\omega^2}. \end{aligned}$$

D2.5.23

- a. Because a positive fraction with a positive denominator is larger when the denominator is smaller, local maximum values of g clearly occur at local minimum values of the denominator.
- b. The denominator is $\sqrt{h(\omega)}$ where $h(\omega) = (\omega_0^2 - \omega^2)^2 + b^2\omega^2$. But obviously $\sqrt{x}$ is minimized when x is minimized, for $x > 0$. So all we need to do is to minimize $h(\omega)$.
- c. $h'(\omega) = 2(\omega_0^2 - \omega^2) \cdot (-2\omega) + 2b^2\omega = 4\omega^3 + 2b^2\omega - 4\omega_0^2\omega = 2\omega(b^2 - 2\omega_0^2 + 2\omega^2)$.
- d. The extrema of $h(\omega)$ occur at the zeros of $h'(\omega)$; those zeros are (for $\omega \geq 0$)

$$\omega = 0, \quad \omega = \sqrt{\frac{2\omega_0^2 - b^2}{2}} = \sqrt{\omega_0^2 - \frac{b^2}{2}}.$$

Now,

$$h''(\omega) = (4\omega^3 + 2b^2\omega - 4\omega_0^2\omega)' = 12\omega^2 + 2b^2 - 4\omega_0^2 = 12\omega^2 - 4(\omega_0^2 - \frac{b^2}{2}).$$

Then

$$\begin{aligned} h''(0) &= -4(\omega_0^2 - \frac{b^2}{2}), \\ h''(\sqrt{\omega_0^2 - \frac{b^2}{2}}) &= 12 \cdot (\omega_0^2 - \frac{b^2}{2}) - 4(\omega_0^2 - \frac{b^2}{2}) = 8(\omega_0^2 - \frac{b^2}{2}). \end{aligned}$$

The second root of $h'(\omega)$ is real and nonzero if $\omega_0^2 > \frac{b^2}{2}$, i.e. if $b < \sqrt{2}\omega_0$. But in this case, h'' is positive at that root from the preceding equation, so that h has a minimum there and thus g is maximized there. Thus if $b < \sqrt{2}\omega_0$, we see that g has a local maximum at $\omega = \sqrt{\omega_0^2 - \frac{b^2}{2}}$. Its maximum value there is

$$\begin{aligned} g(\sqrt{\omega_0^2 - \frac{b^2}{2}}) &= \frac{1}{\sqrt{(\omega_0^2 - \sqrt{\omega_0^2 - \frac{b^2}{2}})^2 + b^2\sqrt{\omega_0^2 - \frac{b^2}{2}}}} \\ &= \frac{1}{\sqrt{\omega_0^2 - \omega_0^2 + \frac{b^2}{2} + b^2\omega_0^2 - \frac{b^4}{2}}} \\ &= \frac{1}{b\sqrt{\frac{1}{2} - \frac{b^2}{2} + \omega_0^2}}. \end{aligned}$$

- e. If, on the other hand, $b \geq \sqrt{2}\omega_0$, then the only real root of $h'(\omega)$ is at $\omega = 0$. However, in this case $h''(0)$ is -4 times a negative number, so it is positive and h is minimized at 0. Thus g is maximized at 0. Further, recall that

$$h'(\omega) = 2\omega(b^2 - 2\omega_0^2 + 2\omega^2) = 2\omega(2(\frac{b^2}{2} - \omega_0^2) + 2\omega^2).$$

Because $b \geq \sqrt{2}\omega_0$, the first term in the sum is positive. The second term in the sum is always positive. Thus h' is positive everywhere, so that h is increasing everywhere and thus g is decreasing everywhere. So g is monotonically decreasing for $\omega \geq 0$, and has a local maximum at $\omega = 0$.

- f. Part (e) corresponds to strong damping, since in this case any oscillations are completely damped out and the curve monotonically decreases.

D2.5.24

- a. Kirchoff's law says that the sum of the voltage changes at each point in the outer circuit must be zero. This says that

$$v_{\text{in}} - RI - \frac{Q}{C} - v_{\text{out}} = 0.$$

since the voltage drop across the resistor is RI and the voltage drop across the capacitor is $\frac{Q}{C}$ (see Section D2.4).

- b. The voltage across the inductor, which is LI' , must also be v_{out} ; using this together with the fact that $I(t) = Q'(t)$ and thus $I'(t) = Q''(t)$ gives

$$v_{\text{in}} - RQ'(t) - \frac{Q}{C} - LQ'' = 0; \quad \text{rearranging gives} \quad Q'' + \frac{R}{L}Q' + \frac{1}{LC}Q = \frac{1}{L}v_{\text{in}} = \frac{1}{L}fe^{i\omega t}.$$

- c. The transfer function for this equation is

$$H(\omega) = \frac{1}{-\omega^2 + i\frac{R}{L}\omega + \frac{1}{LC}} = \frac{LC}{-LC\omega^2 + iRC\omega + 1}.$$

The gain function is

$$g(\omega) = \frac{1}{\sqrt{(\frac{1}{LC} - \omega^2)^2 + \frac{R^2}{L^2}\omega^2}}.$$

For later use, we also note that (after some computation!)

$$g'(\omega) = \frac{\omega((4L - 2CR^2) - 4CL^2\omega^2)}{2CL^2(\frac{R^2}{L^2}\omega^2 + (\omega^2 - \frac{1}{LC})^2)^{3/2}}$$

so that for $\omega > 0$, g has an extremum for

$$\omega = \sqrt{\frac{2L - CR^2}{2CL^2}}.$$

- d. Apply the general forced oscillator solution in the box on page 1225 to the equation in part (b) to see that the solution is

$$Q(t) = H(\omega)\frac{1}{L}fe^{i\omega t}.$$

Thus

$$Q'(t) = H(\omega)\frac{1}{L}f(i\omega e^{i\omega t}) = i\omega Q(t), \quad \text{and} \quad Q''(t) = i\omega Q'(t) = (i\omega)^2 Q(t) = -\omega^2 Q(t).$$

- e. From part (b), $v_{\text{out}} = LQ''(t)$; from that and part (d), we get

$$v_{\text{out}} = LQ'' = -L\omega^2 Q = -L\omega^2 H(\omega)\frac{1}{L}fe^{i\omega t} = -\omega^2 H(\omega)fe^{i\omega t} = -\omega^2 H(\omega)v_{\text{in}}.$$

- f. From the previous part together with our computation of $g(\omega)$ in part (c), we get

$$\left| \frac{v_{\text{out}}}{v_{\text{in}}} \right| = \left| \frac{-\omega^2 H(\omega)v_{\text{in}}}{v_{\text{in}}} \right| = \omega^2 |H(\omega)| = \omega^2 g(\omega) = \frac{\omega^2}{\sqrt{(\frac{1}{LC} - \omega^2)^2 + \frac{R^2}{L^2}\omega^2}}.$$

g. Divide top and bottom of the ratio from part (f) by ω^2 to get

$$\left| \frac{v_{\text{out}}}{v_{\text{in}}} \right| = \frac{1}{\sqrt{(1 - \frac{1}{LC}\omega^{-2})^2 + \frac{R^2}{L^2}\omega^{-2}}}.$$

Then the ratio is monotonically increasing if and only if the denominator is monotonically decreasing. This happens when the formula inside the square root is monotonically decreasing. If $h(\omega)$ is that formula, then

$$\begin{aligned} h'(\omega) &= 2(1 - \frac{1}{LC}\omega^{-2}) \cdot (-2 \cdot \frac{-1}{LC}\omega^{-3}) - 2\frac{R^2}{L^2}\omega^{-3} \\ &= (\frac{4}{LC} - \frac{2R^2}{L^2})\omega^{-3} - \frac{4}{L^2C^2}\omega^{-5} \\ &= \omega^{-5}((\frac{4}{LC} - \frac{2R^2}{L^2})\omega^2 - \frac{4}{L^2C^2}). \end{aligned}$$

Now, $h(\omega)$ is monotonically decreasing if and only if $h'(\omega)$ is everywhere negative. But if the coefficient of ω^2 is positive, then $h'(\omega) > 0$ for large values of ω . Thus $h(\omega)$ is monotonically decreasing, and thus the ratio $\left| \frac{v_{\text{out}}}{v_{\text{in}}} \right|$ is monotonically increasing, if and only if

$$\frac{4}{LC} - \frac{2R^2}{L^2} \leq 0, \quad \text{or} \quad R \geq \sqrt{\frac{2L}{C}}.$$

h. The ratio has an extremum if its derivative has a zero. The numerator of the derivative of the formula in part (f) is (after some computation)

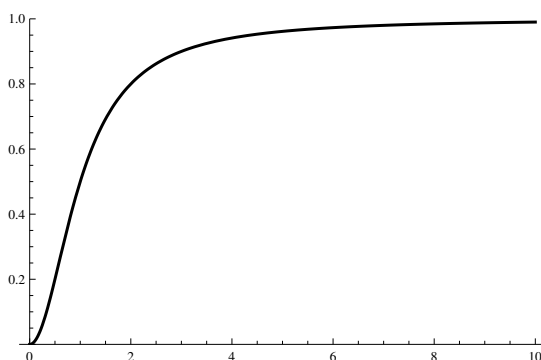
$$\omega((C^2R^2 - 2CL)\omega^2 + 2),$$

which has a root if $C^2R^2 - 2CL < 0$, which occurs if $R < \sqrt{\frac{2L}{C}}$. That root occurs for

$$\omega = \sqrt{\frac{2}{2CL - C^2R^2}}.$$

The other local extremum is at $\omega = 0$.

i. A graph of the ratio of v_{out} to v_{in} for this filter for, say, $L = 1$, $C = 1$, and $R = 2 > \sqrt{2}$ is



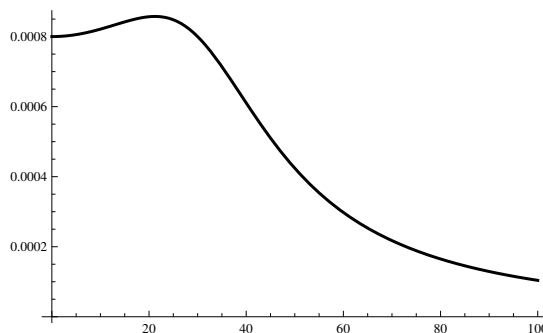
Note that higher frequencies have higher gains, so that low frequencies are removed from the output. The knee of the curve can be adjusted by varying the parameters L , C , and R .

D2.5.25

- a. With $L = 1/5$, $C = 1/250$, and $R = 8$, the gain function is

$$g(\omega) = \frac{1}{\sqrt{(\frac{1}{LC} - \omega^2)^2 + \frac{R^2}{L^2}\omega^2}} = \frac{1}{\sqrt{(1250 - \omega^2)^2 + 1600\omega^2}}.$$

- b. A plot of $g(\omega)$ is

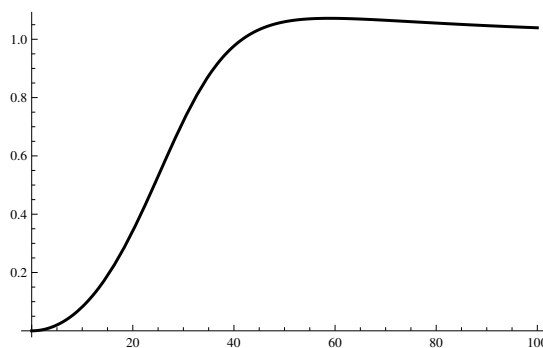


From Exercise 24(c), $g(\omega)$ has a maximum where $\omega = \sqrt{\frac{2L-CR^2}{2CL^2}}$, which for the given values is

$$\sqrt{\frac{\frac{2}{5} - \frac{64}{250}}{\frac{2}{250 \cdot 25}}} = \sqrt{\frac{\frac{36}{250}}{\frac{2}{250 \cdot 25}}} = 15\sqrt{2}.$$

The gain rises to its maximum at $\omega = 15\sqrt{2}$ and then decreases.

- c. A plot of the ratio $\left| \frac{v_{\text{out}}}{v_{\text{in}}} \right|$ is



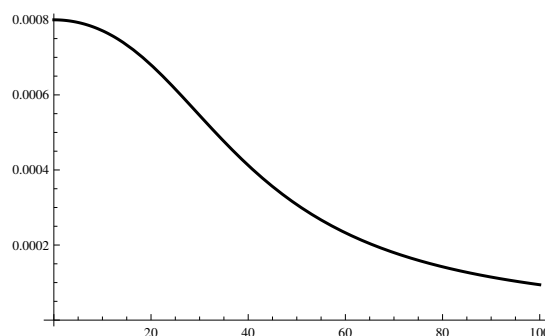
This shows that the lower frequencies, from $\omega = 0$ to about $\omega = 30$, are reproduced with a much lower amplitude than in the input, while higher frequencies are passed through at roughly the same amplitude. Thus higher frequencies are “passed” through.

D2.5.26

- a. With $L = 1/5$, $C = 1/250$, and $R = 12$, the gain function is

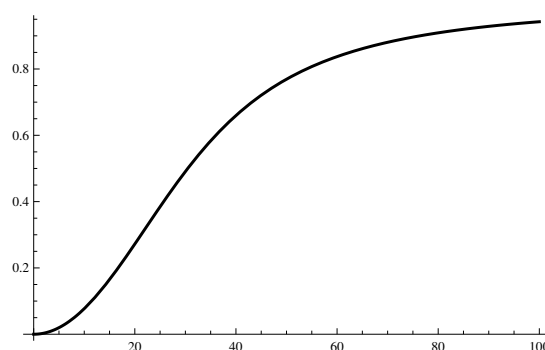
$$g(\omega) = \frac{1}{\sqrt{(\frac{1}{LC} - \omega^2)^2 + \frac{R^2}{L^2}\omega^2}} = \frac{1}{\sqrt{(1250 - \omega^2)^2 + 3600\omega^2}}.$$

- b. A plot of $g(\omega)$ is



From Exercise 24(c), since $2L - CR^2 = \frac{2}{5} - \frac{144}{250} < 0$, we see that $g'(\omega)$ is never zero for $\omega > 0$, so that $g(\omega)$ has no extrema other than $\omega = 0$. The gain is decreasing everywhere.

- c. A plot of the ratio $\left| \frac{v_{\text{out}}}{v_{\text{in}}} \right|$ is



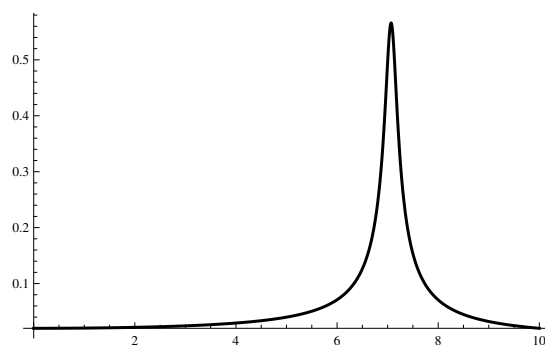
This shows roughly the same behavior as in Exercise 25; however, the rise is much more gradual, so that more lower frequencies will be passed through recognizably, and the falloff is not as sharp.

D2.5.27

- a. With $L = 40$, $C = 1/2000$, and $R = 10$, the gain function is

$$g(\omega) = \frac{1}{\sqrt{(\frac{1}{LC} - \omega^2)^2 + \frac{R^2}{L^2}\omega^2}} = \frac{1}{\sqrt{(50 - \omega^2)^2 + \frac{1}{16}\omega^2}} = \frac{4}{\sqrt{16(50 - \omega^2)^2 + \omega^2}}.$$

- b. A plot of $g(\omega)$ is

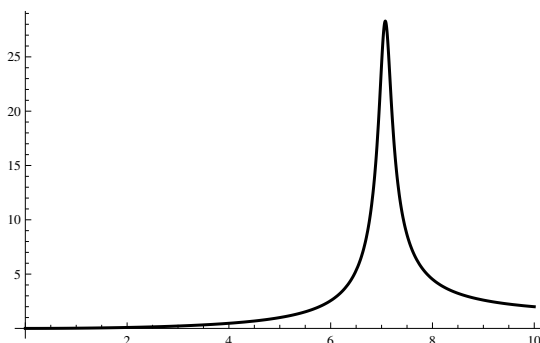


From Exercise 24(c), $g(\omega)$ has a maximum where $\omega = \sqrt{\frac{2L - CR^2}{2CL^2}}$, which for the given values is

$$\sqrt{\frac{80 - \frac{1}{20}}{\frac{8}{5}}} = \sqrt{\frac{\frac{1599}{20}}{\frac{8}{5}}} = \frac{\sqrt{3198}}{8} \approx 7.069.$$

The gain rises sharply to its maximum at $\omega \approx 7.069$ and then decreases sharply.

- c. A plot of the ratio $\left| \frac{v_{\text{out}}}{v_{\text{in}}} \right|$ is



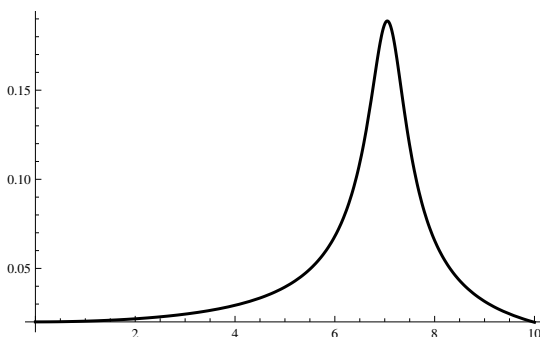
This filter passes frequencies right around its peak with much higher amplitude than those on either side, so it filters out everything but a narrow band around that frequency. Note also the amount of gain in the response curve compared to the input, and how narrow the band is.

D2.5.28

- a. With $L = 40$, $C = 1/2000$, and $R = 30$, the gain function is

$$g(\omega) = \frac{1}{\sqrt{(\frac{1}{LC} - \omega^2)^2 + \frac{R^2}{L^2}\omega^2}} = \frac{1}{\sqrt{(50 - \omega^2)^2 + \frac{9}{16}\omega^2}} = \frac{4}{\sqrt{16(50 - \omega^2)^2 + 9\omega^2}}.$$

- b. A plot of $g(\omega)$ is

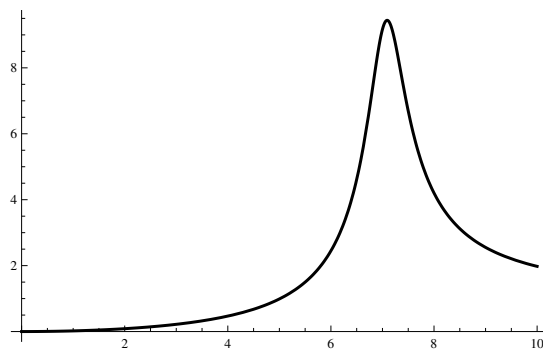


From Exercise 24(c), $g(\omega)$ has a maximum where $\omega = \sqrt{\frac{2L - CR^2}{2CL^2}}$, which for the given values is

$$\sqrt{\frac{80 - \frac{9}{20}}{\frac{8}{5}}} = \sqrt{\frac{\frac{1591}{20}}{\frac{8}{5}}} = \frac{\sqrt{3182}}{8} \approx 7.051.$$

The gain rises sharply to its maximum at $\omega \approx 7.051$ and then decreases sharply.

- c. A plot of the ratio $\left| \frac{v_{\text{out}}}{v_{\text{in}}} \right|$ is



This filter passes frequencies right about its peak with much higher amplitude than those on either side, so it filters out everything but a narrow band around that frequency. However, the rise and fall are not as sharp as in the previous exercise, nor is the gain right around the peak nearly as high.

Chapter D2 Review

1

- a. True. The highest derivative appearing is two, and terms involving y appear by themselves or with coefficients involving only t and constants. Finally, it is homogeneous since every term involves y or one of its derivatives.
- b. False. It is not linear because it involves the term y^2 , which is not linear in y . It is, however, both second order and nonhomogeneous (the latter because of the presence of the term 3).
- c. False. This is a Cauchy-Euler equation; we start with t^p and find p . See the discussion of Cauchy-Euler equations in Section 2.
- d. False. Because the characteristic polynomial of $y'' - 2y' + y = 0$ is $r^2 - 2r + 1 = (r - 1)^2$, the equation has the repeated root 1, so that the general solution of the equation is $c_1 e^t + c_2 t e^t$. For the particular solution, we must start with $A t^2 e^t$.
- e. True. Because the equation is homogeneous, every term involves y or one of its derivatives, so if $y = 0$, every term is zero and the equation is satisfied.

2 The characteristic polynomial is $r^2 + r - 6 = (r + 3)(r - 2)$, which has roots 2 and -3 . Thus the general solution is $c_1 e^{-3t} + c_2 e^{2t}$.

3 The characteristic polynomial is $r^2 - 2r - 8 = (r - 4)(r + 2)$, which has roots -2 and 4. Thus the general solution is $c_1 e^{-2t} + c_2 e^{4t}$.

4 The characteristic polynomial is $r^2 + 8 = (r + 2i\sqrt{2})(r - 2i\sqrt{2})$. Because the real parts of the solutions are zero, the general solution is $c_1 \sin 2\sqrt{2}t + c_2 \cos 2\sqrt{2}t$.

5 The characteristic polynomial is $r^2 + 36 = (r + 6i)(r - 6i)$. Because the real parts of the solutions are zero, the general solution is $c_1 \sin 6t + c_2 \cos 6t$.

6 The characteristic polynomial is $r^2 + 4r + 13$, with roots $-2 \pm 3i$. Thus the general solution is $e^{-2t}(c_1 \sin 3t + c_2 \cos 3t)$.

7 The characteristic polynomial is $r^2 + 4r + 4 = (r + 2)^2$, which has the repeated root -2 . Thus the two linearly independent solutions are e^{-2t} and $t e^{-2t}$, so that the general solution is $c_1 e^{-2t} + c_2 t e^{-2t}$.

8 This is a Cauchy-Euler equation with associated characteristic polynomial $r^2 + (3-1)r + 8 = r^2 + 2r + 8$. The roots of this polynomial are $r = -1 \pm i\sqrt{7}$. Then (see Exercise 64 in Section 2), the general solution is $c_1 t^{-1} \cos(\sqrt{7} \ln t) + c_2 t^{-1} \sin(\sqrt{7} \ln t)$.

9 The characteristic polynomial is $r^2 - 2r + 5$, which has complex roots $1 \pm 2i$. Thus the general solution is $e^t(c_1 \sin 2t + c_2 \cos 2t)$.

10 The characteristic polynomial of the homogeneous equation is $r^2 - 3r - 4 = (r-4)(r+1)$, with roots -1 and 4 . Thus the general solution to the homogeneous equation is $c_1 e^{-t} + c_2 e^{4t}$. Because $2e^{2t}$ is not a solution to the homogeneous equation, we use Ae^{2t} as a trial solution:

$$y'' - 3y' - 4 = 4Ae^{2t} - 3 \cdot 2Ae^{2t} - 4Ae^{2t} = -6Ae^{2t} = 2e^{2t},$$

so that $A = -\frac{1}{3}$. Thus the particular solution is

$$y_p(t) = -\frac{1}{3}e^{2t}.$$

11 The characteristic polynomial of the homogeneous equation is $r^2 + 25 = 0$, with roots $\pm 5i$. Thus the general solution of the homogeneous equation is $c_1 \cos 5t + c_2 \sin 5t$. Because $3 \cos 2t$ is therefore not a solution of the homogeneous equation, we use $A \cos 2t + B \sin 2t$ as a trial solution:

$$\begin{aligned} y'' + 25y &= -4A \cos 2t - 4B \sin 2t + 25A \cos 2t + 25B \sin 2t \\ &= 21A \cos 2t + 21B \sin 2t = 3 \cos 2t. \end{aligned}$$

Solving gives $A = \frac{1}{7}$ and $B = 0$, so that the particular solution is

$$y_p(t) = \frac{1}{7} \cos 2t.$$

12 The characteristic polynomial of the homogeneous equation is $r^2 - r - 2 = (r-2)(r+1)$, which has roots -1 and 2 . Thus the homogeneous equation has a general solution of $c_1 e^{-t} + c_2 e^{2t}$. Because $t^2 + 10$ is not such a solution, we use $At^2 + Bt + C$ as a trial solution:

$$\begin{aligned} y'' - y' - 2y &= (At^2 + Bt + C)'' - (At^2 + Bt + C)' - 2(At^2 + Bt + C) \\ &= 2A - 2At - B - 2At^2 - 2Bt - 2C = -2At^2 - (2A + 2B)t + (2A - B - 2C) \\ &= 3t^2 + 10. \end{aligned}$$

Solving gives $A = -\frac{3}{2}$, $B = \frac{3}{2}$, and $C = -\frac{29}{4}$, so that the particular solution is

$$y_p(t) = -\frac{3}{2}t^2 + \frac{3}{2}t - \frac{29}{4}.$$

13 The characteristic polynomial of the homogeneous equation is $r^2 + 5r - 6 = (r+3)(r-2)$, so that the general solution of the homogeneous equation is $c_1 e^{2t} + c_2 e^{-3t}$. Because neither t nor $2e^{-t}$ is such a solution, we use $At + B + Ce^{-t}$ as a trial solution:

$$\begin{aligned} y'' + 5y' - 6y &= (At + B + Ce^{-t})'' + 5(At + B + Ce^{-t})' - 6(At + B + Ce^{-t}) \\ &= Ce^{-t} + 5A - 5Ce^{-t} - 6At - 6B - 6Ce^{-t} \\ &= -6At + (5A - 6B) - 10Ce^{-t} = t + 2e^{-t}. \end{aligned}$$

Thus $A = -\frac{1}{6}$, $B = -\frac{5}{36}$, and $C = -\frac{1}{5}$, so that the particular solution is

$$y_p(t) = -\frac{1}{6}t - \frac{5}{36} - \frac{1}{5}e^{-t}.$$

14 The characteristic polynomial of the homogeneous equation is $r^2 - r - 6 = (r - 3)(r + 2)$, so that the general solution of the homogeneous equation is $c_1e^{-2t} + c_2e^{3t}$. Because $2e^{-2t}$ satisfies the homogeneous equation, we must use Ate^{-2t} as a trial solution:

$$\begin{aligned} y'' - y - 6y &= (Ate^{-2t})'' - (Ate^{-2t})' - 6Ate^{-2t} \\ &= (Ae^{-2t} - 2Ate^{-2t})' - (Ae^{-2t} - 2Ate^{-2t}) - 6Ate^{-2t} \\ &= -2Ae^{-2t} - 2Ae^{-2t} + 4Ate^{-2t} - Ae^{-2t} - 4Ate^{-2t} \\ &= -5Ae^{-2t} = 2e^{-2t}. \end{aligned}$$

Thus $A = -\frac{2}{5}$, and the particular solution is

$$y_p(t) = -\frac{2}{5}te^{-2t}.$$

15 The characteristic polynomial of the homogeneous equation is $r^2 + 16 = 0$, with complex conjugate roots $\pm 4i$. Thus the general solution of the homogeneous equation is $c_1 \cos 4t + c_2 \sin 4t$. Because $2 \cos 4t$ is a solution of the homogeneous equation, we must use $At \sin 4t + Bt \cos 4t$ as a trial solution:

$$\begin{aligned} y'' + 16y &= (At \sin 4t + Bt \cos 4t)'' + 16At \sin 4t + 16Bt \cos 4t \\ &= (A \sin 4t + 4At \cos 4t + B \cos 4t - 4Bt \sin 4t)' + 16At \sin 4t + 16Bt \cos 4t \\ &= 4A \cos 4t + 4A \cos 4t - 16At \sin 4t - 4B \sin 4t - 4B \sin 4t - 16Bt \cos 4t \\ &\quad + 16At \sin 4t + 16Bt \cos 4t \\ &= -8B \sin 4t + 8A \cos 4t = 2 \cos 4t. \end{aligned}$$

Thus $A = \frac{1}{4}$ and $B = 0$, so that the particular solution is

$$y_p(t) = \frac{1}{4}t \sin 4t.$$

16 The characteristic polynomial of the homogeneous equation is $r^2 - 6r - 7 = (r - 7)(r + 1)$, with real roots -1 and 7 . Thus the general solution of the homogeneous equation is $c_1e^{-t} + c_2e^{7t}$. Because $4e^t$ is not a solution of the homogeneous equation, we use Ae^t as a trial solution:

$$y'' - 6y' - 7y = Ae^t - 6Ae^t - 7Ae^t = -12Ae^t = 4e^t,$$

so that $A = -\frac{1}{3}$. Thus the general solution to the nonhomogeneous equation is

$$y(t) = c_1e^{-t} + c_2e^{7t} - \frac{1}{3}e^t.$$

17 The characteristic polynomial of the homogeneous equation is $r^2 + 4$, with complex conjugate roots $\pm 2i$. Thus the general solution of the homogeneous equation is $c_1 \sin 2t + c_2 \cos 2t$. Because $3 \sin 3t$ is not a solution of the homogeneous equation, we use $A \cos 3t + B \sin 3t$ as a trial solution:

$$\begin{aligned} y'' + 4y &= (A \cos 3t + B \sin 3t)'' + 4A \cos 3t + 4B \sin 3t \\ &= -9A \cos 3t - 9B \sin 3t + 4A \cos 3t + 4B \sin 3t \\ &= -5A \cos 3t - 5B \sin 3t = 3 \sin 3t. \end{aligned}$$

Thus $A = 0$ and $B = -\frac{3}{5}$, so that the general solution to the nonhomogeneous equation is

$$y(t) = c_1 \sin 2t + c_2 \cos 2t - \frac{3}{5} \sin 3t.$$

18 The characteristic polynomial of the homogeneous equation is $r^2 - 2r + 2$, with complex conjugate roots $-1 \pm i$. Thus the general solution of the homogeneous equation is $e^{-t}(c_1 \sin t + c_2 \cos t)$. Because neither 1 nor e^{-t} is a solution of the homogeneous equation, we use $A + Be^{-t}$ as a trial solution:

$$\begin{aligned} y'' - 2y' + 2y &= (A + Be^{-t})'' - 2(A + Be^{-t})' + 2(A + Be^{-t}) \\ &= Be^{-t} + 2Be^{-t} + 2A + 2Be^{-t} \\ &= 2A + 5Be^{-t} = 1 + e^{-t}. \end{aligned}$$

Thus $A = \frac{1}{2}$ and $B = \frac{1}{5}$, so that the general solution to the nonhomogeneous equation is

$$y(t) = e^{-t}(c_1 \sin t + c_2 \cos t) + \frac{1}{2} + \frac{1}{5}e^{-t}.$$

19 The characteristic polynomial of the homogeneous equation is $r^2 + 4r + 5$, with complex conjugate roots $-2 \pm i$. Thus the general solution of the homogeneous equation is $e^{-2t}(c_1 \sin t + c_2 \cos t)$. Because $2 \cos t$ is not a solution of the homogeneous equation, we use $A \sin t + B \cos t$ as a trial solution:

$$\begin{aligned} y'' + 4y' + 5y &= -A \sin t - B \cos t + 4(A \cos t - B \sin t) + 5A \sin t + 5B \cos t \\ &= (4A - 4B) \sin t + (4A + 4B) \cos t = 2 \cos t. \end{aligned}$$

Solving $4A - 4B = 0$ and $4A + 4B = 2$ gives $A = \frac{1}{4}$ and $B = \frac{1}{4}$, so that the general solution to the nonhomogeneous equation is

$$y(t) = e^{-2t}(c_1 \sin t + c_2 \cos t) + \frac{1}{4}(\sin t + \cos t).$$

20 The characteristic polynomial of the homogeneous equation is $r^2 + 4r - 32 = (r + 8)(r - 4)$, with roots 4 and -8 . Thus the general solution of the homogeneous equation is $c_1 e^{4t} + c_2 e^{-8t}$. Because neither 1 nor $2t$ is a solution of the homogeneous equation, we use $At + B$ as a trial solution:

$$\begin{aligned} y'' + 4y' + 5y &= (At + B)'' + 4(At + B)' - 32(At + B) \\ &= 4A - 32At - 32B = -32At + (4A - 32B) = 2t + 1. \end{aligned}$$

Thus $A = -\frac{1}{16}$ and $B = -\frac{5}{128}$, so that the general solution to the nonhomogeneous equation is

$$y(t) = c_1 e^{4t} + c_2 e^{-8t} - \frac{1}{16}t - \frac{5}{128}.$$

21 The characteristic polynomial of the homogeneous equation is $r^2 - 1 = (r - 1)(r + 1)$, with roots ± 1 . Thus the general solution of the homogeneous equation is $c_1 e^{-t} + c_2 e^t$. Because $2e^t$ is a solution of the homogeneous equation, we use Ate^t as a trial solution:

$$y'' - y = (Ate^t)'' - Ate^t = (Ae^t + Ate^t)' - Ate^t = Ae^t + Ae^t + Ate^t - Ate^t = 2Ae^t = 2e^t.$$

Thus $A = 1$, so that the general solution to the nonhomogeneous equation is

$$y(t) = c_1 e^{-t} + c_2 e^t + te^t.$$

22 This is a system with forced undamped oscillations and an external force in the form of a cosine curve, so the equation is

$$y'' + \omega_0^2 y = \frac{F_0}{m} \cos \omega t$$

where m is the mass, F_0 is the amplitude of the external force, ω_0 is the natural frequency of the system, and ω is the frequency of the external force. The general solution to this equation is

$$y(t) = c_1 \sin \omega_0 t + c_2 \cos \omega_0 t + \frac{F_0/m}{\omega_0^2 - \omega^2} \cos \omega t.$$

In our problem, $F_0 = 4$, $m = 4$ and $\omega_0^2 = \frac{k}{m} = \frac{16}{4} = 4$, so that $\omega_0 = 2$, and we have

$$y(t) = c_1 \sin 2t + c_2 \cos 2t + \frac{1}{4 - \omega^2} \cos \omega t.$$

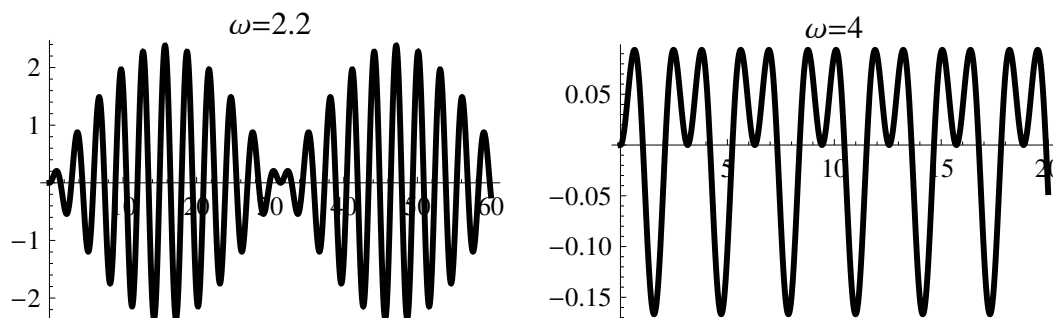
Because $y(0) = c_2 + \frac{1}{4 - \omega^2} = 0$, we get $c_2 = -\frac{1}{4 - \omega^2}$. Also,

$$y'(t) = 2c_1 \cos 2t - 2c_2 \sin 2t - \frac{\omega}{4 - \omega^2} \sin \omega t,$$

so that $y'(0) = 2c_1 = 0$ and thus $c_1 = 0$. So the solution to the system is

$$y(t) = \frac{1}{4 - \omega^2} (\cos \omega t - \cos 2t).$$

Graphs for $\omega = 1.5$ (for $0 \leq t \leq 100$) and for $\omega = 4$ (for $0 \leq t \leq 20$) are:



This clearly shows beats for $\omega = 2.2$ and no beats for $\omega = 4$.

23 From the text, the general differential equation for this situation is

$$y'' + \frac{c}{m}y' + \omega^2 y = 0,$$

where in our case $m = 0.2$ and $\omega^2 = \frac{k}{m} = \frac{20}{0.2} = 100$, so that $\omega = 10$. Thus the equation is

$$y'' + 5cy' + 100y = 0.$$

The characteristic polynomial is $r^2 + 5cr + 100$, with roots

$$\frac{-5c \pm \sqrt{25c^2 - 400}}{2}.$$

- a. When $c = 3$, we have $\sqrt{25c^2 - 400} = \sqrt{-175} = 5i\sqrt{7}$, so that the roots of the characteristic polynomial are the complex conjugates

$$-\frac{5 \cdot 3}{2} \pm \frac{5\sqrt{7}}{2},$$

and thus the general solution is

$$y(t) = e^{-15t/2} \left(c_1 \cos \frac{5\sqrt{7}}{2}t + c_2 \sin \frac{5\sqrt{7}}{2}t \right).$$

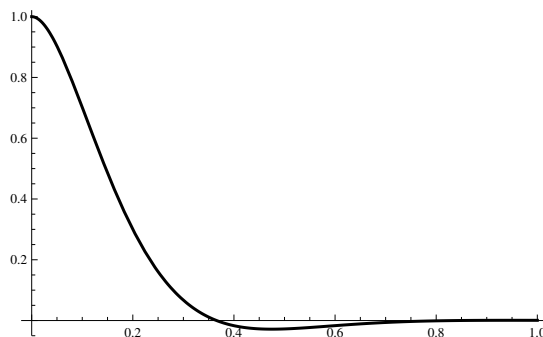
Then $y(0) = c_1 = 1$, so that $c_1 = 1$. Further,

$$\begin{aligned} y'(t) &= (e^{-15t/2} (\cos \frac{5\sqrt{7}}{2}t + c_2 \sin \frac{5\sqrt{7}}{2}t))' \\ &= -\frac{15}{2} e^{-15t/2} (\cos \frac{5\sqrt{7}}{2}t + c_2 \sin \frac{5\sqrt{7}}{2}t) \\ &\quad + e^{-15t/2} (-\frac{5\sqrt{7}}{2} \sin \frac{5\sqrt{7}}{2}t + \frac{5\sqrt{7}}{2} c_2 \cos \frac{5\sqrt{7}}{2}t), \end{aligned}$$

so that $y'(0) = -\frac{15}{2} + \frac{5\sqrt{7}}{2}c_2 = 0$. Thus $c_2 = \frac{3\sqrt{7}}{7}$, and the solution to the system with $c = 3$ is

$$y(t) = e^{-15t/2} \left(\cos \frac{5\sqrt{7}}{2}t + \frac{3\sqrt{7}}{7} \sin \frac{5\sqrt{7}}{2}t \right)$$

with graph



Because the discriminant was negative, this is an underdamped system. There are visible oscillations in the solution, but they damp out pretty quickly because of the high multiple of t in the exponential.

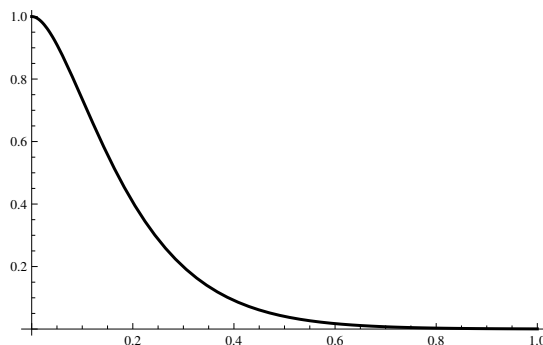
- b. With $c = 4$, we have $\sqrt{25c^2 - 400} = 0$, so that the characteristic polynomial has the repeated root $r = -\frac{5 \cdot 4}{2} = -10$. In this case the general solution is

$$y(t) = c_1 e^{-10t} + c_2 t e^{-10t}.$$

Then $y(0) = c_1 = 1$, so that $c_1 = 1$. Further, $y'(t) = -10c_1 e^{-10t} + c_2 e^{-10t} - 10c_2 t e^{-10t}$, so that $y'(0) = -10c_1 + c_2 = 0$ and thus $c_2 = 10$. Hence the solution to the system when $c = 4$ is

$$y(t) = e^{-10t} + 10t e^{-10t}$$

with graph



Because the discriminant is zero, this system has critical damping.

- c. With $c = 5$, we have $\sqrt{25c^2 - 400} = \sqrt{225} = 15$, so that the characteristic polynomial has the distinct roots

$$\frac{-5 \cdot 5 \pm 15}{2} = \{-5, -20\}.$$

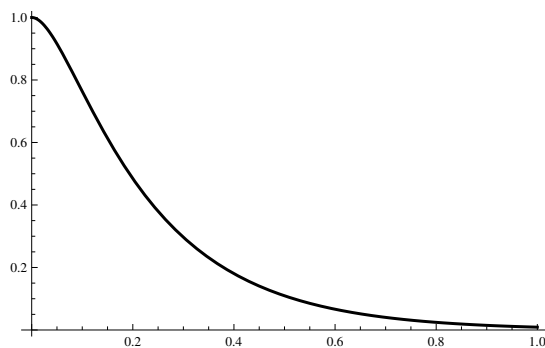
Thus the general solution in this case is

$$y(t) = c_1 e^{-5t} + c_2 e^{-20t}.$$

Then $y(0) = c_1 + c_2 = 1$. Also, $y'(t) = -5c_1e^{-5t} - 20c_2e^{-5t}$, so that $y'(0) = -5c_1 - 20c_2 = 0$. Solving these simultaneous equations gives $c_1 = \frac{4}{3}$ and $c_2 = -\frac{1}{3}$, so that the solution to the system for $c = 5$ is

$$y(t) = \frac{4}{3}e^{-5t} - \frac{1}{3}e^{-20t},$$

with graph



Because the discriminant is positive, this system is overdamped.

24

- a. The general equation is $y'' + by' + \omega_0^2 y = F_0 \cos \omega t$. The solution takes the form of a transient solution, which decays as $t \rightarrow \infty$, and a particular solution to the nonhomogeneous equation, which is a steady-state solution of the form $A \sin \omega t + B \cos \omega t$ (as long as $\omega \neq \omega_0$). The form of the transient solution is dictated by the values of b and ω_0 . For the problem at hand,

$$b = \frac{c}{m} = \frac{40}{10} = 4, \quad \omega_0^2 = \frac{k}{m} = \frac{62.5}{10} = \frac{25}{4}.$$

Thus the equation is $y'' + 4y' + \frac{25}{4}y = 10 \cos 2t$. The characteristic polynomial is $r^2 + 4r + \frac{25}{4}$, with roots $-2 \pm \frac{3}{2}i$, so that the general solution to the homogeneous equation is $y(t) = e^{-2t}(c_1 \cos \frac{3}{2}t + c_2 \sin \frac{3}{2}t)$. Because $10 \cos 2t$ is not a solution to the homogeneous equation, we take $A \sin 2t + B \cos 2t$ as a trial solution, and compute

$$\begin{aligned} y'' + 4y' + \frac{25}{4}y &= (A \sin 2t + B \cos 2t)'' + 4(A \sin 2t + B \cos 2t)' + \frac{25}{4}(A \sin 2t + B \cos 2t) \\ &= -4A \sin 2t - 4B \cos 2t + 8A \cos 2t - 8B \sin 2t + \frac{25}{4}A \sin 2t + \frac{25}{4}B \cos 2t \\ &= \left(\frac{9}{4}A - 8B\right) \sin 2t + \left(8A + \frac{9}{4}B\right) \cos 2t = 10 \cos 2t. \end{aligned}$$

Thus $\frac{9}{4}A - 8B = 0$ and $8A + \frac{9}{4}B = 10$, so that $A = \frac{256}{221}$ and $B = \frac{72}{221}$. It follows that the general solution to the nonhomogeneous equation, incorporating this particular solution, is

$$y(t) = e^{-2t}(c_1 \cos \frac{3}{2}t + c_2 \sin \frac{3}{2}t) + \frac{1}{221}(256 \sin 2t + 72 \cos 2t).$$

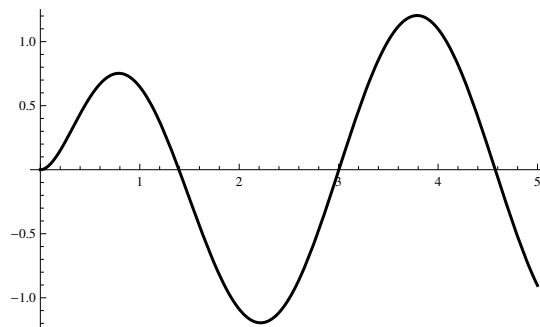
Then $y(0) = c_1 + \frac{72}{221} = 0$, so that $c_1 = -\frac{72}{221}$. Also,

$$\begin{aligned} y'(t) &= -2e^{-2t}(c_1 \cos \frac{3}{2}t + c_2 \sin \frac{3}{2}t) + e^{-2t}(-\frac{3}{2}c_1 \sin \frac{3}{2}t + \frac{3}{2}c_2 \cos \frac{3}{2}t) \\ &\quad + \frac{1}{221}(512 \cos 2t - 144 \sin 2t), \end{aligned}$$

so that $y'(0) = -2c_1 + \frac{3}{2}c_2 + \frac{512}{221} = 0$, so that $c_2 = -\frac{1312}{663}$. Thus the solution to the system is

$$y(t) = e^{-2t}\left(-\frac{72}{221} \cos \frac{3}{2}t - \frac{1312}{663} \sin 2t\right) + \frac{1}{221}(256 \sin 2t + 72 \cos 2t).$$

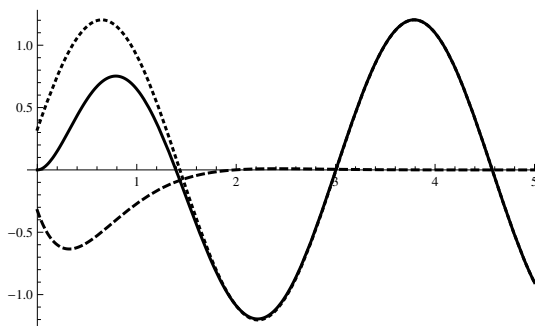
A plot of this solution is



- b. The transient and steady-state solutions are

$$\begin{aligned} y_{\text{transient}} &= e^{-2t}\left(-\frac{72}{221} \cos \frac{3}{2}t - \frac{1312}{663} \sin 2t\right) \\ y_{\text{ss}} &= \frac{1}{221}(256 \sin 2t + 72 \cos 2t). \end{aligned}$$

All three solutions (with $y(t)$ the solid line, $y_{\text{transient}}$ the dashed line, and y_{ss} the dotted line) look like



- c. The transient solution becomes negligible after about 2 seconds, and the curves for $y(t)$ and $y_{ss}(t)$ are indistinguishable after about that time.

25

- a. The general equation for an LCR circuit is

$$I'' + \frac{R}{L}I' + \frac{1}{LC}I = \frac{1}{L}E'(t).$$

In the case given, we have $R = 10$, $L = 0.1$, $C = \frac{1}{240}$, and $E(t) = 200 \sin 60t$, so that $E'(t) = 12000 \cos 60t$ and the differential equation becomes

$$I'' + 100I' + 2400I = 120000 \cos 60t.$$

The homogeneous equation has characteristic polynomial $r^2 + 100r + 2400 = (r + 40)(r + 60)$, with roots -40 and -60 . Thus the general solution to the homogeneous problem is $c_1 e^{-40t} + c_2 e^{-60t}$. We use $A \cos 60t + B \sin 60t$ as a trial solution for the nonhomogeneous problem:

$$\begin{aligned} I'' + 100I' + 2400I &= -3600A \cos 60t - 3600B \sin 60t - 6000A \sin 60t + 6000B \cos 60t \\ &\quad + 2400A \cos 60t + 2400B \sin 60t \\ &= (6000B - 1200A) \cos 60t - (1200B + 6000A) \sin 60t \\ &= 120000 \cos 60t. \end{aligned}$$

Thus (dividing through by 100), we get $60B - 12A = 1200$ and $12B + 60A = 0$. Solving gives for the general solution

$$I(t) = c_1 e^{-40t} + c_2 e^{-60t} - \frac{50}{13} \cos 60t + \frac{250}{13} \sin 60t.$$

We are given that $I(0) = 0$. To determine $I'(0)$, use the fact that $Q(0) = 0$ and evaluate the equation

$$LI' + RI + \frac{1}{C}Q = E(t)$$

from the text at $t = 0$ to get $0.1I'(0) + RI(0) + \frac{1}{C}Q(0) = E(0)$, or $0.1I'(0) = 0$. Thus $I'(0) = 0$ as well. Differentiating $I(t)$ gives

$$I'(t) = -40c_1 e^{-40t} - 60c_2 e^{-60t} + \frac{3000}{13} \sin 60t + \frac{15000}{13} \cos 60t.$$

Thus

$$I(0) = c_1 + c_2 - \frac{50}{13} = 0, \quad I'(0) = -40c_1 - 60c_2 + \frac{15000}{13} = 0.$$

Solving for c_1 and c_2 gives for a solution to the initial value problem

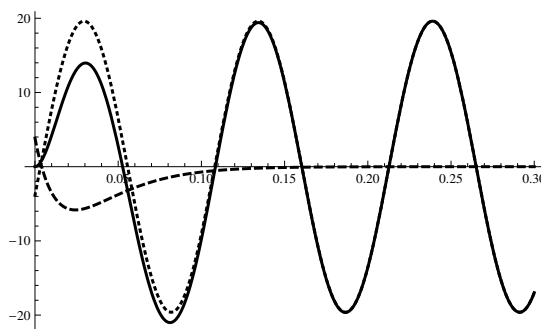
$$I(t) = -\frac{600}{13} e^{-40t} + 50 e^{-60t} - \frac{50}{13} \cos 60t + \frac{250}{13} \sin 60t.$$

b. The transient and steady-state currents are

$$I_{\text{transient}} = -\frac{600}{13}e^{-40t} + 50e^{-60t},$$

$$I_{\text{ss}} = -\frac{50}{13}\cos 60t + \frac{250}{13}\sin 60t.$$

c. All three solutions (with $I(t)$ the solid line, $I_{\text{transient}}$ the dashed line, and I_{ss} the dotted line) look like



d. From the equation in part (a), the gain and phase lag functions are

$$g(\omega) = \frac{1}{\sqrt{((2400 - \omega^2)^2 + 100^2\omega^2)}}, \quad \tan \gamma(\omega) = -\frac{100\omega}{2400 - \omega^2}.$$

Thus for $\omega = 60$ as in the given problem, we get $g(60) = \frac{1}{1200\sqrt{26}} \approx 0.00016343$ and also $\tan \gamma(60) = 5$. Thus γ is the third-quadrant angle with tangent 5, which is $\gamma(60) \approx -1.76819$. So, finally, the solution for the steady-state portion of the equation is

$$I_{\text{ss}} = \frac{1}{1200\sqrt{26}} \cdot 120000 \cos(60t - 1.76819) = \frac{50\sqrt{26}}{13} \cos(60t - 1.76819).$$

To see that this is the same as the answer from part (b), first compute the amplitude of the previous answer:

$$A = \frac{\sqrt{50^2 + 250^2}}{13} = \frac{\sqrt{65000}}{13} = \frac{50\sqrt{26}}{13}.$$

Next, the phase from the answer in part (b) is $-\frac{250/13}{-50/13} = 5$, so we get the same phase angle (since numerator and denominator are both negative, this is a third-quadrant angle). Thus the two solutions are the same.